Tr. J. of Mathematics
21 (1997) , 47 – 53.

Proceedings of 5^{th} Gökova
Geometry-Topology Conference

Exotic structures and adjunction inequality

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1. Introduction

Here we want to reprove and strengthen some old difficult theorems of 4 manifolds by the aid of recently proven modern tools. One of the important recent results of smooth 4-manifolds is Eliashberg's topological description of compact Stein manifolds, that is complex manifolds which admit strictly plurisubharmonic Morse function (PC - manifolds):

Theorem 1.1. *([E]) Let $X = B^4 \cup$ (1-handles) $\cup$ (2-handles) be four-dimensional handlebody with one 0-handle and no 3- or 4-handles. Then*

- *The standard PC-structure on B^4 can be extended over 1-handles so that manifold $X_1 = B^4 \cup$ (1-handles) has pseudo-convex boundary.*
- *If each 2-handle is attached to ∂X_1 along a Legendrian knot with framing one less then Thurston-Bennequin framing of this knot then the symplectic form and complex structure on X_1 can be extended over 2-handles to a symplectic form on X, which makes X a PC manifold.*

Lisca and Matić showed that PC manifolds imbed naturally into Kähler surfaces:

Theorem 1.2. *([LM]) Every PC manifold X can be holomorphically embedded as a domain into a minimal Kähler surface S with ample canonical bundle and $b_2^+(S) > 1$, such that the induced Kähler form on X agrees with the symplectic form of the PC structure.*

Theorem 1.3. *A minimal Kähler surface X, with $b_2^+(X) > 1$ and an ample canonical bundle, can not contain a smoothly embedded 2-sphere $\Sigma \subset X$ with $\Sigma.\Sigma \geq -1$.*

This theorem roughly follows from the fact that Kähler surfaces have nonzero Seiberg-Witten invariants (see [B] and [MF] in case $\Sigma.\Sigma = -1$, and [FS] in case $\Sigma.\Sigma \geq 0$). Also Kähler surfaces satisfy the adjunction inequality of Kronheimer and Mrowka:

Theorem 1.4. *([KM1], [MST]) Let X be a closed smooth 4-manifold with $b_2^+(X) > 1$, with a nonzero Seiberg Witten invariant (e.g. X Kähler surface) corresponding to the line bundle $L \to X$. Let $\Sigma \subset X$ be a compact oriented embedded surface with $\Sigma.\Sigma \geq 0$ when Σ is not a sphere, then*

$$2g(\Sigma) - 2 \geq \Sigma.\Sigma + |c_1(L).\Sigma|$$

The first author was partially supported by an NSF grant DMS-9626204

After discussing the basics, we will show some 4-manifold theorems can be obtained as easy corollaries of these three basic theorems.

2. Definitions

Let us recall some basic facts (e.g. [G], [E]). Any PC-manifold X induces a contact structure on the boundary 3-manifold $Y = \partial X$, which is isomorphic to the restriction of the dual K^* of the canonical line bundle $K \to X$. Furthermore if α is an oriented Legendrian knot in Y (a knot whose tangents lie in the contact planes) bounding an oriented surface F in X, then the "rotational number" is defined to be the relative Chern class $rot(\alpha, F) = c_1(K^*, v)$ of the induced 2-plane bundle $K^* \to F$ with respect to the tangent vector field v of α (i.e. the obstruction to extending v to a section of K^* over F). Also this contact structure gives the so called Thurston-Bennequin framing $tb(\alpha)$.

The simplest example of a PC-manifold is $B^4 \subset \mathbf{C}^2$ with the induced symplectic structure. We choose coordinates in $\mathbf{R}^3 \subset S^3 = \partial B^4$, so that the induced contact structure ξ_0 on $\mathbf{R}^3$ is the kernel of the form $\lambda_0 = dz + xdy$. By Theorem 1.1 this *PC*-structure extends across 1-handles attached to B^4; we draw the attaching balls of each 1-handle on the plane $\{x = 0, z = \text{constant}\}$. Any link L in S^3 can be isotoped to a Legendrian link, we can achieve this by first isotoping L so that all crossings are left handed as in the first picture of Figure 1. For example, we can turn a right handed crossing to a left handed crossing by the local isotopy as in the second picture of Figure 1.

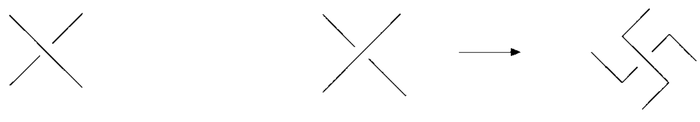

FIGURE 1

Projection of a Legendrian link to the yz-plane has intersections only coming from left handed crossing, no vertical tangencies and all minima and maxima in y-direction are cusps (as the projection of the knot in Figure 3 to the yz-plane). Moreover, every projection with these properties is a projection of some Legendrian link. Rotational number of an oriented Legendrian knot α in S^3 does not depend the surface $F \subset S^3$ it bounds. Invariants $rot(K)$ and $tb(K)$ can be calculated by:

$$rot(\alpha) = 1/2(\text{ Number of "downward" cusps } - \text{ Number of "upward" cusps })$$

$$tb(\alpha) = bb(\alpha) - c(\alpha)$$

where $bb(\alpha)$ is the blackboard (yz-plane) framing of the projection of α, $c(\alpha)$ is the number of right cusps, and "downward" and "upward" cusps are calculated in the obvious way. Because the local isotopy in Figure 1 introduces one right cusp (hence a -1 contribution to calculation of tb(K)) we can incorporate this to "self crossing number" calculation by reading the crossings numbers in a modified way as in in Figure 2. For example in the knot of Figure 3 we can calculate $\mathrm{tb}(K) = 5 - 4 = 1$ and $\mathrm{rot}(K) = 0$. A useful corollary to these theorems is the following generalization of the Bennequin inequality:

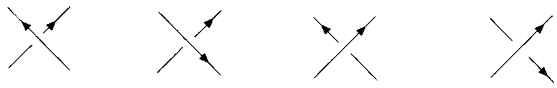

FIGURE 2

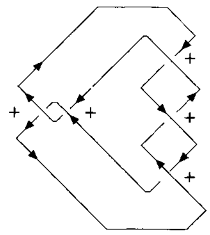

FIGURE 3

Corollary 2.1. *Let X be a PC manifold, $F \subset X$ be a two-dimensional submanifold of X, such that $\alpha = \partial F \subset \partial X$ is Legendrian with respect to induced contact structure and f is framing on α induced by a trivialization of the normal bundle of F in X, then*

$$-\chi(F) \geq [tb(\alpha) - f] + |rot(\alpha)|$$

where $tb(\alpha)$ and $rot(\alpha)$ are the Thurston-Bennequin framing and rotational number of α

To prove this we attach a 2- handle to X along α with the framing $tb(\alpha) - 1$, and apply Theorems 1.1, 1.2, and 1.3 to the resulting manifold and closed surface $F' = F \cup_\partial D$, where D is the core 2-disc of the 2-handle.

3. Applications

Theorem 3.1. *([A1]) Let W be the contractible manifold of Figure 4. Let $f : \partial W \to \partial W$, be the involution induced by an involution of S^3 (as described in [A1]) with $f(\gamma) = \gamma'$, where γ and γ' are circles in ∂W shown on Figure 4. Then $f : \partial W \to \partial W$ does not extend to a diffeomorphism $F : W \to W$ (but it extends to a homeomorphism)*

Proof. By applying Theorem 1.1 to W (the second picture of W in Figure 4) we see that W has a PC structure. Also since γ is slice in W if f extended to a diffeomorphism $F : W \to W$, then γ' would be slice also. But this contradicts the inequality of Theorem 2.1 (here $F = D^2$, $f = 0$, and $tb(\gamma') = 0$). □

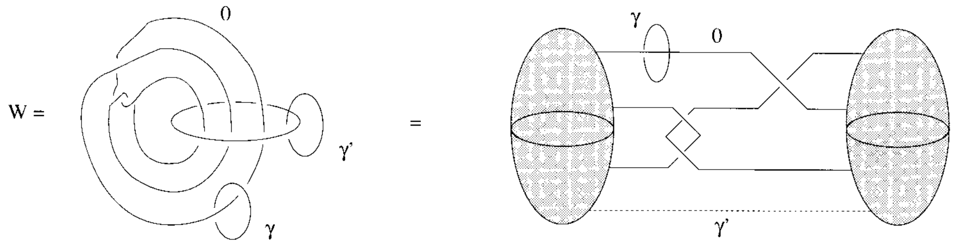

FIGURE 4

Theorem 3.2. *([A2]) Let Q_1 and Q_2 be the manifolds obtained by attaching 2-handles to B^4 along the knots K_1 and K_2 with -1 framings as in Figure 5, then Q_1 and Q_2 are homeomorphic but not diffeomorphic to each other, even interiors are not diffeomorphic to each other.*

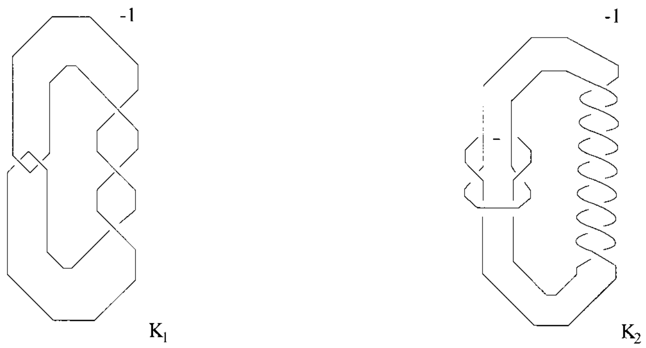

FIGURE 5

Proof. By Theorem 1.1 Q_1 has a *PC* structure, by Theorem 1.2 Q_1 is a domain in a minimal Kähler surface S with $b_2^+(S) > 1$. If Q_1 were diffeomorphic to Q_2, the generator of $H_2(Q_1; \mathbf{Z})$ would be represented by a smooth embedded sphere (since K_2 is a slice knot the generator of $H_2(Q_1; \mathbf{Z})$ represented by a smooth sphere) with -1 self intersection, violating Theorem 1.3. □

Let $K \subset S^3$ be a Legendrian knot, and K_0' be a 0-push off of K (the zero framing is the framing induced from the normal vector field of K in the oriented surface in S^3 bounding K).

Proposition 3.3. *We can move K_0' to a Legendrian knot K_0 by an isotopy which fixes K, such that $tb(K_0) = -|tb(K)|$.*

Proof. $tb(K) = bb(K) - c(K)$. If $bb(K) \leq 0$ then K_0' is just the blackboard push-off of K with $-2bb(K)$ right half twist, but this is a projection of a Legendrian link and $bb(K_0) = bb(K)$ and $c(K_0) = c(K)$ hence $tb(K_0) = tb(K)$ If $tb(K) \leq 0$ and $bb(K) > 0$,

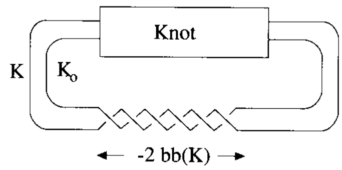

FIGURE 6

then K_0' is the blackboard push-off with $2bb(K)$ left handed half twist. Knot K has $2c(K)$ cusps, and $2c(K) \geq 2bb(K)$. We can produce a Legendrian picture by placing all half twists near cusps as in Figure 7. Thus $tb(K_0) = tb(K)$.

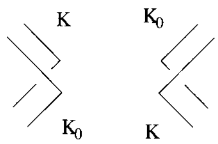

FIGURE 7

If $tb(K) > 0$, then $bb(K) > 0$ and again K_0' is the blackboard push-off with $2bb(K)$ left half twist. By above trick of placing half left twist near cusps, we can get rid of $2c(K)$ left twist, and we are left with $2bb(K) - 2c(K)$ left twist each of which contributes -1 to $tb(K_0)$ as in Figure 8. Hence $tb(K_0) = tb(K) - [\, 2bb(K) - 2c(K)\,] = tb(K) - 2tb(K) = -tb(K)$

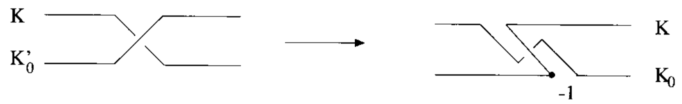

FIGURE 8

□

By Proposition 3.3 we can produce many 0-push offs K_0^i, $i = 1, 2, ..k$ such that $tb(K_0^i) = -|tb(K)|$. So if $tb(K) \leq 0$ then all K_0^i have same tb, if $tb(K) > 0$ then all K_0^i but the original knot K has the same tb.

Theorem 3.4. *If $K \subset S^3$ a Legendrian knot with $tb(K) \geq 0$, then all iterated positive Whitehead doubles of $Wh_n(K)$ are not slice. In fact if $Q^r_n(K)$ is the manifold obtained by attaching a 2-handle to B^4 along $Wh_n(K)$ with a framing $-1 \leq r \leq 0$, then there is no smoothly embedded 2-sphere in $Q^r_n(K)$ representing the generator of $H_2(Q^r_n(K); \mathbf{Z})$.*

Proof. The first positive Whitehead double is obtained by connecting K and K_0 by a left handed cusp as in Figure 9 which contributes $+1$ to tb, hence

$$tb(Wh(K)) = tb(K) + tb(K_0) + 1 = tb(K) - tb(K) + 1 = 1.$$

So by iteration we get $tb(Wh_n(K)) = 1$. But Corollary 2.1 says that any slice knot L must have $tb(L) \leq -1$.

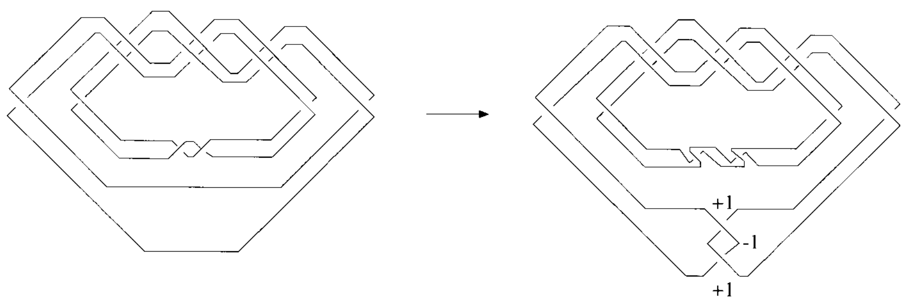

FIGURE 9

For the second part, observe that since $r \leq 0 = tb(Wh_n(K)) - 1$, by Theorem 1.1 $Q^r_n(K)$ is PC. By Theorem 1.2 we can imbed $Q^r_n(K)$ into Kähler surface S. Then by Theorem 1.3 we can not have a smoothly embedded 2-sphere $\Sigma \subset Q^r_n(K) \subset S$ representing $H_2(Q^r_n(K); \mathbf{Z})$, since $\Sigma.\Sigma = r \geq -1$. □

Remark 3.1. L.Rudolf has previously shown that $Wh_n(K)$ are not slice if $tb(K) \geq 0$ ([R]).

Remark 3.2. Proof of Proposition 3.2 gives: If K'_r is the r-framing push off of a knot K, then K'_r can be isotoped to a Legendrian knot K_r fixing K with $(tb(K_r) - r) = -|tb(K) - r|$

Remark 3.3. If ξ is the contact structure on $\Sigma = \partial W$ induced by the PC manifold W of Theorem 3.1, and $f^*(\xi)$ is the "pull-back" contact structure on Σ, then it follows that the contact structures ξ and $f^*(\xi)$ are homotopic through 2-plane fields but not isotopic through contact structures ([AM]).

References

[A1] S. Akbulut, A Fake compact contractible 4-manifold. *J. Diff. Geom* Vol. 33 (1991) pp. 335-356.
[A2] S. Akbulut, An exotic 4-manifold *J. Diff. Geom* Vol. 33 (1991) pp. 357-361.
[AM] S. Akbulut and R. Matveyev, A note on Contact Structures (to appear Pac. Jour. of Math).

[B] R. Brussee, The Kähler class and the C^∞ properties of Kähler surfaces, New York Journal of Mathematics volume 2 1996 (also in alg-geom 9503004).

[G] R.Gompf, Handlebody construction of Stein surfaces (preprint).

[E] Y. Eliashberg, Topological characterization of Stain manifolds of dimension > 2. *International J. of Math.* Vol. 1, No 1 (1990) pp.29-46.

[FS] R. Fintushel and R. Stern, Immersed spheres in 4-manifolds and the immersed Thom Conjecture, Proceedings of the 3^{rd} Gokova Geometry-Topology Conference, 27-39 (1994).

[KM1] P. Kronheimer and T. Mrowka, On genus of embedded surfaces in the projective plane, Math. Res. Letters 1 (1994), 797-808.

[KM2] P. Kronheimer and T. Mrowka, Monopoles and contact structures (to appear).

[LM] P. Lisca and G. Matić, Tight contact structures and Seiberg Witten invariants.

[MF] J. Morgan and R. Friedman, Algebraic surfaces and Seiberg-Witten invariants (preprint).

[MST] J. Morgan, Z.Szabo, C.H. Taubes, The generalized Thom conjecture (to appear).

[R] L. Rudolf, An obstruction to sliceness via contact geometry and "classical" gauge theory, Invent.math. 119, 155-163 (1995).

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