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A NOTE ON FINITE HYPERBOLIC PLANES OBTAINED FROM PROJEKTIVE PLANES

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Abstract

Let Π be a finite projective plane of order n and $\mathcal{M}$ be a set, $|\mathcal{M}| = m$, of any lines of Π which contains three non-concurrent lines. Consider the hyperbolic plane Π_m obtained from Π by removing all lines (including all points on them) of $\mathcal{M}$. In this paper, we obtain larger values than the known maximum value of m and determine the line classes of some hyperbolic planes of type Π_m. Furthermore we give an answer to a question in Bumcrot [1] about hyperbolic planes containing two-point liens.

1. Preliminary Definitions and Propositions

An incidence structure is an ordered triple of sets $(\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P} \cap \mathcal{L} = \emptyset, I \subset \mathcal{P} \times \mathcal{L}$. For P in $\mathcal{P}$ and l in $\mathcal{L}, P\ I\ l$ is read "P is on l". We also write this $l\ I\ P$.

Definition 1.1. *A linear space is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$ satisfying:*

L1. Each two distinct points are on exactly one line.

L2. Each line is on at least two points.

Definition 1.2. *A hyperbolic plane is a linear space $(\mathcal{P}, \mathcal{L}, I)$ satisfying:*

H1. Through each point P, not on a line l, there pass at least two lines not meeting l.

H2. There exist at least four points, no three of which are collinear.

H3. If a subset S of $\mathcal{P}$ contains three non-collinear points and contains all points on the lines through pairs of distinct points of S, then S contains all points of $\mathcal{P}$.

If $S = (\mathcal{P}, \mathcal{L}, I)$ is a linear space, we define as usual $\nu = |\mathcal{P}|, b = |\mathcal{L}|$, where $|\ |$ denotes cardinality. For each point P and line l of S, let

$$\begin{aligned} r(P) &= |\{l \in \mathcal{L} :\ P\ I\ l\}| \\ k(l) &= |\{P \in \mathcal{P} :\ P\ I\ l\}| \end{aligned}$$

If ν is finite, then all of these numbers as well as $|I|$ are finite, in this case we say S is finite. For finite S we further define

$$\begin{aligned} k_m &= \min\{k(l) \ : \ l \in \mathcal{L}\} \\ k_M &= \max\{k(l) \ : \ l \in \mathcal{L}\} \\ r_m &= \min\{r(P) \ : \ P \in \mathcal{P}\} \\ r_M &= \max\{r(P) \ : \ P \in \mathcal{P}\} \end{aligned}$$

The following proposition is easy consequence of axiom H1.

Proposition 1.1 (Bumcrot [1]). *If a two-dimensional linear space S contains three distinct points P_1, P_2, P_3 such that $k(P_iP_j) = 2$ for $1 \leq i < j \leq 3$, then S is not a hyperbolic plane.*

Proposition 1.2 (Bumcrot [1]). *Any finite linear space satisfying:*

1. $r_m \geq k_M + 2$
2. $k_m(k_{m-1}) \geq r_M$

is a hyperbolic plane.

2. Some Hyperbolic Planes Obtained From Projective Planes by Removing Some Lines.

Let $\Pi = (\mathcal{P}, \mathcal{L}, I)$ be a finite projective plane of order n, and $\mathcal{M}$ be a set of lines in Π satisfying:

C. Every line of Π meets lines of $\mathcal{M}$ in at least two distinct points.

Let Q be the set of all points of $\mathcal{P}$ that are on at least one line of $\mathcal{M}$. Then the substructure

$$\Pi_m = (\mathcal{P} - Q, \mathcal{L} - \mathcal{M}, I \cap (\mathcal{P} - Q) \times (\mathcal{L} - \mathcal{M}))$$

is a hyperbolic plane, if $3 \leq m \leq n + \frac{1}{2}(1 - \sqrt{4n+5})$ and $n \geq 5$, where $m = |\mathcal{M}|$.

Here, the inequality $m \leq n + \frac{1}{2}(1 - \sqrt{4n+5})$ is a sufficient but not necessary condition. In fact, when $\mathcal{M}$ consists of lines such that no three of them are concurrent, denife a *corner point* as an intersection point of any two lines in $\mathcal{M}$. If the minimum number of corner points on any line of Π_m is r and

$$3 \leq m \leq n + r + \frac{1}{2}(1 - \sqrt{4n+5}), \quad n \geq 5,$$

then Π_m is a hyperbolic plane (see Kaya-Özcan [2]).

Recall that in this case, if n is odd, $m \leq n + 1$ and if n is even, $m \leq n + 2$. Therefore there are two cases:

Case I: Let n be odd. If $m = n+1$, that is, the elemnts of $\mathcal{M}$ are tangent lines of an oval $\mathcal{O}$, then $\Pi_m = \Pi_{n+1}^{\mathcal{O}}$ is the hyperbolic plane model of Ostrom [4].

If n elements of $\mathcal{M}$ are removed from Π, then $\Pi_n^{\mathcal{O}}$ is not a hyperbolic plane, since each tangent line in $\Pi_n^{\mathcal{O}}$ contains only one point in $\Pi_n^{\mathcal{O}}$. On the other hand, if m tangent lines of $\mathcal{O}$ are removed from Π, where $3 \leq m \leq n-1$, then $\Pi_m^{\mathcal{O}}$ is a hyperbolic plane which can be easily shown.

Case II. Let n be even. If $m = n+2$, then it is known that Π_{n+2} is hyperbolic plane (Olgun [3]). However, Π_{n+1}, Π_n and Π_{n-1} are not hyperbolic planes since Π_{n+1} contains a line which has no point in Π_{n+1}, Π_n contains two lines, each of which has only one point in Π_n and Π_{n-1} contains three lines which form a triangle and every one of which has only two points in Π_{n-1}. (See Prop.1.1).

On the other hand, one can easily show that Π_m is a hyperbolic plane for $3 \leq m \leq n-2$. Now we give a proposition which will show taht the proposition is still true for larger values of m.

Proposition 2.1. *Let Π be a projective plane of order n and $\mathcal{M}$ be a set of m lines that contains $n+1$ or $n+2$ lines which no three concurrent, according as n is odd or even, respectively. Denote by Π_m the structure obtained from Π by removing all lines (including all points on them) of $\mathcal{M}$.*

If $n+1 \leq m \leq \dfrac{3n-\sqrt{4n+5}}{2}$ *for* $n \geq 7$, *when n is odd*

or

$n+2 \leq m \leq \dfrac{3n+3-\sqrt{4n+5}}{2}$ *for* $n \geq 8$, *when n is even,*

then Π_m is a hyperbolic plane.

Proof. We give the proof for n odd.

L1. Clearly any two distinct points in Π_m, as points of Π, are on exactly one line in Π_m.

L2. Any line of Π_m is on at least

$$n+1-\left[\frac{n-1}{2}+\left(m-2\left(\frac{n-1}{2}\right)\right)\right]=\frac{3n+1}{2}-m$$

points in Π_m since it has at most $\frac{n-1}{2}$ corner points.

$$\frac{3n+1}{2}-m=\frac{3n+1}{2}-\frac{3n-\sqrt{4n+5}}{2}=\frac{1+\sqrt{4n+5}}{2}>2, (n \geq 7) \text{ even if}$$

$m = \dfrac{3n-\sqrt{4n+5}}{2}$. So, any line of Π_m has more than two points. Thus, L1 and L2 imply that Π_m is a linear space.

In Π_m, since $r_m = r_M = n+1$ and $k_M \leq \dfrac{n+3}{2}$,

$$r_m > k_M + 2 \tag{1}$$

Since $k_m = \dfrac{3n+1}{2} - m$,

$$k_m(k_m - 1) = \frac{3n+1-2m}{2}\left(\frac{3n+1-2m}{2} - 1\right) = \frac{3n+1-2m}{2} \cdot \frac{3n-1-2m}{2}$$

Thus $k_m(k_m - 1) \geq n+1$, since $m \leq \frac{3n-\sqrt{4n+5}}{2}$. Therefore we have

$$k_m(k_m - 1) \geq r_M \tag{2}$$

It follows from Proposition 1.2 the inequalities (1)-(2) imply that the linear space Π_m is a hyperbolic plane.

If n is even the proof is similar. □

Open question: Let Π_m be the hyperbolic plane obtained in Proposition 2.1. Is there any value of m which is larger than obtained in Proposition 2.1?

The classification of sets of lines with respect to the number of points on each line of a hyperbolic plane obtained from Π by removing some lines, in particular, no three of them are concurrent is an important subject. In the mean time, the lines of some hyperbolic planes of this type have been classified by some authors. For example, line classes of the hyperbolic planes $\Pi_3, \Pi_4, \Pi_5, \Pi_6$ and Π_7, have been determined. Also the line classes of the hyperbolic planes $\Pi^{\mathcal{O}}_{n-1}$ and $\Pi^{\mathcal{O}}_{n-2}$ are examined in Özcan-Olgun-Kaya [5].

The Classification of Lines.

Let Π_m be a hyperbolic plane obtained from Π by removing m lines such that no three of them are concurrent. Then the lines of Π_m are classified as follows.

The set of lines of Π_m, every one of which contains exactly s corner points, is called a class and denoted by C_s.

Unless otherwise stated, Π_m will be understood as defined above.

Now, we recall, without proof, two results on line classes of the hyperbolic planes $\Pi_3, \Pi_4, \Pi_5, \Pi_6$ and Π_7 given in Özcan-Olgun-Kaya [5].

Corollary 2.1. *For any hyperbolic plane Π_m with $m \in \{3,4,5\}$,*

(i) $q_2 = \dfrac{1}{2}\binom{m}{2}\binom{m-2}{2}$

(ii) $q_1 = \binom{m}{2}(n-1) - \binom{m-2}{2}$

(iii) $q_0 = n^2 + n\left(1 - \binom{m-2}{2}\right) + \binom{m-1}{2} + \frac{1}{2}\binom{m}{2}\binom{m-2}{2}$

where $q_s = |C_s|, s = 0, 1, 2$.

Corollary 2.2. *The number of lines in C_0, C_1, C_2 of any hyperbolic plane of type Π_6 or Π_7 can be determined in terms of the number of lines in C_3 as follows:*

$$\begin{aligned} q_2 &= 45 - 3q_3 & & q_2 = 105 - 3q_3 \\ q_1 &= 15(n-7) + 3q_3 & \text{or} \quad & q_1 = 21(n-11) + 3q_3 \\ q_0 &= n^2 - 14n + 55 - q_3 & & q_0 = n^2 - 20n + 120 - q_3 \end{aligned}$$

respectively.

Proposition 2.2. *Let Π be a projective plane of even order n. Let $\mathbf{H}$ be a dual hyperoval in Π and Π_m be a hyperbolic plane obtained from Π by removing m lines, $m \in \{n-2, n-3\}$, of $\mathbf{H}$. Then there exist three line classes in Π_{n-2} and Π_{n-3}, namely, $C_{\frac{n-6}{2}}, C_{\frac{n-4}{2}}$ and $C_{\frac{n-2}{2}}$ which are in Π_{n-2} and $C'_{\frac{n-8}{2}}, C'_{\frac{n-2}{2}}$ and $C'_{\frac{n-4}{2}}$ which are in Π_{n-3}. Furthermore,*

$$q_{\frac{n-6}{2}} = q_0, q_{\frac{n-4}{2}} = q_1, \; q_{\frac{n-2}{2}} = q_2 \text{ and } q'_{\frac{n-8}{2}} = q'_0, q'_{\frac{n-6}{2}} = q'_1, q'_{\frac{n-4}{2}} = q'_2,$$

where q_s, is related to Π_4 and q'_s is related to Π_5 as in Corollary 2.1, $(s = 0, 1, 2)$.

Sketch of proof. The proof of the first part is straightforward. For the second part, consider the lines which complete m lines ($m \in \{n-2, n-3\}$, to $n+2$ lines of **H.** Then, the classes $C_{\frac{n-6}{2}}, C_{\frac{n-4}{2}}, C_{\frac{n-2}{2}}$ of Π_{n-2} and the classes C_0, C_1, C_2 of Π_4 are the same, that is, $C_{\frac{n-6}{2}} = C_0, C_{\frac{n-4}{2}} = C_1$ and $C_{\frac{n-2}{2}} = C_2$.

Likewise, the classes $C'_{\frac{n-8}{2}}, C'_{\frac{n-6}{2}}, C'_{\frac{n-4}{2}}$ of Π_{n-3} and the classes C'_0, C'_1, C'_2 of Π_5 are the same, that is, $C'_{\frac{n-8}{2}} = C'_0, C'_{\frac{n-6}{2}} = C'_1$ and $C'_{\frac{n-4}{2}} = C'_2$.

(Recall that q_s and $q'_s, s = 0, 1, 2$ is known for $m = 4$ and $m = 5$, from Corollary 2.1., respectively). Thus the proof is finished. □

Proposition 2.3. *Let Π be a projective plane of order n, with n even, $\mathbf{H}$ be a dual hyperoval in Π and Π_m be a hyperbolic plane obtained from Π by removing m lines $m \in \{n-4, n-5\}$ of $\mathbf{H}$, then there exist for line classes in Π_{n-4} and Π_{n-5}. Then there exist for line classes in Π_{n-4} and Π_{n-5}, namely, $C_{\frac{n-i}{2}}$, $i \in \{4, 6, 8, 10\}$ which are in Π_{n-4} and $C'_{\frac{n-j}{2}}, j \in \{6, 8, 10, 12\}$ which are in Π_{n-5}.*

Furthermore

$q_{\frac{n-6}{2}} = 45 - 3q_{\frac{n-4}{2}}, q_{\frac{n-8}{2}} = 15(n-7) + 3q_{\frac{n-4}{2}}, q_{\frac{n-10}{2}} = n^2 - 14n + 55 - q_{\frac{n-4}{2}}$ *in* Π_{n-4}, *and* $q'_{\frac{n-8}{2}} = 105 - 3q'_{\frac{n-6}{2}}, q'_{\frac{n-10}{2}} = 21(n-11) + 3q'_{\frac{n-6}{2}}, q'_{\frac{n-12}{2}} = n^2 - 20n + 120 - q'_{\frac{n-6}{2}}$ *in* Π_{n-5}.

Sketch of proof. The proof of the first part is straightforward. For the second part, consider again the lines which complete m lines, $m \in \{n-4, n-5\}$, to $n+2$ lines belong to **H**. Then, the classes $C_{\frac{n+2}{2}-(6-k)} = C_{\frac{n-10}{2}+k}$ of Π_{n-4} and the classes C_k of $\Pi_6, k \in \{0,1,2,3\}$ are completely the same. Therefore, one can easily write

$$q_{\frac{n-6}{2}} = 45 - 3q_{\frac{n-4}{2}}, q_{\frac{n-8}{2}} = 15(n-7) + 3q_{\frac{n-4}{2}}, q_{\frac{n-10}{2}} = n^2 - 14n + 55 - q_{\frac{n-4}{2}}$$

from Corollary 2.2 The proof for Π_{n-5} is similar to above.

3. Some Models of the Hyperbolik Planes Containing two-Point Lines

The following proposition constitutes an answer for question in Bumcrot [1]: "How many two-point lines can be on a given point in a hyperbolic plane?".

Proposition 3.1. *Let* Π *be a projective plane of order* n. *Remove* $n-1$ *lines (including all points on them) from* Π *such that* $n-2$ *of them are concurrent. Denote this substructure as* Π_{n-1}. *If* Π *is not a Fano plane, that is, the diagonal points of any complete quadrangle in* Π *are not collinear and* $n \geq 5$, *then* Π_{n-1} *is a hyperbolic plane.*

Proof. Let l_i, $l, (i = 1, 2, 3, ..., n-2)$, be removed lines from Π to obtain the hyperbolic plane Π_{n-1} and the lines l_i, $(i = 1, 2, ..., n-2)$ be concurrent at the point Q_0. Denote, $l_i \bigwedge l = Q_i$ $(i = 1, 2, ..., n-2)$, and the other points of l as Q_{n-1}, Q_n, Q_{n+1}.

It is clear that any line of Π_{n-1} on a point $Q_j, (j = n-1, n, n+1)$, but not on Q_0 is a line including only two points of Π_{n-1} (two-point line). So, there exist exactly two two-point lines on any point of Π_{n-1}.

We now start with the satisfaction of the hyperbolic plane conditions.

L1. Clearly, any two distinct points in Π_{n-1} as points of Π are on exactly one line in Π_{n-1}.

L2. Let $\Pi_{n-1} = (\mathcal{P}', \mathcal{L}', I')$ and $d \in \mathcal{L}'$.

If d does not contain $l_i \bigwedge l$ for $1 \leq i \leq n-2$, then $k(d) = 2$ or $k(d) = n-1$. Hence $k_m = 2$. Therefore, each line of Π_{n-1} contains at least two points.

H1. Π_{n-1} has $b = n^2 + 2$ lines, $3n-3$ of them have degree $2, n^2 - 3n + 2$ of them have degree of 3, and the remaining three of them have degree $n-1$. So through each point which is not on a line d there exists at least two lines that do not meet d.

H2. All lines, which are of degree $n-1$ do not meet in Π_{n-1}. So there exist at least four points with no three collinear.

H3. Let S be a subset of $\mathcal{P}'$.

Suppose that S contains three non-collinear points and all point on the lines through pairs of any distinct points of S.

If S contains three distinct non-collinear points A, B, C such that $k(AB) = k(AC) = 2$ in Π_{n-1} and $AQ_nI'Q_0$ then $\{A, Q_0, Q_{n-1}, Q_{n+1}\}$ is a complete quadrangle in Π and the diagonal points, Q_n, B, C of this quadrangle in Π are not collinear, since Π is not a Fano plane. So the line BC is not on Q_n. Therefore $k(BC) = 3$ in Π_{n-1} and $|S| \geq 4$.

Suppose first that $|S| = 4$ and S consists of all points on the triangle $\{A, B, C\}$. $k(BC) = 3$. Therefore S contains all points of the line AD.

If $EI'AD$ then S contains all points of the line BE and the line CE. So $k(BE) = k(CE) = 3$.

If $GI'BE$ and $FI'CE$, then $GI'l_{n-1}$ and $FI'l_{n+1}$. Therefore S contains all points of $\mathcal{P}'$. The proof for $|S| > 4$ is entirely similar to the proof for $|S| = 4$. □

Remark: Consider the hyperbolic plane $\Pi^{\mathcal{O}}_{n-1}$ in the case I of section 2. $\Pi^{\mathcal{O}}_{n-1}$ has exactly two two-point lines meeting at a point of $\Pi^{\mathcal{O}}_{n-1}$ (two tangent lines, not removed, of $\mathcal{O}$).

References

[1] Bumcrot, R.J.: "Finite hyperbolic spaces". Atti Convegno. Geom. Comb. e. sue Appl. Perugia, 113-130 (1971).

[2] Kaya, R., Özcan, E.: "On the Construction of Bolyai-Lobackevsky planes from projective planes". Rendiconti del seminario Mathematico di Brescia, 7, 427-434 (1984).

[3] Olgun, Ş.: "A note on some finite hyperbolic planes". Journal of Sci. and Arts of Gazi Univ. V.3. Number 2, 31-36 (1991).

[4] Ostrom, T.G.: "Ovals and finite Bolyai-Lobachevsky planes". Amer. Math. Monthly, 69, 899-901 (1961).

[5] Özcan, E., Olgun, Ş., Kaya, R.: "On the line classes in some finite hyperbolic planes". Comm. Fac. Sci. Univ. Ankara, Series A1, V. 38, 7-13 (1989).

PROJEKTİF DÜZLEMLERDEN ELDE EDİLEN SONLU HİPERBOLİK DÜZLEMLER ÜZERİNE BİR NOT

Özet

Π mertebesi n olan sonlu bir projektif düzlem, $\mathcal{M}$ de Π nin noktadaş olmayan üç doğru kapsayan herhangi doğrular cümlesi, $|\mathcal{M}| = m$, olsun. Π den $\mathcal{M}$ nin tüm doğrularının üzerlerindeki tüm noktalarıyla birlikte atılmasıyla elde edilen Π_m hiperbolik düzlemi için m nin bilinen değerlerinden daha büyük olan bazı değerler elde edildi. Π_m tipi bazı hiperbolik düzlemlerin doğru sınıfları belirlendi. Bundan başka Bumcrot[1] de iki noktalı doğrular kapsayan hiperbolik düzlemlere dair bir soruya cevap verildi.

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