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ON HIGH ORDER RIESZ TRANSFORMATIONS GENERATED BY A GENERALIZED SHIFT OPERATOR

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Abstract

In this paper, we determine high order Riesz transformations by using generalized shift operators and giving some of their properties

1. Introduction

Let us consider Riesz Transformation as

$$(R_j f)(x) = \lim_{\epsilon \to 0} c_n \int\limits_{|x-y|\geq\epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)dy \qquad (j = 1, 2, ..., n)$$

where $c_n = \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}$. In this transformations the difference $x - y$ can be regarded as an ordinary shift operation.

In 1987, Riesz transformations have been determined by considering generalized shift operator by I. Aliev [1].

In this study, we determine the high order Riesz transformations generated by generalized shift operators that are ordinary shift according to first $n-2$ variables and are also R^+ shift according to the last two variable [5].

2. Background and Notation

Let $\mathbf{R}_n^{++} = \{x = (x_1, x_2, ...x_{n-1}, x_n) : x_{n-1} \geq 0,\ x_n \geq 0\}$. Let the space of testing functions $\mathcal{Z}_+(\mathbf{R}_n^+) = \mathcal{Z}_+$ be class of all C^∞ functions φ on $\mathbf{R}_n^+$ (i.e. the partial derivatives of φ exist and are continuous) such that

$$\sup_{x\in\mathbf{R}_n^+} |x^\beta (D^\alpha \varphi)(x)| < \infty$$

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[2] This paper is dedicated to Ord.Prof.Dr. *C ARF*

for all n-tuples $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ of nonnegative integers, $D^\alpha = \partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2} \cdot \partial x_n^{\alpha_n}$. The dual space of $\mathcal{Z}_+$ is denoted by $\mathcal{Z}'_+$. We define $\mathcal{L}_{p,\nu_1,\nu_2}$ space

$$\mathcal{L}_{p,\nu_1,\nu_2} = \left\{ f(x) : ||f(x)||_{p,\nu_1,\nu_2} = \left(\int\limits_{R_n^{++}} |f(x)|^p x_{n-1}^{2\nu_1} x_n^{2\nu_2} dx \right)^{\frac{1}{p}} < \infty \right\}$$

where $\nu_1, \nu_2 > 0$ and $1 \leq p < \infty$. The principal value of $f(x)$ is

$$(v.p\ f, \varphi) = \lim_{\epsilon \to 0} \int\limits_{\substack{0<\varepsilon<|x| \\ 0 \leq x_n < \infty \\ 0 \leq x_{n-1} < \infty}} f(x)\varphi(x) x_{n-1}^{2\nu_1} x_n^{2\nu_2} dx, \qquad \varphi \in \mathcal{Z}_+.$$

Let Δ_B be the Laplacean-Bessel operators,

$$\Delta_B = \sum_{k=1}^{n-2} \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2\nu_1}{y}\frac{\partial}{\partial y} + \frac{2\nu_2}{z}\frac{\partial}{\partial z}, \qquad \nu_1, \nu_2 > 0$$

If

$$\frac{d^2u}{dr^2} + \frac{2\nu_1}{r}\frac{du}{dr} + \lambda^2 u = 0 \quad u(0) = 1, \quad u'(0) = 0 \quad (\lambda > 0) \tag{1}$$

and

$$\frac{d^2u}{dr^2} + \frac{2\nu_2}{r}\frac{du}{dr} + \lambda^2 u = 0 \quad u(0) = 1, \quad u'(0) = 1 \quad (\lambda > 0) \tag{2}$$

then solutions of equations (1) and (2) are denoted by $j_{\nu_1-\frac{1}{2}}(\lambda r)$ and $j_{\nu_2-\frac{1}{2}}(\lambda r)$ respectively. The Fourier-Bassel transformations can be defined by

$$[F_B\varphi](y) \int\limits_{R_n^{++}} \varphi(x) e^{-i<x'',y''>} j_{\nu_1-\frac{1}{2}}(x_{n-1}y_{n-1}) x_{n-1}^{2\nu_1} j_{\nu_2-\frac{1}{2}}(x_n y_n) x_n^{2\nu_2} dx,$$

$$\varphi \in \mathcal{Z}_+.$$

and its invers transformations can be given by

$$[F_B^{-1}\varphi](y) = [F_B\varphi](-y), \qquad \varphi \in \mathcal{Z}_+.$$

where,
$< x'', y'' >= x_1y_1 + x_2y_2 + \cdots + x_{n-2}y_{n-2}$ and

$c_\nu = (2\pi)^{-\frac{n-2}{2}} 2^{-\nu_1-\nu_2+1}\Gamma(\nu_1+\frac{1}{2})^{-1}\cdot\Gamma(\nu_2+\frac{1}{2})^{-1}$
The generalized shift operator is denoted by T_x^y [6], [2],

$$T_x^y\varphi(x) = c_\nu \int_0^\pi\int_0^\pi \varphi\big[x''-y'' = \sqrt{x_{n-1}^2+y_{n-1}^2-2x_{n-1}y_{n-1}\cos\alpha_1},$$
$$\sqrt{x_n^2+y_n^2-2x_ny_n\cos\alpha_2}\big]\sin^{2\nu_1-1}\alpha\sin^{2\nu_2-1}\alpha d\alpha_1 d\alpha_2$$

where $x''y''\in R_{n-2}$ and $c_\nu = \dfrac{\Gamma(\nu_1+\frac{1}{2})}{\Gamma(\nu_1)\Gamma(\frac{1}{2})}\dfrac{\Gamma(\nu_2+\frac{1}{2})}{\Gamma(\nu_2)\Gamma(\frac{1}{2})}$.

3. High Order Riesz Transformations Generated By A Generalized Shift Operator

In this section, we consider generalized shift operator which has ordinary shift according to first $n-2$ terms and are also R^+-shift according to the last two terms. We study relations between the Fourier-Bessel operator and this generalized shift operator. We give the Fourier-Bessel transformation of homogeneous polinomial which holds Laplacean-Bessel equations. Finally, we define Riesz transformations related to the shift operators and so we show that this Riesz transformations holds the condition of classical Riesz transformation [5].

Theorem 3.1. *If $P_k(x) = P_k(x_1,x_2,...,x_{n-1}^2,x_n^2)$, is a homogeneous polynomial with order k which holds $\Delta_B P_k(x) = 0$ Laplecean Bessel equations, then*

$$\big[P_k(x)e^{-|x|^2}\big]^{\hat{}}(y) = 2^{-(k+\nu_1+\nu_2+\frac{n}{2})}i^k P_k(y)e^{\frac{-|y|^2}{4}}$$

where $\hat{}$ denotes Fourier-Bassel transformations.

Proof. By. [3] and [7], [8],

if $x'',y''\in R_{n-2}$ and $\alpha>0$, then

$$(2\pi)^{\frac{2-n}{2}}\int_{R_{n-2}} e^{-\alpha|x''|^2-i(x''\cdot y'')}dx'' = \left(\frac{\pi}{\alpha}\right)^{\frac{n-2}{2}} e^{\frac{-|y''|^2}{4\alpha}} \tag{3}$$

If $\nu>-1,\alpha>0$ and $J_\nu(br)$ is Bessel function, then

$$\int_0^\infty e^{-\alpha r^2} r^{\nu+1}J_\nu(br)dr = \frac{b^\nu}{(2\alpha)^{\nu+1}}e^{\frac{-b^2}{4\alpha}} \tag{4}$$

Since $J_\nu(r) = [2^\nu\Gamma(\nu+1]^{-1}r^\nu j_\nu(r)$, by (3) and (4) we have

$$F_B\big(e^{-\alpha|x|^2}\big)(y) = e^{\frac{-|y|^2}{4\alpha}}(2\alpha)^{-\frac{n+2\nu_1+2\nu_2}{2}}, \qquad y\in R_n^{++}$$

Hence

$$\int\limits_{R_n^{++}} e^{|x|^2+2i(x''\cdot y'')} \quad j_{\nu_1-\frac{1}{2}}(x_{n-1}2y_{n-1})x_{n-1}^{2\nu_1}\cdot j_{\nu_2-\frac{1}{2}}(x_n\,2y_n)x_n^{2\nu_2}dx$$
$$= \frac{\Gamma(\nu_2+\frac{1}{2})\Gamma(\nu_1+\frac{1}{2})}{2^2}\pi^{\frac{n-2}{2}}e^{-|y|^2} \tag{5}$$

If we apply differential operator

$$P_k(\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_2},...,\frac{\partial}{\partial t_{n-2}},Bt_{n-1},Bt_n)$$

to (5), then we obtain

$$\int\limits_{R_n^{++}} P_k(x)e^{|x|^2+2i(x''\cdot t'')} \quad j_{\nu_1-\frac{1}{2}}(x_{n-1}2t_{n-1})x_{n-1}^{2\nu_1}\cdot j_{\nu_2-\frac{1}{2}}(x_n2t_n)x_n^{2\nu_2}dx$$
$$= Q(t)\frac{\Gamma(\nu_2+\frac{1}{2})\Gamma(\nu_1+\frac{1}{2})}{2^2}\pi^{\frac{n-2}{2}}e^{-|t|^2}$$

where $Q(t)$ is a polynomial and

$$B_{t_k}=\frac{\partial^2}{\partial t_k^2}+\frac{2\nu}{t_k}\frac{\partial}{\partial t_k} \qquad k=n-1,n.$$

Using the following formula [7].

$$j_{\nu-\frac{1}{2}}(r)=\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)\Gamma\frac{1}{2})}\int\limits_0^\pi e^{i\,r\cos\alpha}(\sin\alpha)^{2\nu-1}d\alpha$$

we have

$$Q(t)=2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_1+\frac{1}{2})]^{-1}\frac{\Gamma(\nu_1+\frac{1}{2})}{\Gamma(\nu_1)\Gamma(\frac{1}{2})}\frac{\Gamma(\nu_2+\frac{1}{2})}{\Gamma(\nu_2)\Gamma(\frac{1}{2})}\int\limits_{R_n^{++}} P_k(x)e^{|t|^2-|t|^2+2i(x''.t'')}$$
$$\cdot\left(\int\limits_0^\pi e^{2ix_nt_n\cos\varphi}(\sin\varphi)^{2\nu-1}d\varphi\int\limits_0^\pi e^{2ix_{n-1}t_{n-1}\cos\alpha}(\sin\alpha)^{2\nu-1}d\alpha\right)x_{n-1}^{2\nu_1}x_n^{2\nu_2}dx$$
$$=2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_2+\frac{1}{2})]^{-1}\int\limits_{R_n^{++}} P_k(x)x_{n-2}^{2\nu_1}x_n^{2\nu_2}dx$$
$$\frac{\Gamma(\nu_1+\frac{1}{2})}{\Gamma(\nu_1)\Gamma(\frac{1}{2})}\frac{\Gamma(\nu_2+\frac{1}{2})}{\Gamma(\nu_2)\Gamma(\frac{1}{2})}\int\limits_0^\pi\int\limits_0^\pi e^{-|x''-it''|^2}e^{-(x_n^2+i^2t_n^2-2ix_nt_n\cos\varphi)}(\sin\varphi)^{2\nu_1-1}$$

$$e^{-(x_{n-1}^2+i^2t_{n-1}^2-2ix_{n-1}t_{n-1}\cos\alpha)}(\sin\alpha)^{2\nu_2-1}d\varphi d\alpha$$

By the properties of T_x^y, we obtain

$$Q(t)=2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_2+\frac{1}{2})]^{-1}\cdot\int\limits_{R_n^{++}}P_k(x)\left[T_{x_n,x_{n-1}}^{-it}e^{-|x'|^2}\right]x_n^{2\nu_2}x_{n-1}^{2\nu_1}dx$$

$$Q(-it)=2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_2+\frac{1}{2})]^{-1}\int\limits_{R_n^{++}}[T_{x_n,x_{n-1}}^{-t}P_k(x)]e^{-|x'|^2}x_n^{2\nu_2}x_{n-1}^{2\nu_1}dx$$

If $x=r\theta$ $(0<r<\infty,\theta\in S^+=\{|x|=1,\ x_{n-1},x_n\geq 0\})$, then

$$Q(-it)=c_{\nu^*}\int\limits_0^\infty r^{2\nu_1+2\nu_2+n-1}\left(\int\limits_{S+}T_{r\theta,r\theta'}^{-t}P_k(r\theta,r\theta')\theta_n^{2\nu_2}\theta_{n-1}^{2\nu_1}\right)e^{-r^2}dr \qquad (6)$$

where $c_{\nu^*}=2^2[\pi^{\frac{n-2}{2}}\Gamma(\nu_1+\frac{1}{2})\Gamma(\nu_2+\frac{1}{2})]^{-1}$.
Applying the mean value theorem for $\Delta_B u=0$, [4] to (6), we have

$$Q(-it)=\frac{2}{\Gamma(\nu_1+\nu_2+\frac{n}{2})}P_k(t)\int\limits_0^\infty r^{2\nu_1+2\nu_2+n-1}e^{-r^2}dr=P_k(t)$$

and

$$\int\limits_{R_n^{++}}P_k(x)e^{|x|^2+2i(x''.t'')}j_{\nu_1-\frac{1}{2}}(x_{n-1}2t_{n-1})x_{n-1}^{2\nu_1}j_{\nu_2-\frac{1}{2}}(x_n2t_n)x_n^{2\nu_2}dx$$

$$=P_k(it)e^{-|t|^2}\pi^{\frac{n-2}{2}}\frac{\Gamma(\nu_2+\frac{1}{2})\Gamma(\nu_1+\frac{1}{2})}{2^2}$$

Since P_k are homogeneous, we have

$$\left[P_k(x)e^{-|x|^2}\right]^{\hat{}}=2^{-(k+\nu_1+\nu_2+\frac{n}{2})}i^kP_k(y)e^{\frac{-|y|^2}{4}}$$

and so the theorem is proved □

Now we theorem is proved.

Lemma 3.2. *If*

$$\int\limits_{S^+}f(\theta_1,\theta_2,...,\theta_{n-1},\theta_n)\theta_{n-1}^{2\nu_1}\theta_n^{2\nu_2}dS^+=0$$

then

$$\frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2-\epsilon}} \to v.p\,\frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2}}$$

for $\epsilon \to 0$, where

$$(v.pf.\varphi) = \lim_{\epsilon\to 0} \int\limits_{\substack{0<\epsilon<|x| \\ 0\le x_n<\infty \\ 0\le x_{n-1}<\infty}} f(x)\varphi(x)x_{n-1}^{2\nu_1}x_n^{2\nu_2}dx \quad \varphi \in Z_+$$

Proof. The proof follows immediatly from the representation.

$$\int\limits_{R_n^+} \frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2-\epsilon}}\varphi(x)x_{n-1}^{2\nu_1}x_n^{2\nu_2}dx$$

$$= \int\limits_{|x|\le 1} \frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2-\epsilon}}[\varphi(x)-\varphi(0)]x_{n-1}^{2\nu_1}x_n^{2\nu_1}dx$$

$$+ \int\limits_{|x|>1} \frac{f\left(\frac{x}{|x|}\right)}{|x|^{n+2\nu_1+2\nu_2-\epsilon}}\varphi(x)x_{n-1}^{2\nu_1}x_n^{2\nu_2}dx.$$

By considering Theorem 3.1 and Lemma 3.2, we have □

Theorem 3.3. *Let P_k be homogeneous polynomial with order k, then*

$$\left[v.p\frac{P_k}{|x|^{k+n+2\nu_1+2\nu_2}}\right]^{\hat{}}(y) = 2^{\frac{n+2\nu_1+2\nu_2}{2}} i^k \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{k+n+2\nu_1+2\nu_2}{2})}\frac{P_k(y)}{|y|^k}$$

where $\hat{}$ denotes Fourier-Bassel transformations.

Proof. Let us consider

$$F_B[f(\alpha x)](t) = \alpha^{-n-2\nu_1-2\nu_2}F_B[f(x)](\frac{t}{\alpha}) \tag{7}$$

By (7) and Theorem 3.1 we have

$$[P_k(x)e^{-\alpha|x|^2}]^{\hat{}}(y) = (2\alpha)^{-(k+\nu_1+\nu_2+\frac{n}{2})}i^k e^{\frac{-|y|^k}{4\alpha}}P_k(y)$$

If $\varphi \in \mathcal{Z}_+$, then

$$\int\limits_{R_n^+} P_k(x) e^{\frac{-\alpha|x|^2}{4\alpha}} \varphi(x) x_n^{2\nu_2} x_{n-1}^{2\nu_1} dx = (2\alpha)^{-(k+\nu_1+\nu_2+\frac{n}{2})} i^k \cdot \int\limits_{R_n^+} P_k(x) e^{-\alpha|x|^2} \varphi(x) x_n^{2\nu_2} x_{n-1}^{2\nu_1} dx \tag{8}$$

If we apply $\alpha^{\frac{k+n+2\nu_1+2\nu_2-\epsilon-2}{2}}$ to (8), integrate with respect to α from 0 to ∞ and use

$$\int\limits_0^\infty e^{-\alpha|x|^2} \alpha^{\frac{k+n+2\nu_1+2\nu_2-\epsilon}{2}-1} d\alpha = \Gamma\left(\frac{k+n+2\nu_1+2\nu_2-\epsilon}{2}\right) |x|^{-(k+n+2\nu_1+2\nu_2-\epsilon)}$$

then we have

$$\Gamma\left(\frac{k+n+2\nu_1+2\nu_2-\epsilon}{2}\right) \int\limits_{R_n^+} \frac{P_k(x)}{|x|^{-(k+n+2\nu_1+2\nu_2-\epsilon)}} \varphi(x) x_n^{2\nu_2} x_{n-1}^{2\nu_1} dx$$

Hence

$$2^{-\frac{2\nu_1+2\nu_2+n}{2}+\epsilon} \Gamma\left(\frac{k+\varepsilon}{2}\right) i^k \int\limits_{R_n^+} \frac{P_k(x)}{|x|^{k+\epsilon}} \varphi(x) x_n^{2\nu_2} x_{n-1}^{2\nu_1} dx$$

Therefore

$$[\frac{P_k(x)}{|x|^{k+n+2\nu_1+2\nu_2-\epsilon}}]\hat{}(y) = 2^{-\frac{2\nu_1+2\nu_2+n}{2}+\epsilon} i^k \cdot \frac{\Gamma\left(\frac{k+\epsilon}{2}\right)}{\Gamma\left(\frac{k+n+2\nu_1+2\nu_2-\epsilon}{2}\right)} \cdot \frac{P_k(y)}{|y|^{k+\epsilon}}$$

By Lemma 3.2, we have

$$[p.v \frac{P_k(x)}{|x|^{k+n+2\nu_1+2\nu_2-\epsilon}}]\hat{}(y) = 2^{\frac{-2\nu_1+2\nu_2+n}{2}} i^k \cdot \frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+n+2\nu_1+2\nu_2}{2}\right)} \cdot \frac{P_k(y)}{|y|^k}$$

Thus, the proof is complete. □

Now we give the Riesz transformations generated by generalized shift operators what we call Riesz-Bessel transformations.

Definition 3.4. *Let T_x^y be generalized shift operator. Then*

$$\begin{aligned}(R_B^{(k)} f)(\xi) &= c_k(n,\nu_1,\nu_2) \left(p.v \frac{P_k(x)}{|x|^{k+n+2\nu_1+2\nu_2}} \star f\right)(\xi) \\ &\equiv c_k(n,\nu_1,\nu_2) \lim_{\epsilon\to 0} \int\limits_{\substack{0<\epsilon<|x| \\ 0<x_n<\infty}} \frac{P_k(x)}{|x|^{k+n+2\nu_1+2\nu_2}} T_\xi^x f(\xi) x_{n-1}^{2\nu_1} x_n^{2\nu_2} dx\end{aligned}$$

is called Riesz-Bessel transformation with order k for $f(x) \in \mathcal{Z}_+$, where

$$c_k(n,\nu_1,\nu_2)2^{\frac{n+2\nu_1+2\nu_2}{2}}\Gamma\left(\frac{k+n+2\nu_1+2\nu_2}{2}\right)\left[\Gamma\left(\frac{k}{2}\right)\right]^{-1} \quad k=1,2,...$$

Let $P_k(x)$ be homogeneous polynomial with order k and $\Delta_B P_k = 0$. By Theorem 3.3, we have

$$\left(R_B^{(k)}f\right)^{\hat{}}(\xi) = i^k \frac{P_k(\xi)}{|\xi|^k} f^{\hat{}}(\xi) \tag{9}$$

We note that in (9) $i^k P_k|\xi|^{-k}$ is a factor corresponding to the transformation $R_B^{(k)}$.

Now consider Riesz-Bessel transformations with order one. Note that the number of these transformations is $n-2$ and $(j=1,2,...,n-2)$

$$(R_{B_i}f)(\xi) = c_1(n,\nu_1,\nu_2)\lim_{\epsilon\to 0}\int\limits_{\substack{0<\epsilon<|x| \\ 0<x_n<\infty \\ 0<x_{n-1}<\infty}} \frac{x_j}{|x|^{1+n+2\nu_1+2\nu_2}} T_\xi^x f(\xi) x_{n-1}^{2\nu_1} x_n^{2\nu_2}dx \tag{10}$$

where,

$$c_1(n,\nu_1,\nu_2) = 2^{\frac{n+2\nu_1+2\nu_2}{2}}\Gamma\left(\frac{1+n+2\nu_1+2\nu_2}{2}\right)\frac{1}{\sqrt{\pi}}$$

We have also two Riesz-Bessel transformations with order 2 such that

$$(R_{B_i}^2 f)(\xi) = c_2(n,\nu_1,\nu_2)\lim_{\epsilon\to 0}\int\limits_{\substack{0<\epsilon<|x| \\ 0<x_n<\infty \\ 0<x_{n-1}<\infty}} \frac{P_2(x)}{|x|^{2+n+2\nu_1+2\nu_2}} T_\xi^x f(\xi) x_{n-1}^{2\nu_1} x_n^{2\nu_2}dx \tag{11}$$

where,

$$c_2(n,\nu_1,\nu_2) = 2^{\frac{n+2\nu_1+2\nu_2}{2}}\Gamma\left(\frac{2+n+2\nu_1+2\nu_2}{2}\right)$$

and corresponding polynomial to P_k are

$$P_2(x) = \frac{2\nu_1+2\nu_2+2}{\frac{n}{2}+\nu_1+\nu_2}|x|^2 - 2x_{n-1}^2 - 2x_n^2$$

and

$$P_2'(x) = -\frac{2\nu_1+2\nu_2+2}{n+2\nu_1+2\nu_2}|x|^2 + x_{n-1}^2 + x_n^2$$

It can be easily shown that $\Delta_B P_2 = 0$ and $\Delta_B P_2' = 0$. Then, by (9) we have

$$(R_{B_n}f)^{\hat{}}(\xi) = \left(\frac{2\nu_1+2\nu_2+2}{\frac{n}{2}+\nu_1+\nu_2} - \frac{2\xi_{n-1}^2}{|\xi|^2} - \frac{2\xi_n^2}{|\xi|^2}\right) f^{\hat{}}(\xi)$$

and

$$(R_{B_{n-1}} f)\hat{}(\xi) = \left(-\frac{2\nu_1 + 2\nu_2 + 2}{n + 2\nu_1 + 2\nu_2} + \frac{\xi_{n-1}^2}{|\xi|^2} + \frac{\xi_n^2}{|\xi|^2} \right) f\hat{}(\xi)$$

We note that

(i)

$$\sum_{j=1}^{n-2} (R_{B_j})^2 + R_{B_{n-1}} + R_{B_n} = \left(-1 + \frac{2\nu_1 + 2\nu_2 + 2}{2\nu_1 + 2\nu_2 + n} \right) E$$

wehere E is the identity operator in $L_{p,\nu}(R_n^+)$.

(ii) If $1 \leq j, k \leq n - 2$ then

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = -R_{B_j} + R_{B_k} \Delta_B f, \qquad f \in \mathcal{Z}_+$$

(iii)

$$[(B_{x_n} + B_{x_{n-1}})f](y) = \left[\frac{2\nu_1 + 2\nu_2 + 2}{2\nu_1 + 2\nu_2 + n} - R_{B_n} - R_{B_{n-1}} \right] \Delta_B f$$

where $B_{x_n} = \dfrac{\partial^2}{\partial x_n^2} + \dfrac{2\nu}{x_n} \dfrac{\partial}{\partial x_n}$.

Remark By using the same method, this proof also can be given in general for this transformations generated by generalized shift operators that are ordinary shift operator according to m variables and are also R^+-shift operators with respect to k variables. Here $m + k = n$.

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GENELLEŞTİRİLMİŞ ÖTELEME OPERATÖRÜ İLE ELDE EDİLEN YÜKSEK MERTEBELİ RİESZ DÖNÜŞÜMLERİ

Özet

Bu çalışmada Genelleştirilmiş Öteleme Operatörü yardımıyla yüksek mertebeli Riesz dönüşümleri ve bu dönüşümlerin bazı özellikleri verilmiştir.

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