Tail empirical process for some long memory sequences

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Abstract

This paper describes limiting behaviour of tail empirical process associated with long memory stochastic volatility models. We show that such process has dichotomous behaviour, according to an interplay between a Hurst parameter and a tail index. In particular, the limit may be non-Gaussian and/or degenerate, indicating an influence of long memory. On the other hand, tail empirical process with random levels never suffers from long memory. This is very desirable from a practical point of view, since such the process may be used to construct Hill estimator of the tail index. To prove our results we need to establish several new results for regularly varying distribution functions, which may be of independent interest.

1 Introduction

The goal of this article is to study weak convergence results for the tail empirical process associated to some long memory sequences. Besides theoretical interests on its own, the results are applicable in different statistical procedures based on several extremes. The similar problem was studied in case of independent, identically distributed random variables in [11], or for weakly dependent sequences in [10], [9], [8], [17].

Our set-up is as follows. Assume that $\{X_i\,,\ i\in\mathbb{Z}\}$, is a stationary Gaussian process with the unit variance and covariance

$$\rho_{i-j}=\mathrm{cov}(X_i,X_j)=|i-j|^{2H-2}\ell_0(|i-j|)\ , \tag{1}$$

where $H\in[1/2,1)$ is the Hurst exponent and ℓ_0 is a slowly varying function at infinty, i.e. $\lim_{t\to\infty}\ell_0(tx)/\ell_0(x)=1$ for all $x>0$. The case $H=1/2$ can be informally thought of as the case of weakly dependent Gaussian sequences. We shall consider a stochastic volatility process defined as

$$Y_i=\sigma(X_i)Z_i, \qquad i\in\mathbb{Z},$$

where $\sigma(\cdot)$ is a nonnegative, deterministic function and that $\{Z,Z_i,\,i\in\mathbb{Z}\}$, is a sequence of i.i.d. random variables, independent of the process $\{X_i\}$. We note, in particular, that if $\mathbb{E}[Z^2] < \infty$ and $\mathbb{E}[Z] = 0$, then the Y_is are uncorrelated, no matter what are the assumptions on dependence structure of the underlying Gaussian sequence.

Stochastic volatility models have become popular in financial time series modeling. In particular, if $H \in (1/2, 1)$, these models are believed to capture two standardized feature of financial data: long memory of squares or absolute values, and heteroscedascity. If $\sigma(x) = \exp(x)$, then the model is called in econometrics literature *Long Memory in Stochastic Volatility* and was introduced in [3]. For an overview of stochastic volatility models with long memory we refer to [6].

Let $F = F_i$, $i \geq 1$, be the marginal distribution of Y_i. In extreme value literature the following assumption on F is commonly made: with $u_n, n \geq 1$, $u_n \to \infty$, and $\sigma_n, n \geq 1$, its associated conditional tail distribution function

$$T_n(x) := \frac{\bar{F}(u_n + \sigma_n x)}{\bar{F}(u_n)}, \qquad x \geq 0,\ n \geq 1, \tag{2}$$

satisfies

$$T_n(x) \to T(x) = (1+x)^{-1/\gamma}\ . \tag{3}$$

For the stochastic volatility model, we will need a further specification. Let F_Z be the marginal distribution of the noise sequences. We will assume that for some $\alpha \in (0, \infty)$,

$$\bar{F}_Z(z) = \mathbb{P}(Z > x) = x^{-\alpha}\ell(x)\ , \tag{4}$$

where ℓ is again a slowly varying function. Having (4) and $\mathbb{E}[\sigma^{\alpha+\epsilon}(X_1)] < \infty$ for some $\epsilon > 0$, we conclude by Breiman's Lemma [4] (see also [16, Proposition 7.5]) that

$$\bar{F}(x) = \mathbb{P}(Y_1 > x) = \mathbb{P}(\sigma(X_1)Z_1 > x) \sim \mathbb{E}[\sigma^\alpha(X_1)]\mathbb{P}(Z_1 > x)\ , \quad \text{as } x \to \infty.$$

Consequently, $\bar{F}(\cdot)$ satisfies (3) with $\sigma_n = u_n$ and $\gamma = 1/\alpha$.

Similarly to [17], we define the tail empirical distribution function and the tail empirical process, respectively, as

$$\tilde{T}_n(s) = \frac{1}{n\bar{F}(u_n)} \sum_{j=1}^{n} \mathbb{1}_{\{Y_j > u_n + u_n s\}}\ ,$$

and

$$e_n(s) = \tilde{T}_n(s) - T_n(s)\ ,\ s \in [0, \infty)\ . \tag{5}$$

From [17] we conclude that under appropriate mixing and other conditions on a stationary sequence Y_i, $i \geq 1$, the tail empirical process converges weakly and the limiting covariance is affected by dependence. In our case, the results [17] do not seem applicable. In fact, it will be shown that we have two different modes of convergence. If u_n is *big*, then $\sqrt{n\bar{F}(u_n)}$ is the proper scaling factor and the limiting process is Gaussian with the same covariance structure as in case of i.i.d. random variables Y_i. Otherwise, if u_n is *small*, then the limit is affected by long memory of the Gaussian sequence. A scaling is different and the limit may be non-normal. These results are presented in Section 2.1. Note that the similar dichotomous phenomenon was observed in a context of sums of extreme values associated with long memory moving averages, see [14] for more details. On the other hand, this dichotomous behaviour is in contrast with convergence of point processes based on stochastic volatility models with regularly varying innovations, [5], where (long range) dependence does not affect the limit.

The process $e_n(\cdot)$ is rather not practical, since the parameter u_n depends on the unknown distribution F. Also, u_n being *big* or *small* depends on a delicate balance between the tail index α and the Hurst parameter H. In order to overcome this, we consider as in [17] a process with random levels. There, we set $k = n\bar{F}(u_n)$ and replace the deterministic level u_n by $Y_{n-k:n}$, where $Y_{n:n} \geq Y_{n-1:n} \geq \cdots \geq Y_{1:n}$ are the increasing order statistics of the sample $Y_1, \ldots, Y_n$. The number k can be thought as the number of extremes used in a construction of the tail empirical process. It turns out, that if the number of extremes is *small* (which corresponds to a *big* u_n above), then the limiting process changes as compared to the one associated with $e_n(\cdot)$, but the speed of convergence remains the same. This has been already noticed in [17] in weakly dependent case. On the other hand, if k is *big*, then the scaling from $e_n(\cdot)$ is no longer correct (see Corollary 2.5). In fact, the process with random levels has faster rates of convergence and we claim in Theorem 2.6 that the rate of convergence and the limiting process are not affected at all by long memory, provided that a technical second order regular variation condition is fulfilled. The reader is referred to Section 2.2. On the other hand, it should be pointed out that our results are for *the* long memory stochastic volatility models. It is not clear for us whether such phenomena will be valid for example for subordinated long memory Gaussian sequences with infinite variance.

The results for the tail empirical process $e_n(\cdot)$ allow to obtain asymptotic normality and non-normality of intermediate quantiles, as described in Corollary 2.4. On the other hand, the tail empirical process with random levels allows to study the Hill estimator of the tail index α (Section 2.3). Consequently, as it is shown in Corollary 2.7, long memory does not have influence on its asymptotic behaviour. These theoretical observations are justified by simulations in Section 3.

Last but not least, we have some contribution to theory of regular variation. To establish our results in random level case, we need to work under second order regular variation condition. Consequently, one has to establish Breiman's-type lemma, that such condition is transferable from $\bar{F}_Z$ to $\bar{F}$. This has been done in Section 2.4.

2 Results

2.1 Tail empirical process

Let us define a function G_n on $(-\infty, \infty) \times [0, \infty)$ by

$$G_n(x, s) = \frac{\mathbb{P}(\sigma(x)Z_1 > (1+s)u_n)}{\mathbb{P}(Z_1 > u_n)} \ . \tag{6}$$

By Breiman's Lemma and the regular variation of $\bar{F}_Z$, we conclude that for each $s \in [0, 1]$, this function converges pointwise to $T(s)G(x)$, where $G(x) = \sigma^\alpha(x)$. A stronger convergence can actually be proved (see Section 4.6 for a proof).

Lemma 2.1. *If (4) holds and $\mathbb{E}[\sigma^{\alpha+\epsilon}(X)] < \infty$ for some $\epsilon > 0$, then*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{s\ge 0} |G_n(X, s) - \sigma^\alpha(X)T(s)|^p\right] = 0 \tag{7}$$

for all p such that $p\alpha < \alpha + \epsilon$.

In order to introduce our assumptions, we need to define the Hermite rank of a function. Recall that Hermite polynomials H_m, $m \ge 0$, form an orthonormal basis of the set of functions h such that $\mathbb{E}[h^2(X)] < \infty$, where X denotes a generic standard Gaussian random variable (independent of all other random variables considered here), and have the following properties:

$$\mathbb{E}[H_m(X)] = 0 \ , \ m \ge 1 \ , \ \mathrm{cov}(H_j(X), H_k(X)) = \delta_{j,k} k!$$

where $\delta_{j,k}$ is Kronecker's delta, equal to 1 if $j = k$ and zero otherwise. Then h can be expanded as

$$h = \sum_{m=0}^{\infty} \frac{c_m}{m!} H_m \ ,$$

with $c_m = \mathbb{E}[h(X)H_m(X)]$ and the series is convergent in the mean square. The smallest index $m \ge 1$ such that $c_m \ne 0$ is called the Hermite rank of h. Note that with this definition, the Hermite rank is always at least equal to one and the Hermite rank of a function h is the same as that of $h - \mathbb{E}[h(X)]$.

Let $J_n(m, s)$ denote the Hermite coefficients of the function G_n. Since the Hermite polynomials H_m are in $L^r(\mu)$ for all $r \ge 1$, Lemma 2.1 implies that the Hermite coefficients $J_n(m, s)$ converge to $J(m)T(s)$, where $J(m)$ is the m-th Hermite coefficient of G, uniformly with respect to $s \ge 0$. This implies that for large n, the Hermite rank of $G_n(\cdot, s)$ is not bigger than the Hermite rank of G. In order to simplify the proof of our results, we will use the following assumption, which is not very restrictive.

Assumption (H) Denote by $J_n(m,s)$, $m \geq 1$, the Hermite coefficients of $G_n(\cdot,s)$ and let $q_n(s)$ be the Hermite rank of $G_n(\cdot,s)$. Define

$$q_n = \inf_{s\geq 0} q_n(s) ,$$

the Hermite rank of the class of functions $\{G_n(\cdot,s), s \geq 0\}$. In other words, the number q_n is the smallest m such that $J_n(m,s) \neq 0$ for at least one s. Furthermore, let q be the Hermite rank of G. We assume that $q_n = q$ for n large enough.

Remark. Since for large enough n it holds that $q_n(s) \leq q$ for all s, the assumption is fulfilled, for example, when G has Hermite rank 1 (as is the case for the function $x \to \mathrm{e}^x$), or if the function σ is even with the Hermite rank 2.

In order to prove tightness, we will also need the following condition.

$$\exists C > 0 , \quad \forall t \geq s \geq 1 , \quad \frac{\mathbb{P}(sy < Y \leq ty)}{\mathbb{P}(Y > y)} \leq C(t-s) . \tag{8}$$

The result for the general tail empirical process is as follows.

Theorem 2.2. *Assume* (H) *with* $q(1-H) \neq 1/2$, *(1), (4), and that there exists* $\epsilon > 0$ *such that*

$$0 < \mathbb{E}[\sigma^{2\alpha+\epsilon}(X_1)] < \infty . \tag{9}$$

(i) If $n\bar{F}(u_n)\rho_n^q \to 0$ *as* $n \to \infty$*, then* $\sqrt{n\bar{F}(u_n)}\, e_n$ *converges weakly in the sense of finite dimensional distributions to the Gaussian process* $W \circ T$*, where* W *is the standard Brownian motion. If moreover (8) holds, then the convergence holds in* $D([0,\infty))$*.*

(ii) If $n\bar{F}(u_n)\rho_n^q \to \infty$ *as* $n \to \infty$ *then* $\rho_n^{-q/2} e_n$ *converges weakly in the sense of finite dimensional distributions to the process* $(\mathbb{E}[\sigma^\alpha(X_1)])^{-1} J(q) T L_q$*, where the random variable* L_q *is defined in (30). If moreover (8) holds, then the convergence holds in* $D([0,\infty))$*.*

Remarks

- We rule out the borderline case $q(1-H) = 1/2$ for the sake of brevity and simplicity of exposition. It can be easily shown that if $q(1-H) = 1/2$, then $\sqrt{n\bar{F}(u_n)}e_n$ converges to $W \circ T$ provided $1/\bar{F}(u_n)$ tends to infinity faster than a certain slowly varying function (e.g. if $u_n = n^\gamma$ for some $\gamma > 0$), even though it may hold in this case that $n\rho_n^q \to \infty$. The reason is that the variance of the partial sums of $G(X_k)$ is of order n times a slowly varying function which dominates $\ell_0^q(n)$.

- Here $D([0,\infty)$ is endowed with Skorohod's J_1 topology, which is checked by applying the Komogorov-Cencov criterion see [1, Theorem 15.6]. Since the limiting processes have almost surely continuous paths, this convergence implies uniform convergence on compact sets of $[0,\infty)$. See also [20].

- The meaning of the above result is that for u_n *big*, long memory does not play any role. However, if u_n is *small*, long memory comes into play and the limit is degenerate. Furthermore, in the case of Theorem 2.2, *small* and *big* depends on the relative behaviour of the tail of Y_1 and the memory parameter. Note that the condition $n\bar{F}(u_n)\rho_n^q \to \infty$ implies that $1 - 2q(1-H) > 0$, in which case the partial sums of the subordinate process $\{G(X_i)\}$ weakly converge to the Hermite process of order q (see Section 4.1). The cases (i), (ii) will be referred to as the limits *in the i.i.d. zone* and *in the LRD zone*, respectively.

- Condition $\mathbb{E}[\sigma^{\alpha+\epsilon}(X_1)] < \infty$ is standard when one deals with regularly varying tails. However, here we need the condition $\mathbb{E}[\sigma^{2\alpha+\epsilon}(X_1)] < \infty$, which is needed for Hermite expansion in the proof of limit in Section 4.3.1 and tightness.

- The result should be extendable to general, not necessary Gaussian, long memory linear sequences. Instead of limit theorems and covariance bounds in Section 4.1, one can use limit theorems from [13], and the covariance bounds of [12, Lemma 3].

- The condition (8) is unprimitive since it is expressed in terms of Y. It holds if $\mathbb{E}[\sigma^{\alpha+\epsilon}(X)] < \infty$ for some $\epsilon > 0$ and Z satisfies the following condition

$$\exists C > 0\ , \quad \forall y \geq 1\ , \quad \forall t \geq s > 0\ , \quad \frac{\mathbb{P}(sy < Z \leq ty)}{\mathbb{P}(Z > y)} \leq C(s \wedge 1)^{-\alpha-1-\epsilon}(t-s)\ . \qquad (10)$$

  This condition holds in particular if Z has an ultimately monotone density, which is then necessarily regularly varying at infinity with index $-\alpha-1$ by the monotone density Theorem, see [2, Theorem 1.7.2]. See Section 2.4 for a second order regular condition that implies (10).

- Let u_n be such that $n\bar{F}(u_n) \to 1$, as $n \to \infty$. Then with some $r_n \to \infty$,

$$n\sum_{j=1}^{r_n} \mathbb{P}(\sigma(X_0)Z_0 > u_n, \sigma(X_j)Z_j > u_n) \sim n\bar{F}^2(u_n)\sum_{j=1}^{r_n} \mathrm{cov}(\sigma^\alpha(X_0), \sigma^\alpha(X_j))$$
$$\sim \bar{F}(u_n)\mathrm{var}\left(\sum_{j=1}^{r_n}\sigma^\alpha(X_j)\right)/r_n \sim \bar{F}(u_n)q_n^{1-q(2-2H)}\ell_0^{2q}(q_n)\ .$$

  Consequently, Case (i) guarantees Leadbetter's condition $D'(u_n)$. Therefore (see [17, Section 4]) the condition C3 of Rootzen is fulfilled and in principle our result in the i.i.d. zone could be concluded from [17], provided we can verify also Rootzen's conditions C1, C2 and $\beta(u_n)$ mixing, which could be more difficult, than proving convergence via our approach. Nevertheless, results in the LRD zone are not related to Rootzen's results for weakly dependent sequences.

2.2 Random levels

Similarly to [17], we consider the case of random levels. Let $\Rightarrow$ denote weak convergence in $D([0,\infty))$. Define the increasing function U on $[1,\infty)$ by $U(t) = F^{\leftarrow}(1-1/t)$, where $F^{\leftarrow}$ is the left-continuous inverse of F. Let k denote a sequence of integers depending on n, where the dependence in n is omitted from the notation as customary, and such that

$$\lim_{n\to\infty} k = \lim_{n\to\infty} n/k = \infty \, . \tag{11}$$

Such a sequence is usually called an intermediate sequence. Define $u_n = U(n/k)$. If F is continuous, then $n\bar{F}(u_n) = k$, otherwise, since $\bar{F}$ is regularly varying, it holds that $\lim_{n\to\infty} k^{-1} n\bar{F}(u_n) = 1$. Thus, we will assume without loss of generality that $k = n\bar{F}(u_n)$ holds. Then the statements of Theorem 2.2 may be written respectively as

$$\sqrt{k}(\tilde{T}_n - T_n) \Rightarrow W \circ T \, , \tag{12}$$

$$\rho_n^{-q/2}(\tilde{T}_n - T_n) \Rightarrow \frac{J(q)}{\mathbb{E}[\sigma^\alpha(X_1)]} \, T \cdot L_q \, . \tag{13}$$

Let us rewrite the statements of (12), (13) as

$$w_n(\tilde{T}_n - T_n) \Rightarrow w \, ,$$

where

$$w_n = \sqrt{k} \quad \text{if} \quad \lim_{n\to\infty} k\rho_n^q = 0 \, , \tag{14}$$

$$w_n = \rho_n^{-q/2} \quad \text{if} \quad \lim_{n\to\infty} k\rho_n^q = \infty \, , \tag{15}$$

and $w = W \circ T$ if (14) holds (i.i.d. zone) and $w = (\mathbb{E}[\sigma^\alpha(X_1)])^{-1} J(q) T L_q$ if (15) holds (LRD zone).

We now want to center the tail empirical process at T instead of T_n. To this aim, we introduce an unprimitive second order condition.

$$\lim_{n\to\infty} w_n \|T_n - T\|_\infty = 0 \, , \tag{16}$$

where

$$\|T_n - T\|_\infty = \sup_{t\geq 1} \left| \frac{\mathbb{P}(\sigma(X)Z > u_n t)}{\mathbb{P}(\sigma(X)Z > u_n)} - t^{-\alpha} \right| \, .$$

The following result is a straightforward corollary of Theorem 2.2.

Corollary 2.3. *Under the assumptions of Theorem 2.2, if moreover (16) holds, then* $w_n(\tilde{T}_n - T)$ *converges weakly in* $D([0,\infty))$ *to the process* w.

Let $Y_{n:1} \leq \cdots \leq Y_{n:n}$ be the increasing order statistics of $Y_1, \dots, Y_n$. The former result and Verwaat's Lemma [16, Proposition 3.3] yield the convergence of the intermediate quantiles.

Corollary 2.4. *Under the assumptions of Corollary 2.3, $w_n(Y_{n:n-k} - u_n)/u_n$ converges weakly to $\gamma w(1)$.*

Define

$$\hat{T}_n(s) = \frac{1}{k} \sum_{j=1}^{n} \mathbb{1}_{\{Y_j > Y_{n-k:n}(1+s)\}} .$$

In this section we consider the *practical* process

$$\hat{e}_n^*(s) = \hat{T}_n(s) - T(s), \qquad s \in [0, \infty) .$$

For the process $\hat{e}_n^*(\cdot)$, the previous results yield the following corollary.

Corollary 2.5. *Assume* (H), *(1), (4), (8), (9) and (16). Then $w_n \hat{e}_n^*$ converges weakly in $D([0,\infty))$ to $w - T \cdot w(0)$, i.e.*

- *If $\lim_{n\to\infty} k\rho_n^q = 0$, then*
$$\sqrt{k}\hat{e}_n^* \Rightarrow B \circ T \tag{17}$$
where B is the Brownian bridge.
- *If $\lim_{n\to\infty} k\rho_n^q \to \infty$, then*
$$\rho_n^{-q/2} \hat{e}_n^* \Rightarrow 0 .$$

The convergence of $w_n(\hat{T}_n - T)$ to $w - T \cdot w(0)$ is standard. The surprising result is that in the LRD zone the limiting process is 0, because the limiting process of $w_n(\hat{T}_n - T_n)$ has the degenerate form $T \cdot L_q$ (up to constants). In fact, as we will see below, there is no dichotomy for the process with random levels, and the rate of convergence of $\hat{e}_n^*$ is the same as in the i.i.d. case.

To proceed, we need to introduce a more precise second order conditions on the distribution function F_Z of Z. Several types of second order assumptions have been proposed in the literature. We follow here [7].

Assumption (SO) There exists a bounded non increasing function η^* on $[0,\infty)$, regularly varying at infinity with index $-\alpha\beta$ for some $\beta \geq 0$, and such that $\lim_{t\to\infty} \eta^*(t) = 0$ and there exists a measurable function η such that for $z > 0$,

$$\mathbb{P}(Z > z) = cz^{-\alpha} \exp \int_1^z \frac{\eta(s)}{s} \, \mathrm{d}s , \tag{18}$$

$$\exists C > 0 , \quad \forall s \geq 0 , \quad |\eta(s)| \leq C\eta^*(s) . \tag{19}$$

If (18) and (19) hold, we will say that $\bar{F}_Z$ is second order regularly varying with index $-\alpha$ and rate function η^*, in shorthand $\bar{F}_Z \in 2RV(-\alpha, \eta^*)$.

Theorem 2.6. *Assume* (H), *(1), (4),* (SO) *with rate function* η^* *regularly varying at infinity with index* $-\alpha\beta$ *and there exists* $\epsilon > 0$ *such that*

$$0 < \mathbb{E}[\sigma^{2\alpha(\beta+1)+\epsilon}(X_1)] < \infty .$$

If

$$\lim_{n\to\infty} \sqrt{k}\eta^*(U(n/k)) = 0 , \tag{20}$$

then $\sqrt{k}\hat{e}_n^*$ *converges weakly in* $D([0,\infty))$ *to* $B \circ T$*, where* B *is the Brownian bridge (regardless of the behaviour of* $k\rho_n^q$*).*

Remark 1. Before, in Corollaries 2.3 and 2.5, we needed conditions (8) and (16) to establish weak convergence. In Theorem 2.6, they are replaced with (S0) and (20); see Section 2.4 for more details.

The behaviour described in Theorem 2.6 is quite unexpected, since the process with *estimated* levels $Y_{n-k:n}$ has a faster rate of convergence than the one with the deterministic levels u_n. A similar phenomenon was observed in the context of LRD based empirical processes with estimated parameters. We refer to [15] for more details.

2.3 Tail index estimation

A natural application of the asymptotic results for tail empirical process $\hat{e}_n^*$ is the asymptotic normality of the Hill estimator of the extreme value index γ defined by

$$\hat{\gamma}_n = \frac{1}{k}\sum_{i=1}^{k} \log\left(\frac{Y_{n-i+1:n}}{Y_{n-k:n}}\right) = \int_0^\infty \frac{\hat{T}_n(s)}{1+s}\,\mathrm{d}s .$$

Since $\gamma = \int_0^\infty (1+s)^{-1}T(s)\,\mathrm{d}s$, we have

$$\hat{\gamma}_n - \gamma = \int_0^\infty \frac{\hat{e}_n^*(s)}{1+s}\,\mathrm{d}s .$$

Thus we can apply Theorem 2.6 to obtain the asymptotic distribution of the Hill estimator.

Corollary 2.7. *Under the assumptions of Theorem 2.6,* $\sqrt{k}(\hat{\gamma}_n - \gamma)$ *converges weakly to the centered Gaussian distribution with variance* γ^2*.*

It is known that the above result gives the best possible rate of convergence for the Hill estimator (see [7]). The surprising result is that it is possible to achieve the i.i.d. rates regardless of H.

2.4 Second order conditions

Whereas the transfer of the tail index of Z to Y is well known, the transfer of the second order property seems to have been less investigated. We state this in the next proposition, as well as the rate of convergence of T_n to T and G_n to $G \times T$.

Proposition 2.8. *If $\bar{F}_Z \in 2RV(-\alpha, \eta^*)$, where η^* is regularly varying at infinity with index $-\alpha\beta$, for some $\beta \geq 0$, and if*

$$\mathbb{E}[\sigma^{\alpha(\beta+1)+\epsilon}(X)] < \infty \ , \tag{21}$$

for some $\epsilon > 0$, then $\bar{F} \in 2RV(-\alpha, \eta^*)$, (8) holds and

$$\|T_n - T\|_\infty = O(\eta^*(u_n)) \ . \tag{22}$$

Moreover, for any $p \geq 1$ such that $p\alpha(\beta+1) < \alpha(\beta+1) + \epsilon$,

$$\mathbb{E}\left[\sup_{s \geq 0} |G_n(X,s) - \sigma^\alpha(X)T(s)|^p\right] = O(\eta^*(u_n)^p) \ . \tag{23}$$

Examples The most commonly used second order assumption is that $\eta^*(s) = O(s^{-\alpha\beta})$ for some $\beta > 0$. Then

$$\bar{F}_Z(x) = cx^{-\alpha}(1 + O(x^{-\alpha\beta})) \quad \text{as } x \to \infty \ , \tag{24}$$

for some constant $c > 0$. Then, $\|T_n - T\|_\infty = O((k/n)^\beta)$, and the second order condition (16) becomes

$$\lim_{n\to\infty} k \left(\frac{k}{n}\right)^{2\beta} = 0 \ , \quad \text{if} \quad \lim_{n\to\infty} k\rho_n^q = 0 \tag{25}$$

and

$$\lim_{n\to\infty} \rho_n^{-q} \left(\frac{k}{n}\right)^{2\beta} = 0 \quad \text{if} \quad \lim_{n\to\infty} k\rho_n^q = \infty \ . \tag{26}$$

Condition (25) holds if both $k \ll n^{(2\beta)/(2\beta+1)}$ and $k \ll n^{2(1-H)}$. The central limit theorem with rate $\sqrt{k}$ holds if $k \asymp n^\gamma$ with

$$\gamma < 2(1-H) \vee \frac{2\beta}{2\beta+1} \ .$$

Condition (26) holds if $n^{2(1-H)} \ll k \ll n^{1-(1-H)/\beta}$. This may happen only if

$$\beta > \frac{1-H}{2H-1}$$

or equivalently

$$1 > H > \frac{1+\beta}{2\beta+1} \ .$$

As $\beta \to 0$, only for very long memory processes (i.e. H close to 1) will the LRD zone be possible.

The extreme case is the case $\beta = 0$, i.e. η^* slowly varying. For instance, if $\eta^*(x) = 1/\log(x)$ (for x large), then the tail $\bar{F}(x) = x^{-\alpha}\log(x)$ belongs to $2RV(-\alpha, \eta^*)$ and $U(t) \sim \{t\log(t)/\alpha\}^{-1/\alpha}$. The second order condition (16) holds if

$$k^{1/2}\log^{-1}(n) \to 0 \; .$$

If this condition holds, then $k\rho_n^q \to 0$ for any $H > 1/2$ and the LRD zone never arises (i.e. it is dominated by bias).

3 Numerical results

We conducted some simulation experiments to illustrate our results. We used `R` functions `HillMSE()` and `HillPlot` available on the authors webpages.

Our experiment deals with Mean Squared Error.

1. Using `R-fracdiff` package we simulated fractional Gaussian noise $X_i(d)$ with parameters $d = 0, 0.2, 0.4, 0.45$. Here, $d = H - 1/2$, so that $d = 0$ corresponds to i.i.d. case.

2. We simulated $n = 1000$ i.i.d. Pareto random variables Z_i with parameters $\alpha = 1$ and 2.

3. We set $Y_i(d) = \exp(X_i(d))Z_i$.

4. Hill estimator was constructed for different number of extremes.

5. This procedure was repeated 10000 times.

6. The results are displayed on Figure 1. On each plot, we visualise Mean Square Error (with the true centering) w.r.t. the number of extremes. Solid lines represent different LRD parameters, starting with $d = 0$ (at the bottom).

We note that for small number of extremes there is no influence of LRD parameter. Also, there is no difference for large number of extremes. This agrees with our findings in Corollary 2.7. It seems that long memory appears for average number of extremes, which does not seem to agree with Corollary 2.7. This can be explained as follows: for small number of extremes i.i.d. type of behaviour dominates (see $R_n(\cdot)$ in (32); for large number of extremes long memory of the term S_n in (32) starts to dominate. It contributes in the asymptotic behaviour of $e_n(\cdot)$, but is reduced in the asymptotic behaviour of $\hat{e}_n^*(\cdot)$. For average number of extremes influence of long memory may not be strong enough to force the aforementioned reduction.

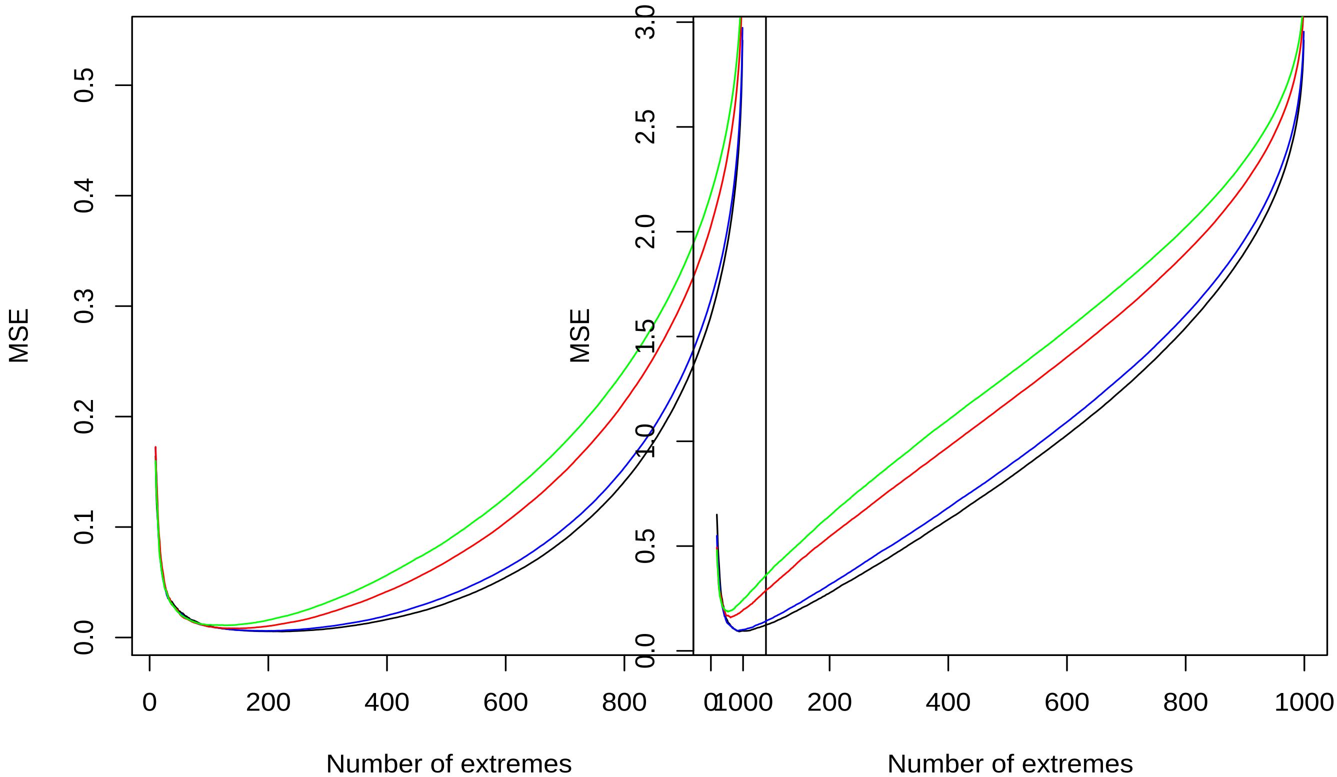

Figure 1: $\alpha = 1$ (left panel), $\alpha = 2$ (right panel)

4 Proofs

4.1 Gaussian long memory sequences

Recall that each function $G(\cdot)$ in $L^2(\mathrm{d}\mu)$, with $\mu(\mathrm{d}x) = (2\pi)^{-1/2}\exp(-x^2/2)\,\mathrm{d}x$ can be expanded as

$$G(X) = \mathbb{E}[G(X)] + \sum_{m=1}^{\infty} \frac{J(m)}{m!} H_m(X) ,$$

where $J(m) = \mathbb{E}[G(X)H_m(X)]$ and X is a standard Gaussian random variable. The smallest $q \geq 1$ such that $J(q) \neq 0$ is called the Hermite rank of G. We have

$$\mathbb{E}[G(X_0)G(X_k)] = \mathbb{E}[G(X_0)] + \sum_{m=q}^{\infty} \frac{J^2(m)}{m!} \rho_k^m , \tag{27}$$

where $\rho_k = \mathrm{cov}(X_0, X_k)$. Thus, the asymptotic behaviour of $\mathbb{E}[G(X_0)G(X_k)]$ is determined by the leading term ρ_n^q. In particular, if $1-q(1-H) > 1/2$, which implies that $n^2\rho_n^q \to \infty$,

$$\mathrm{var}\left(\sum_{j=1}^{n} G(X_j)\right) \sim \frac{J^2(q)}{q!} \frac{n^2\rho_n^q}{1-2q(1-H)} \tag{28}$$

and

$$\frac{1}{n\rho_n^{q/2}} \sum_{j=1}^{n} G(X_j) \stackrel{\mathrm{d}}{\to} J(q) L_q , \tag{29}$$

where

$$L_q = (q!(1-2q(1-H))^{-1/2} Z_{H,q}(1) \tag{30}$$

and $Z_{H,q}$ is the so-called Hermite or Rosenblatt process of order q, defined as a q-fold stochastic integral

$$Z_{H,q}(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}t(x_1+\dots+x_q)}-1}{x_1+\dots+x_q} \prod_{i=1}^{q} x_i^{-H+1/2} W(\mathrm{d}x_1)\dots W(\mathrm{d}x_q) ,$$

where W is an independently scattered Gaussian random measure with Lebesgue control measure. For more details, the reader is referred to [19]. On the other hand, if $1-q(1-H) < 1/2$, then

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} G(X_j) \xrightarrow{\mathrm{d}} \mathcal{N}(0, \Sigma_0^2), \tag{31}$$

where $\Sigma_0^2 = \mathrm{var}(G(X_0)) + 2\sum_{j=1}^{\infty} \mathrm{cov}(G(X_0), G(X_j)) < \infty$.

4.2 Decomposition of the tail empirical process

The main ingredient of the proof of our results will be the following decomposition. Let $\mathcal{X}$ be the σ-field generated by the Gaussian process $\{X_n\}$.

$$\begin{aligned} e_n(s) &= \frac{1}{n\bar{F}(u_n)} \sum_{j=1}^{n} \left\{ \mathbb{1}_{\{Y_j > (1+s)u_n\}} - \mathbb{P}(Y_j > (1+s)u_n | \mathcal{X}) \right\} \\ &\quad + \frac{1}{n\bar{F}(u_n)} \sum_{j=1}^{n} \left\{ \mathbb{P}(Y_j > (1+s)u_n | \mathcal{X}) - \bar{F}(u_n) \right\} \\ &=: R_n(s) + S_n(s) \ . \end{aligned} \tag{32}$$

Conditionnaly on $\mathcal{X}$, R_n is the sum of independent random variables, so it will be referred to as the i.i.d. part; the term S_n is the partial sum process of a subordinated Gaussian process, so it will be referred to as the LRD part.

4.3 Proof of Theorem 2.2

We first prove convergence of the finite dimensional distributions of the i.i.d. and LRD parts, then prove tightness and asymptotic independence.

4.3.1 Finite dimensional limits

Let $\stackrel{\mathrm{d}}{\to}$ denote weak convergence of finite dimensional distributions. It will be shown in Section 4.3.1 and 4.3.1, respectively, that for each $m \geq 1$ and $s_l \in [0,1]$, $l = 1, \dots, M$, $s_1 < \cdots < s_M$,

$$\sqrt{n\bar{F}(u_n)}\,(R_n(s_1), R_n(s_l) - R_n(s_{l-1}), l = 2, \dots, M) \\ \stackrel{\mathrm{d}}{\to} (\mathcal{N}(0, T(s_1)), \mathcal{N}(0, T(s_l) - T(s_{l-1})), l = 2, \dots, M) \ , \quad (33)$$

where the normal random variables are independent, and

$$\rho_n^{-q}(S_n(s_1), \dots, S_n(s_M)) \stackrel{\mathrm{d}}{\to} \frac{J(q)}{\mathbb{E}[\sigma^\alpha(X_1)]}(T(s_1), \dots, T(s_M))L_q \ , \quad (34)$$

if $1 - q(1-H) > 1/2$. On the other hand, if $1 - q(1-H) < 1/2$, then the second term $S_n(\cdot)$ is of smaller order than the first one, $R_n(\cdot)$.

The i.i.d. limit

Define

$$L_{n,j}(x,s) = \mathbb{1}_{\{\sigma(x)Z_j > (1+s)u_n\}} - \mathbb{P}(\sigma(x)Z_1 > (1+s)u_n) \ .$$

Then

$$R_n(s) = \sum_{j=1}^n L_{n,j}(X_j, s) \ .$$

Set $L_{n,j}(x) = L_{n,j}(x,0)$ and $V_n^{(m)}(x) = \mathbb{E}[L_{n,j}^m(x)]$. Note that $\mathbb{E}[V_n^{(1)}(X_j)] = 0$ and

$$V_n^{(2)}(x) = \mathbb{P}(\sigma(x)Z_1 > u_n) - \mathbb{P}^2(\sigma(x)Z_1 > u_n) \ .$$

Let $R_n := R_n(0)$. Therefore, for fixed t,

$$\begin{aligned} &\log \mathbb{E}\left[\mathrm{e}^{\mathrm{i}t\sqrt{n\bar{F}(u_n)}R_n} | \mathcal{X}\right] \\ &= \sum_{j=1}^n \log \mathbb{E}\left[\exp\left(\frac{\mathrm{i}t}{\sqrt{n\bar{F}(u_n)}}\{\mathbb{1}_{\{Y_j > u_n\}} - \mathbb{P}(Y_j > u_n \mid \mathcal{X})\}\right) \mid \mathcal{X}\right] \\ &= \sum_{j=1}^n \log \mathbb{E}\left[1 - \frac{\mathbb{i}t}{\sqrt{n\bar{F}(u_n)}}L_{n,j}(X_j) - \frac{t^2}{2n\bar{F}(u_n)}L_{n,j}^2(X_j) + L_{n,j}^3(X_j)O\left(\frac{1}{(n\bar{F}(u_n))^{3/2}}\right) \mid \mathcal{X}\right] \\ &= \frac{-t^2}{2n\bar{F}(u_n)}\sum_{j=1}^n V_n^{(2)}(X_j) + o\left(\frac{1}{n\bar{F}(u_n)}\right)\sum_{j=1}^n V_n^{(2)}(X_j) + O\left(\frac{1}{(n\bar{F}(u_n))^{3/2}}\right)\sum_{j=1}^n |V_n^{(3)}(X_j)| \ . \end{aligned} \quad (35)$$

We will show that

$$\frac{1}{n\bar{F}(u_n)}\sum_{j=1}^{n} V_n^{(2)}(X_j) \stackrel{\text{P}}{\to} 1, \tag{36}$$

given that $\mathbb{E}[\sigma^{\alpha+\delta}(X_1)] < \infty$. This also shows that the second term in (35) is negligible. Furthermore, since for sufficiently large n and $\delta > 0$ (cf. (56)),

$$|V_n^{(3)}(x)| \le C\mathbb{P}(\sigma(x)Z_1 > u_n) \le C(\sigma(x)\vee 1)^{\alpha+\delta}P(Z_1 > u_n) \ ,$$

the expected value of the last term in (35) is

$$O\left(\frac{nP(Z_1 > u_n)}{(n\bar{F}(u_n))^{3/2}}\right)\mathbb{E}[1\vee\sigma^{\alpha+\delta}(X_1)] \ .$$

Consequently, the last term in (35) converges to 0 in L^1 and in probability. Therefore, on account of (36) and the negligibility, we obtain,

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}t\sqrt{n\bar{F}(u_n)}R_n}|\mathcal{X}\right] \stackrel{\text{P}}{\to} -t^2/2 \tag{37}$$

and from bounded convergence theorem we conclude (33) (for $M = 1$ and $s = 0$). It remains to prove (36). By Lemma 2.1, for each $j \ge 1$, $G_n(X_j, s)$ converges in probability and in L^1 to $\sigma^\alpha(X_j)$. Therefore,

$$\lim_{n\to\infty}\mathbb{E}\left[\left|\frac{1}{n}\sum_{j=1}^{n}\frac{\mathbb{P}(\sigma(X_j)Z_1 > u_n \mid \mathcal{X})}{\mathbb{P}(Z_1 > u_n)} - \sigma^\alpha(X_j)\right|\right] = 0 \ . \tag{38}$$

Next, since $\sigma^\alpha(X_j)$, $j \ge 1$, is ergodic, we have

$$\frac{1}{n}\sum_{j=1}^{n}\sigma^\alpha(X_1) \stackrel{\text{P}}{\to} \mathbb{E}[\sigma^\alpha(X_1)] \ . \tag{39}$$

Thus, (38), (39) and Breiman's Lemma yields

$$\frac{1}{n\bar{F}(u_n)}\sum_{j=1}^{n}\mathbb{P}(\sigma(X_j)Z_1 > u_n \mid \mathcal{X}) \stackrel{\text{P}}{\to} 1 \ . \tag{40}$$

Write now

$$\frac{1}{n\bar{F}(u_n)}\sum_{j=1}^{n}V_n^{(2)}(X_j) = 1 + o_P(1) + \frac{1}{n\bar{F}(u_n)}\sum_{j=1}^{n}\mathbb{P}^2(\sigma(X_j)Z_1 > u_n \mid \mathcal{X}) \ .$$

By Lemma 2.1, we have, for some $\delta > 0$ small enough,

$$\frac{1}{n\bar{F}(u_n)}\sum_{j=1}^{n}\mathbb{P}^2(\sigma(X_j)Z_1 > u_n \mid \mathcal{X}) \le C\mathbb{P}(Z > u_n)\frac{1}{n}\sum_{j=1}^{n}(\sigma(X_j)\vee 1)^{2\alpha+\delta} \stackrel{\text{P}}{\to} 0 \ . \tag{41}$$

This proves (36) and (33) follows with $M = 1$ and $s_1 = 0$. The case of a general $M \ge 1$ is obtained analogously.

Long memory limit

Recall the definition (6) of $G_n(\cdot, s)$ and that $G(x) = \sigma^\alpha(x)$. Define

$$J_n(m, s) = \mathbb{E}[H_m(X_1)G_n(X_1, s)], \quad J(m) = \mathbb{E}[H_m(X_1)G(X_1)],$$

the Hermite coefficients of $G_n(\cdot, s)$ and $G(\cdot)$, respectively. Let q be the Hermite rank of $G(\cdot)$. We write (recall Assumption (H)),

$$\begin{aligned}
&\sum_{j=1}^{n}(G_n(X_j, s) - \mathbb{E}[G_n(X_j, s)]) \\
&\quad = \sum_{j=1}^{n}\sum_{m=q}^{\infty}\frac{T(s)J(m)}{m!}H_m(X_j) + \sum_{j=1}^{n}\sum_{m=q}^{\infty}\frac{J_n(m, s) - T(s)J(m)}{m!}H_m(X_j) \\
&\quad =: T(s)S_n^* + \tilde{S}_n(s) \,, \qquad (42)
\end{aligned}$$

with $S_n^* = \sum_{j=1}^n G(X_j)$. On account of Rozanov's equality (27), we have that the variance of the second term is

$$\begin{aligned}
\mathrm{var}(\tilde{S}_n(s)) &= \sum_{i,j=1}^{n}\sum_{m=q}^{\infty}\frac{(J_n(m, s) - T(s)J(m))^2}{m!}\mathrm{cov}^m(X_i, X_j) \\
&\le \sum_{i,j=1}^{n}|\mathrm{cov}^q(X_i, X_j)|\sum_{m=q}^{\infty}\frac{(J_n(m, s) - T(s)J(m))^2}{m!} \\
&= \|G_n(\cdot, s) - T(s)G(\cdot)\|^2_{L^2(d\mu)}\sum_{i,j=1}^{n}|\mathrm{cov}^q(X_i, X_j)| \\
&\le Cn^2\rho_n^q\,\|G_n(\cdot, s) - T(s)G(\cdot)\|^2_{L^2(d\mu)} \,. \qquad (43)
\end{aligned}$$

Since $\mathbb{E}[\sigma^{2\alpha+\delta}(X)] < \infty$, by Lemma 2.1, $G_n(\cdot, s)$ converges $T(s)G(\cdot)$ in $L^2(d\mu)$, uniformly with respect to s. We conclude that the second term on the right handside of (42) is $o_P\left(n\rho_n^{q/2}\right)$, i.e. it is asymptotically smaller than the first term. Furthermore,

$$S_n(s) = \frac{P(Z_1 > u_n)}{n\bar{F}(u_n)}\sum_{j=1}^{n}(G_n(X_j, s) - \mathbb{E}[G_n(X_j, s)]) \,, \qquad (44)$$

so that via (29) and (57)

$$\rho_n^{q/2}S_n(s) \xrightarrow{\mathrm{d}} \frac{J(q)T(s)}{\mathbb{E}[\sigma^\alpha(X_1)]}L_q \,, \qquad (45)$$

if $1 - q(1 - H) > 1/2$. Consequently, (34) holds for $M = 1$. The multivariate case follows immediately. On the other hand, if $1 - q(1 - H) < 1/2$, then via (31) and (57),

$$\sqrt{n}\sup_{s\in[0,1]} S_n(s) \xrightarrow{\mathrm{d}} \frac{1}{\mathbb{E}[\sigma^\alpha(X_1)]}\mathcal{N}(0, \Sigma_0^2) \,,$$

which proves negligibility with respect to the term $R_n(\cdot)$.

4.3.2 Asymptotic independence

In this section we prove asymptotic independence of $R_n(\cdot)$ and $S_n(\cdot)$. We will carry out a proof for the joint characteristic function of $(R_n, S_n) = (R_n(0), S_n(0))$. Extension to multivariate case is straightforward. On account of (37), (45) and the bounded convergence theorem, we have

$$\begin{aligned} & \mathbb{E}\left[\exp\left\{\mathrm{i}s\sqrt{n\bar{F}(u_n)}R_n + \mathrm{i}t\rho_n^{-q/2}S_n\right\}\right] \\ & = \mathbb{E}\left[\mathbb{E}[\exp\{\mathrm{i}s\sqrt{n\bar{F}(u_n)}R_n\} \mid \mathcal{X}]\exp\left(\mathrm{i}t\rho_n^{-q/2}S_n\right)\right] \\ & \to \exp(-s^2/2)\,\psi_{L_q}\left(\frac{J(q)}{\mathbb{E}[\sigma^\alpha(X_1)]}t\right) \quad \text{as } n\to\infty\,, \end{aligned}$$

where $\psi_{L_q}(\cdot)$ is the characteristic function of L_q. This proves asymptotic independence.

4.3.3 Tightness

We prove tightness for $R_n(s)$, $s \ge 0$. Let

$$L_{n,j}(x,s,t) = L_{n,j}(x,t) - L_{n,j}(x,s)\,.$$

The random variables $L_{n,j}(X_j,s,t)$ are conditionally independent given $\mathcal{X}$. Therefore,

$$\begin{aligned} & \mathbb{E}[|R_n(s_2)-R_n(s_1)|^4] = \mathbb{E}\left[\mathbb{E}[|R_n(s_2)-R_n(s_1)|^4 \mid \mathcal{X}]\right] \\ & = \frac{1}{(n\bar{F}(u_n))^2}\mathbb{E}\left[\mathbb{E}\left[\left(\sum_{j=1}^n L_{n,j}(X_j,s_1,s_2)\right)^4 \mid \mathcal{X}\right]\right] \\ & = \frac{1}{(n\bar{F}(u_n))^2}\sum_{i,j=1}^n \mathbb{E}\left[\mathbb{E}[L_{n,i}^2(X_i,s_1,s_2)\mid\mathcal{X}]\mathbb{E}[L_{n,j}^2(X_j,s_1,s_2)\mid\mathcal{X}]\right] \\ & = \frac{1}{n\bar{F}(u_n)^2}\mathbb{E}[L_{n,1}^4(X_1,s_1,s_2)] + \frac{1}{\bar{F}(u_n)^2}\{\mathbb{E}[L_{n,1}^2(X_1,s_1,s_2)]\}^2\,. \end{aligned} \tag{46}$$

Applying assumption (8), we have, for $p = 2, 4$

$$\mathbb{E}[|L_{n,1}(X_1,s_1,s_2)|^p] \le C|s_2-s_1|\bar{F}(u_n)\,,$$

which, together with (46), yields

$$\mathbb{E}[|R_n(s_2)-R_n(s_1)|^4] \le \frac{C|s_2-s_1|}{n\bar{F}(u_n)} + C|s_2-s_1|^2\,.$$

Arguing as in [18, Theorem 2.1 and Remark 2.1], this proves the tightness of $R_n(\cdot)$.

To prove the tightness of $S_n(s)$, recall that by (42) we have $S_n(s) = T(s)S_n^* + \tilde{S}_n(s)$ and the finite dimensional distributions of $n\rho_n^{q/2}\tilde{S}_n$ converge weakly to zero. We thus only have to prove the tightness of $\tilde{S}_n$ which is done by computing second moments of the increments. By the same arguments leading to (43), we have

$$\begin{aligned}\operatorname{var}(\tilde{S}_n(s) - \tilde{S}_n(s')) &\leq Cn^2\rho_n^q \|G_n(\cdot, s_2) - G_n(\cdot, s_1) - (T(s_2) - T(s_1))G(\cdot)\|_{L^2(d\mu)}^2 \\ &\leq Cn^2\rho_n^q \{\|G_n(\cdot, s_2) - G_n(\cdot, s_1)\|_{L^2(\mu)}^2 + |T(s_2) - T(s_1)|^2\} \ .\end{aligned}$$

By condition (16), we have

$$\mathbb{E}[|G_n(X_1, s_1) - G_n(X_1, s_2)|^2] \leq C|s_2 - s_1|^2$$

and thus tightness follows.

4.4 Proof of Corollary 2.5 and Theorem 2.6

Denote $\bar{T}_n = T_n - T$ and $\xi_n = \frac{Y_{n-k:n} - u_n}{u_n} = \tilde{T}_n^{\leftarrow}(1)$. Then $\tilde{T}_n(\xi_n) = 1$, and we have

$$1 = e_n(\xi_n) + T_n(\xi_n) = e_n(\xi_n) + \bar{T}_n(\xi_n) + T(\xi_n) \ .$$

Thus,

$$T(\xi_n) - 1 = -e_n(\xi_n) - \bar{T}_n(\xi_n) \ . \tag{47}$$

For any $s \geq 0$, $\hat{T}_n(s) = \tilde{T}_n(s + \xi_n(1+s))$ and $T(s + \xi_n(1+s)) = T(s)T(\xi_n)$, thus

$$\begin{aligned}\hat{e}_n^*(s) &= e_n(s + \xi_n(1+s)) + \bar{T}_n(s + \xi_n(1+s)) + T(s + \xi_n(1+s)) - T(s) \\ &= e_n(s + \xi_n(1+s)) + T(s)\{T(\xi_n) - 1\} + \bar{T}_n(s + \xi_n(1+s)) \ .\end{aligned}$$

Plugging (47) into this decomposition of $\hat{e}_n^*$, we get

$$\hat{e}_n^*(s) = e_n(s + \xi_n(1+s)) - T(s)e_n(\xi_n) + \bar{T}_n(s + \xi_n(1+s)) - T(s)\bar{T}_n(\xi_n) \ . \tag{48}$$

In order to prove Corollary 2.5, we write

$$w_n\hat{e}_n^*(s) = w_n\{e_n(s + \xi_n(1+s)) - T(s)e_n(\xi_n)\} + O(w_n\|T_n - T\|_\infty) \ . \tag{49}$$

Since the convergence in Theorem 2.2 is uniform, and by Corollary 2.4 $\xi_n = o_P(1)$, the first term in (49) converges in $D([0,\infty))$ to $w - T \cdot w(0)$. Under the second order condition (16), the second term is $o(1)$. This concludes the proof of Theorem 2.5.

We now prove Theorem 2.6. In order to study the second-order asymptotics of $w_n\hat{e}_n^*(s)$, we need precise expansion for $e_n(s + \xi_n(1+s))$ and $e_n(\xi)$. For this we will use the expansions of the tail empirical process in Section 4.3.1. Since $\bar{F}(u_n) = k/n$, using (32), (42) and (44), we have

$$e_n(s) = R_n(s) + \frac{\bar{F}_Z(u_n)}{n\bar{F}(u_n)} T(s) S_n^* + \frac{\bar{F}_Z(u_n)}{n\bar{F}(u_n)} \tilde{S}_n(s) , \tag{50}$$

which, noting again that $T(s + \xi_n(1+s)) = T(s)T(\xi_n)$, yields

$$\begin{aligned} e_n(s + \xi_n(1+s)) - T(s)e_n(\xi_n) &= R_n(s + \xi_n(1+s)) - T(s)R_n(\xi_n) \\ &\quad + \frac{\bar{F}_Z(u_n)}{n\bar{F}(u_n)} \{\tilde{S}_n(s + \xi_n(1+s)) - T(s)\tilde{S}_n(\xi_n)\} \end{aligned}$$

and

$$\begin{aligned} \hat{e}_n^*(s) &= R_n(s + \xi_n(1+s)) - T(s)R_n(\xi_n) \\ &\quad + \frac{\bar{F}_Z(u_n)}{n\bar{F}(u_n)} \{\tilde{S}_n(s + \xi_n(1+s)) - T(s)\tilde{S}_n(\xi_n)\} + \bar{T}_n(s + \xi_n(1+s)) - T(s)\bar{T}_n(\xi_n) . \end{aligned} \tag{51}$$

Similarly to (43), and utilising $\bar{F}_Z(u_n)/\bar{F}(u_n) = O(1)$,

$$\text{var}\left(\frac{\bar{F}_Z(u_n)}{n\bar{F}(u_n)} \tilde{S}_n(s)\right) \le C\{\rho_n^q \vee \ell_1(n)n^{-1}\} \|G_n(\cdot, s) - T(s)G(\cdot)\|_{L^2(\mu)}^2 .$$

Using the second order Assumption (SO) through (23), we obtain

$$\text{var}\left(\frac{\bar{F}_Z(u_n)}{n\bar{F}(u_n)} \tilde{S}_n(s)\right) = O\left(\{\rho_n^q \vee \ell_1(n)n^{-1}\} \eta^*(u_n)^2\right) = o\left(\eta^*(u_n)^2\right) . \tag{52}$$

Using (50) in the representation (48) and since Proposition 2.8 implies that $\|T_n - T\|_\infty = O(\eta^*(u_n))$, we obtain:

$$\hat{e}_n^*(s) = R_n(s + \xi_n(1+s)) - T(s)R_n(\xi_n) + O_P(\eta^*(u_n)) .$$

Since we have already proved that the convergence of $\sqrt{k}R_n$ is uniform, we obtain that $\sqrt{k}e_n^*$ converges in the sense of finite dimensional distribution to $B \circ T$, where B is the Brownian bridge, if the second order condition (20) holds. To prove tightness, we only have to prove that $k^{1/2}n^{-1}S_n$ converges uniformly to zero on compact sets. For $s \ge 0$ and $x \in \mathbb{R}$, denote $\bar{G}_n(x, s) = G_n(x, s) - T(s)G(x)$ and recall that we have shown in Section 4.3.3 that

$$n^{-2}\text{var}(\tilde{S}_n(s) - \tilde{S}_n(s')) \le C\|\bar{G}_n(\cdot, s_2) - \bar{G}_n(\cdot, s_1)\|_{L^2(d\mu)}^2 .$$

Applying (61), we get

$$n^{-2}\text{var}(\tilde{S}_n(s) - \tilde{S}_n(s')) \le C(\eta^*(u_n))^2 \mathbb{E}\left[(\sigma(x) \vee 1)^{2\alpha(\beta+1)+\epsilon}\right] (s - s')^2 ,$$

which proves that $k^{1/2}n^{-1}\tilde{S}_n$ converges uniformly to zero on compact sets.

4.5 Proof of Corollary 2.7

Using the decomposition (51), and the identity $\int_0^\infty (1+s)^{-1} T(s)\,\mathrm{d}s = \gamma$, we have

$$
\begin{aligned}
\hat{\gamma}_n - \gamma = \int_0^\infty \frac{\hat{e}_n^*(s)}{1+s}\,\mathrm{d}s = \int_0^\infty \frac{R_n(s+\xi_n(1+s))}{1+s}\,\mathrm{d}s - \gamma R_n(\xi_n) \\
+ \frac{\bar{F}_Z(u_n)}{n\bar{F}(u_n)} \int_0^\infty \frac{\tilde{S}_n(s+\xi_n(1+s))}{1+s}\,\mathrm{d}s - \gamma \frac{\bar{F}_Z(u_n)}{n\bar{F}(u_n)} \tilde{S}_n(\xi_n) \qquad (53) \\
+ \int_0^\infty \frac{\bar{T}_n(s+\xi_n(1+s))}{1+s}\,\mathrm{d}s - \gamma \bar{T}_n(\xi_n)\ . \qquad (54)
\end{aligned}
$$

We must prove that the terms in (53) and (54) are $O_P(\eta^*(u_n))$ and that

$$
\sqrt{k}\int_0^\infty (1+s)^{-1} R_n(s+\xi_n(1+s))\,\mathrm{d}s \xrightarrow{\mathrm{d}} \int_0^\infty \frac{W \circ T(s)}{1+s}\,\mathrm{d}s = \gamma \int_0^1 \frac{W(t)}{t}\,\mathrm{d}t\ . \qquad (55)
$$

To prove (55), we follow the lines of [16, Section 9.1.2]. We must prove that we can apply continuous mapping. To do this, it suffices to establish that for any $\delta > 0$ we have

$$
\lim_{M\to\infty} \limsup_{n\to\infty} A_{n,M} = 0\ ,
$$

where

$$
A_{n,M} = \mathbb{P}\left(\sqrt{k} \int_M^\infty \left| \frac{1}{k} \sum_{j=1}^n \left(1_{\{Y_j > u_n s\}} - P\left(Y_j > u_n s | \mathcal{X}\right)\right) \right| \frac{ds}{s} > \delta \right)\ .
$$

By Markov's inequality, conditional independence and Potter's bound [2, Theorem 1.5.6] , we have, for some $\epsilon > 0$,

$$
A_{n,M} \le C\frac{\sqrt{n}}{\sqrt{k}} \int_M^\infty \frac{\mathbb{P}^{1/2}(Y > u_n s)}{s}\,\mathrm{d}s \le C\sqrt{\frac{n\bar{F}(u_n)}{k}} \int_M^\infty s^{-1-\alpha/2+\epsilon}\,\mathrm{d}s \le CM^{-\alpha/2+\epsilon} \to 0
$$

as $M \to \infty$, since $k = n\bar{F}(u_n)$. This proves (55). To get a bound for (54), we use (58) which yields, for all $t \ge 0$,

$$
|\bar{T}_n(t)| \le C\eta^*(u_n)(1+t)^{-\alpha+\rho\pm\epsilon}\ .
$$

Thus $\bar{T}_n(\xi_n) = O_P(\eta^*(u_n))$ and $|\bar{T}_n(s+\xi_n(1+s))| \le C\eta^*(u_n)(1+s)^{-\alpha+\rho+\epsilon}(1+\xi_n)^{-\alpha}$, thus

$$
\int_0^\infty \frac{|T_n(s+\xi_n(1+s))|}{1+s}\,\mathrm{d}s = O_P(\eta^*(u_n))\ .
$$

We finally bound (53).

$$
\int_0^\infty \frac{n^{-1}\tilde{S}_n(s+\xi_n(1+s))}{1+s}\,\mathrm{d}s = \int_{\xi_n}^\infty \frac{n^{-1}\tilde{S}_n(u)}{1+u}\,\mathrm{d}u\ .
$$

Since $\xi_n = o_P(1)$, we can write

$$\mathbb{P}\left(k^{1/2}\int_{\xi_n}^{\infty} \frac{n^{-1}\tilde{S}_n(u)}{1+u}\,\mathrm{d}u > \epsilon\right) \le \mathbb{P}(\xi_n > 1) + \mathbb{P}\left(k^{1/2}\int_{1}^{\infty} \frac{n^{-1}|\tilde{S}_n(u)|}{1+u}\,\mathrm{d}u > \epsilon\right)$$
$$\le o(1) + \frac{k^{1/2}}{n\epsilon}\int_1^\infty \frac{\mathbb{E}^{1/2}[\tilde{S}_n^2(s)]}{1+s}\,\mathrm{d}s$$

Applying (43) and (67) yields

$$\int_1^\infty \frac{n^{-1}\mathbb{E}^{1/2}[\tilde{S}_n^2(s)]}{1+s}\,\mathrm{d}s \le C\rho_n^{q/2}\eta^*(u_n)\int_0^\infty s^{-\alpha(\beta+1)/2+\epsilon-1}\,\mathrm{d}s = o_P(k^{-1/2}) .$$

Thus the first term in (53) is $o_P(k^{-1/2})$, and so is the second term since $k^{1/2}n^{-1}\tilde{S}_n$ converges uniformly to zero on compact sets. This concludes the proof of Corollary 2.7.

4.6 Second order regular variation

The main tool in the study of the tail of the product YZ is the following bound. For any $\epsilon > 0$, there exists a constant C such that, for all $y > 0$,

$$\frac{\mathbb{P}(yZ_1 > x)}{\mathbb{P}(Z_1 > x)} \le C(1 \vee y^{\alpha+\epsilon}) . \tag{56}$$

This bound is trivial if $y < 1$ and follows from Potter's bounds if $y > 1$.

Proof of Lemma 2.1. By Breiman's Lemma, we know that for any sequence u_n such that $u_n \to \infty$,

$$\lim_{n\to\infty} G_n(x,s) = \lim_{n\to\infty} \frac{\mathbb{P}(\sigma(x)Z_1 > (1+s)u_n)}{\mathbb{P}(Z > u_n)}) = \sigma^\alpha(x)(1+s)^{-\alpha} = \sigma^\alpha(x)T(s) . \tag{57}$$

If $\mathbb{E}[\sigma^{\alpha+\epsilon}(X)] < \infty$, then the bound (56) implies that the convergence (57) holds in $L^p(\mu)$ for any p such that $p\alpha < \alpha+\epsilon$, uniformly with respect to s, i.e.

$$\lim_{n\to\infty} \mathbb{E}[\sup_{s\ge 0}|G_n(X,s) - \sigma^\alpha(X)T(s)|^p] = 0 .$$

□

Before proving Proposition 2.8, we need the following lemma which gives a non uniform rate of convergence.

Lemma 4.1. *If (4), (18) and (19) hold, if η^* is regularly varying at infinity with index ρ, for some $\rho \le 0$, then (8) holds and for any $\epsilon > 0$, there exists a constant C such that*

$$\forall t \ge 1\ , \quad \forall z > 0\ , \quad \left|\frac{\mathbb{P}(Z > zt)}{\mathbb{P}(Z > t)} - z^{-\alpha}\right| \le C\eta^*(t)z^{-\alpha+\rho}(z \vee z^{-1})^\epsilon . \tag{58}$$

Proof. Since η^* is decreasing, using the bound $|\mathrm{e}^u - 1| \le u\mathrm{e}^{u_+}$ with $u_+ = \max(u, 0)$, we have, for all $z > 0$,

$$\begin{aligned}\left|\frac{\mathbb{P}(Z > zt)}{\mathbb{P}(Z > t)} - z^{-\alpha}\right| &= z^{-\alpha}\left|\exp\int_1^z \frac{\eta(ts)}{s}\,\mathrm{d}s - 1\right| \\ &\le Cz^{-\alpha}\int_{z\wedge 1}^{z\vee 1}\frac{\eta^*(st)}{s}\,\mathrm{d}s\ \exp\int_{z\wedge 1}^{z\vee 1}\frac{\eta^*(st)}{s}\,\mathrm{d}s \\ &\le Cz^{-\alpha}\log(z)\,\eta^*(t(z\wedge 1))\ \exp\int_{z\wedge 1}^{z\vee 1}\frac{\eta^*(st)}{s}\,\mathrm{d}s \\ &\le Cz^{-\alpha}(z\wedge 1)^{\rho-\epsilon/2}\,\eta^*(t)\ \exp\int_{z\wedge 1}^{z\vee 1}\frac{\eta^*(st)}{s}\,\mathrm{d}s\ . \end{aligned} \tag{59}$$

We now distinguish three cases. Recall that η^* is decreasing.

- If $z \ge 1$, then $z \to \exp\int_1^z s^{-1}\eta^*(s)\,\mathrm{d}s$ is a slowly varying function by Karamata's representation Theorem, and is $O(z^{\epsilon/2})$ for any $\epsilon > 0$. Plugging this bound into (59) yields (58).
- If $z < 1$ and $tz \ge 1$, then

$$\exp\int_z^1 \frac{\eta^*(st)}{s}\,\mathrm{d}s = \exp\int_1^{1/z}\frac{\eta^*(stz)}{s}\,\mathrm{d}s \le \exp\int_1^{1/z}\frac{\eta^*(s)}{s}\,\mathrm{d}s = O(z^{-\epsilon/2})$$

  for any $\epsilon > 0$ by the same argument as above and this yields (58).
- If $tz < 1$, then $t^r \le z^{-r}$ for any $r > 0$ and $t^{\rho-\epsilon} = O(\eta^*(t))$ for any $\epsilon > 0$. Thus

$$\begin{aligned}\left|\frac{\mathbb{P}(Z > zt)}{\mathbb{P}(Z > t)} - z^{-\alpha}\right| &\le \frac{1}{\mathbb{P}(Z > t)} + z^{-\alpha} \le Ct^{\alpha+\epsilon/2} + z^{-\alpha} \le Cz^{-\alpha-\epsilon/2} \\ &\le Cz^{-\alpha+\rho-\epsilon}t^{\rho-\epsilon/2} \le Cz^{-\alpha+\rho-\epsilon}\eta^*(t)\ .\end{aligned}$$

This concludes the proof of (58). We now prove (10). In order to prove Theorem 2.6, we will prove that for any $\epsilon > 0$, there exists a constant C such that for all $t \ge 1$ and $b > a > 0$,

$$\left|\frac{\mathbb{P}(at < Z \le bt)}{\mathbb{P}(Z > t)} - (a^{-\alpha} - b^{-\alpha})\right| \le C\eta^*((a\wedge 1)t)(a\wedge 1)^{-\alpha-1-\epsilon}(b-a)\ . \tag{60}$$

Since η^* is decreasing, the bound (10) is a consequence of (60). Applying (56), the latter yields the bound

$$\left|\frac{\mathbb{P}(at < Z \le bt)}{\mathbb{P}(Z > t)} - (a^{-\alpha} - b^{-\alpha})\right| \le C\eta^*(t)(a\wedge 1)^{-\alpha+\rho-\epsilon}(b-a)\ . \tag{61}$$

which is used in the proof of Theorem 2.6. Let ℓ be the function slowly varying at infinity that appears in (4), defined on $[0,\infty)$ by $\ell(t) = t^\alpha \mathbb{P}(Z > t)$. Assumption (SO) implies that

$$\ell(t) = \ell(1) \exp \int_1^t \eta(s) \, \frac{\mathrm{d}s}{s} \tag{62}$$

where the function η is measurable and bounded. This implies that the function ℓ is the solution of the equation

$$\ell(t) = \ell(1) + \int_1^t \eta(s)\ell(s) \, \frac{\mathrm{d}s}{s} \; . \tag{63}$$

Conversely, if ℓ satisfies (63) then (62) holds. We first prove the following useful bound. For any $\epsilon > 0$, there exists a constant C such that for any $t \geq 1$ and all $a > 0$,

$$\frac{\ell(at)}{\ell(t)} \leq C a^{\pm\epsilon} \; , \tag{64}$$

where we denote $a^{\pm\epsilon} = \max(a^\epsilon, a^{-\epsilon})$. Indeed, if $at \geq 1$, then, η^* being decreasing, we have

$$\frac{\ell(at)}{\ell(t)} \leq C \exp \int_{a\wedge 1}^{a \vee 1} \frac{\eta^*(ts)}{s} \, \mathrm{d}s \leq C \exp \int_1^{a\vee(1/a)} \frac{\eta^*(ts)}{s} \, \mathrm{d}s \leq C a^{\pm\epsilon} \; ,$$

since the latter function is slowly varying by Karamata's representation theorem. If $at < 1$, then $\ell(at) \leq 1$ and $\ell^{-1}(t) = o(t^\epsilon) = o(a^{-\epsilon})$. This proves (64). Next, applying (63) and (64), for any $\epsilon > 0$ and $0 < a < b$, we have

$$\begin{aligned} \left| \frac{\ell(bt)}{\ell(at)} - 1 \right| &= \left| \int_a^b \eta(st) \, \frac{\ell(st)}{\ell(at)} \, \frac{\mathrm{d}s}{s} \right| \leq C a^{\pm\varepsilon} \left| \int_a^b \eta(st) \, \frac{\ell(st)}{\ell(t)} \, \frac{\mathrm{d}s}{s} \right| \\ &\leq C\eta^*(at) \int_a^b s^{\pm 2\epsilon - 1} \, \mathrm{d}s \leq C\eta^*(at) \, a^{\pm\epsilon-1}(b-a) \; . \end{aligned} \tag{65}$$

Applying (64) and (65), we also obtain

$$\left| \frac{\ell(at)}{\ell(t)} - 1 \right| \leq C\eta^*((a\wedge 1)t) \, a^{\pm\epsilon} \; . \tag{66}$$

For $\epsilon > 0$ and $0 < a < b$, we have

$$\begin{aligned} \frac{\mathbb{P}(at < Z \leq bt)}{\mathbb{P}(Z > t)} - (a^{-\alpha} - b^{-\alpha}) &= a^{-\alpha} \left\{ \frac{\ell(at)}{\ell(t)} - 1 \right\} - b^{-\alpha} \left\{ \frac{\ell(bt)}{\ell(t)} - 1 \right\} \\ &= (a^{-\alpha} - b^{-\alpha}) \left\{ \frac{\ell(at)}{\ell(t)} - 1 \right\} - b^{-\alpha} \frac{\ell(at)}{\ell(t)} \left\{ \frac{\ell(bt)}{\ell(at)} - 1 \right\} \; , \end{aligned}$$

which yields

$$\left| \frac{\mathbb{P}(at < Z \leq bt)}{\mathbb{P}(Z > t)} - (a^{-\alpha} - b^{-\alpha}) \right| \leq C\eta^*((a\wedge 1)t) a^{\alpha - 1 \pm\epsilon}(b-a) \; .$$

□

Proof of Proposition 2.8. Define the function $\bar{\sigma}$ by $\bar{\sigma}(x) = \sigma(x) \vee 1$. Applying (58) with $(1+s)/\sigma(x)$ instead of z and u_n for t, we get

$$|G_n(x,s) - \sigma^\alpha(x)T(s)| = \left| \frac{\mathbb{P}(\sigma(x)Z > u_n(1+s))}{\mathbb{P}(Z > u_n)} - \sigma^\alpha(x)T(s) \right| \\ \leq C\eta^*(u_n)\bar{\sigma}(x)^{\alpha(\beta+1)+\epsilon}(1+s)^{-\alpha(\beta+1)+\epsilon} \,. \quad (67)$$

This implies, for all p such that $\mathbb{E}[\sigma^{p\alpha(\beta+1)+\epsilon}(X)] < \infty$, that

$$\mathbb{E}\left[\sup_{s\geq 1} |G_n(X,s) - T(s)\sigma^\alpha(X)|^p\right] = O(\{\eta^*(u_n)\}^p) \,.$$

This proves (23) which in turn implies (22) since $T_n(s) = \frac{\bar{F}(u_n)}{\bar{F}_Z(u_n)}\mathbb{E}[G_n(X,s)]$. In order to prove that $\bar{F}_Y \in 2RV(-\alpha,\eta^*)$, denote $\tilde{\ell}(y) = y^\alpha\mathbb{P}(Y > y)$. We will prove that there exists a measurable function $\tilde{\eta}$ such that (63) holds with $\tilde{\ell}$ and $\tilde{\eta}$. Denote $\xi = \sigma(X)$. Applying (63) and using the independence of ξ and Z, we have

$$\begin{aligned} \tilde{\ell}(y) &= \mathbb{E}[\xi^\alpha\ell(y/\sigma)] = \ell(1)\mathbb{E}[\xi^\alpha] + \mathbb{E}\left[\xi^\alpha\int_1^{y/\xi}\eta(s)\ell(s)\frac{\mathrm{d}s}{s}\right] \\ &= \ell(1)\mathbb{E}[\xi^\alpha] + \mathbb{E}\left[\xi^\alpha\int_\xi^y\eta(s/\xi)\ell(s/\xi)\frac{\mathrm{d}s}{s}\right] \\ &= \mathbb{E}\left[\xi^\alpha\left\{\ell(1) - \int_1^\xi\eta(s/\xi)\ell(s/\xi)\frac{\mathrm{d}s}{s}\right\}\right] + \mathbb{E}\left[\xi^\alpha\int_1^y\eta(s/\xi)\ell(s/\xi)\frac{\mathrm{d}s}{s}\right] \\ &= \mathbb{E}\left[\xi^\alpha\left\{\ell(1) + \int_{1/\xi}^1\eta(s)\ell(s)\frac{\mathrm{d}s}{s}\right\}\right] + \int_1^y\mathbb{E}[\xi^\alpha\eta(s/\xi)\ell(s/\xi)]\frac{\mathrm{d}s}{s} \\ &= \mathbb{E}\left[\xi^\alpha\ell(1/\xi)\right] + \int_1^y\mathbb{E}[\xi^\alpha\eta(s/\xi)\ell(s/\xi)]\frac{\mathrm{d}s}{s} = \tilde{\ell}(1) + \int_1^t\tilde{\eta}(s)\tilde{\ell}(s)\frac{\mathrm{d}s}{s} \,, \end{aligned}$$

where we have defined

$$\tilde{\eta}(s) = \frac{\mathbb{E}[\xi^\alpha\eta(s/\xi)\ell(s/\xi)]}{\mathbb{E}[\xi^\alpha\ell(s/\xi)]} = \frac{\mathbb{E}[\xi^\alpha\eta(s/\xi)\ell(s/\xi)/\ell(s)]}{\mathbb{E}[\xi^\alpha\ell(s/\xi)/\ell(s)]} \,.$$

The denominator of the last expression is bounded away from zero. Indeed, let $\epsilon > 0$ be such that $\mathbb{P}(\xi \geq \epsilon) > 0$. Then

$$\mathbb{E}[\xi^\alpha\ell(s/\xi)/\ell(s)] = \frac{\mathbb{P}(\xi Z > s)}{\mathbb{P}(Z > s)} \geq \frac{\mathbb{P}(\xi \geq \epsilon)\mathbb{P}(Z > s/\epsilon)}{\mathbb{P}(Z > s)} \,.$$

Since Z has a regularly varying tail, it holds that $\inf_{s\geq 0}\mathbb{P}(Z > s/\epsilon)/\mathbb{P}(Z > s) > 0$. This proves our claim. Thus, applying (56) with the regularly varying function η^*, we get, for $\epsilon > 0$ such that $\exp[\xi^{\alpha-\rho+\epsilon}] < \infty$,

$$|\tilde{\eta}(s)| \leq C\eta^*(s)\,\mathbb{E}[\xi^\alpha\{\eta^*(s/\xi)/\eta^*(s)\}\{\ell(s/\xi)/\ell(s)\}] \leq C\eta^*(x)\,\mathbb{E}[\xi^\alpha(\xi \vee 1)^{-\rho+\epsilon}] \,.$$

Thus $\tilde{\ell}$ satisfies equation (63) with $\tilde{\eta}$ such that $|\eta| \leq C\eta^*$, thus $Y \in 2RV(-\alpha,\eta^*)$. Since Condition (SO) implies (8), this concludes the proof of Proposition 2.8. □

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