On Beta-Product Convolutions

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Abstract: Let R be a positive random variable independent of S which is beta distributed. In this paper we are interested on the relation between the distribution function of R and that of RS. For this model we derive first some distributional properties, and then investigate the lower tail asymptotics of RS when R is regularly varying at 0, and vice-versa. The applications we present in this paper concern a) the simplicity of Dirichlet distributions, b) asymptotics of the sample minima of elliptical distributions, and c) the effect of the scaling on the asymptotics of aggregated risks.

Key words and phrases: Weyl fractional-order integral operator; Williamson d-transform; random scaling; weighted Dirichlet distribution; elliptical distribution; Archimedean copula; product convolution; max-domain of attraction; asymptotic independence; asymptotics of sample minima; lower tail asymptotics, risk aggregation.

1 Introduction

Let R be a positive random variable independent of $S \in (0,1)$ almost surely. In this paper we discuss the random scaling model

$$W \stackrel{d}{=} RS, \tag{1.1}$$

with W the scaled version of R ($\stackrel{d}{=}$ stands for equality of the distribution functions). In order to derive distributional properties of W we need to specify the distribution function of S; a tractable instance with various applications is the simple case that S is a beta distributed random variable.

Numerous investigations and application of (1.1) and related models are available in the literature. We mention some recent contributions, see also the references therein: Tang and Tsitsiashvili (2003, 2004), Gomes et al. (2004), Galambos and Simonelli (2004), Maulik and Resnick (2004), Denisov and Zwart (2005), Nadarajah (2005), Nadarajah and Kotz (2005), Jessen and Mikosch (2006), Tang (2006, 2008), Pakes (2007), Pakes and Navarro (2007), Charpentier and Segers (2007, 2008, 2009), Hashorva (2008, 2009a,b, 2010), Hashorva and Pakes (2009), McNeil and Nešlehová (2009a,b), Hashorva et al. (2010), Liu and Tang (2010).

In a financial or insurance framework, the random scaling model (1.1) appears naturally with W the deflated risk arising from some loss or investment R which is independent from the random scaling/deflating factor S. Other prominent applications in the literature concern modeling of network data, see e.g., D'Auria and Resnick (2006, 2008).

If the random variable R is not directly observable, but the distribution of S is known, and W is observable, a natural question arising from (1.1) is the recovery of the distribution function (df) of R, or its estimation. Such a question arises for instance when estimating the true claim cost of a glass insurance coverage. Indeed, if $R_i, i \geq 1$ models the losses payed to claims reported from some glass coverage of a motor portfolio, the insurer

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is interested in the estimation of the true claim cost W_i. However, these costs are typically deflations of R_i, where the deflator S_i explains the presence of fraud or other effects. In this setup R_i is not observed.
In certain cases the df of the scaling random variable is known, or it can be estimated, which prompts the insurer to attempt to recover the df of the true losses. This is possible when the df of W is a beta-product convolution, i.e., the scaling random variable S is beta distributed.

The principal aim of this paper is the investigation of the lower tail asymptotics of W if that of R is known, and vice-versa. Recent results and applications concerning the relation of upper tail asymptotics are presented in Hashorva and Pakes (2009) and Hashorva et al. (2010).

Our working assumption on R is that its df belongs to the min-domain of attraction of some univariate extreme value distribution function. We focus particularly on the case S is beta distributed. This model is interesting since it allows us to recover distributional and asymptotical properties of R when those of W are known, and vice-versa.

Distributional properties and results such as lower (upper) tail asymptotics of beta-product convolutions are of certain importance for insurance application when dealing for instance with the modeling of small (large) claims which are typically affected by some random deflation factor. In fact, from the financial point of view, insurance companies do not suffer from small claims but from the large ones, however, understanding of small claims is important for at least two reasons: a) claim handling is expansive even for zero-losses or very small ones, b) the choice of deductibles and the calculation of pure premiums can be significantly improved if the effect of inflation/deflation on small claims is adequately modeled. In finance, modeling of the effect of a deflator, which can practically ruin an investment, is very important.

We present in this paper three applications of our results:
a) First we answer the following question: Are multivariate Dirichlet distributions simple? In the recent paper McNeil and Nešlehová (2009) it is shown that L_1-Dirichlet distributions are closely related to Archimedean copula, which is being widely used in finance and insurance, see e.g., Embrechts et al. (2001), Juri and Wüthrich (2002, 2003), Müller and Scarsini (2005), Charpentier and Segers (2007, 2008, 2009), Embrechts et al. (2009) and the references therein.

In fact the Archimedean copula is defined only in terms of a univariate survival function, see e.g., McNeil and Nešlehová (2009b) or Charpentier and Segers (2009). Our first application shows that the whole distribution function of a weighted Dirichlet random vector is also defined solely by a univariate survival function, which is actually a consequence of the beta-independent splitting property (see e.g., Hashorva (2008)) and the random scaling model behind these distributions. For those not familiar with multivariate Dirichlet distributions we mention that a prominent subclass is that of elliptical ones with the Gaussian and Kotz type distributions being two canonical examples.

b) Our second application concerns the asymptotic behaviour of the minima of elliptical random vectors; we show that the multivariate sample minima is attracted by some multivariate df with independent components, provided that the associated random radius has df being regularly varying at 0.

c) In our last application we consider the aggregation of two risks with special dependence structure similar to that of bivariate elliptical random vectors. Aggregation of risks is of special importance in insurance and in various statistical applications, see the recent contributions Alink et al. (2004), Denuit et al. (2005), Albrecher et al. (2006), Dhaene et al. (2008), Asmussen and Rojas-Nandaypa (2008), Embrechts and Puccetti (2008, 2009a,b), Asmussen et al. (2009), Embrechts et al. (2009), Foss and Richards (2009), Geluk and Tang (2009), Geluk and Tang (2009), Mitra and Resnick (2009), Valdez et al. (2009).

Brief outline of the rest of the paper: We proceed with a section dedicated to notation and preliminary results. In Section 3 we discuss some distributional properties of the beta random scaling model.
Lower tail asymptotics for W and R is investigated in Section 4. The above mentioned applications are placed in Section 5. Proofs of all the results are relegated to Section 6. In the Appendix (last section) we present some basic properties of the Weyl fractional-order integral operator.

2 Preliminaries

In this section we present our notation and discuss briefly the Weyl fractional-order integral operator and regularly varying functions. Throughout the paper α, β are two positive constants, and $B_{\alpha,\beta}$ denotes a beta random variable with density functions

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0,1),$$

where $\Gamma(\cdot)$ is the gamma function.

In the following R is a positive random variable with df H (we write $R \sim H$) and $H_{\alpha,\beta}$ is the df of W with stochastic representation (1.1). We assume throughout this paper that $H(0) = 0$, i.e., the lower endpoint of H equals 0. In the sequel the upper endpoint of H is denoted by ω with $\omega \in (0, \infty]$. In this setup the df of W is said to be a product convolution distribution given in terms of H and the df of S. As mentioned in the Introduction when S is beta distributed with parameters α, β, (write $H_{\alpha,\beta}$ for the df of W), then the relation between H and $H_{\alpha,\beta}$ is very tractable due to the role of the Weyl fractional-order integral operator. We refer to $H_{\alpha,\beta}$ alternatively as a beta-product convolution.
Next, we introduce the aforementioned operator acting on real-valued measurable functions h defined on $(0, \infty)$. For a given constant $\beta \in (0, \infty)$ the Weyl fractional-order integral operator I_β is defined by

$$(I_\beta h)(x) \quad := \quad \frac{1}{\Gamma(\beta)} \int_x^\infty (y-x)^{\beta-1} h(y)\, dy, \quad x > 0.$$

Since $I_\beta h_\lambda$ with $h_\lambda(x) = \exp(-\lambda x), x > 0, \lambda \in (0,1)$ is well-defined and

$$(I_\beta h_\lambda)(x) \quad = \quad \frac{1}{\Gamma(\beta)} \int_x^\infty (y-x)^{\beta-1} \exp(-\lambda y)\, dy = \exp(-\lambda x), \quad x > 0 \tag{2.1}$$

h_λ is a fixed point of I_β. Now, if for any $\varepsilon > 0$ we have

$$\int_\varepsilon^\infty x^{\beta-1} |h(x)|\, dx < \infty,$$

which is abbreviated by $h \in \mathcal{I}_\beta$, then $(I_\beta h)(x)$ is almost surely finite for all $x \in (0, \infty)$.
It follows easily that

$$H_{\alpha,\beta}(x) \quad = \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} x^\alpha (I_\beta p_{-\alpha-\beta} H)(x), \quad x \in (0, \omega), \quad \text{with } p_s(x) := x^s, s \in I\!R \tag{2.2}$$

showing the importance of the Weyl fractional-order integral operator in the setup of beta random scaling.

When $\alpha = 1$ we have $\boldsymbol{P}\{B_{1,\beta} > s\} = (1-s)^\beta, s \in (0,1)$, hence (2.2) simplifies to

$$\overline{H}_{\alpha,\beta}(x) = \int_x^\infty (1 - x/y)^\beta\, dH(y), \quad \overline{H}_{\alpha,\beta} := 1 - H_{\alpha,\beta}.$$

This leads us to the introduction of the Weyl-Stieltjes fractional-order integral operator $\mathcal{J}_{\beta,g}$ with $g:(0,\infty)\to I\!R$ a measurable weight function defined by

$$(\mathcal{J}_{\beta,g}H)(x) \quad := \quad \frac{1}{\Gamma(\beta)}\int_x^\infty (y-x)^{\beta-1}g(y)\,dH(y), \quad x\in(0,\omega).$$

With this notation we have

$$\overline{H}_{1,\beta}(x) \quad = \quad \Gamma(\beta+1)(\mathcal{J}_{\beta+1,p_{1-\beta}}H)(x), \quad x\in(0,\omega). \tag{2.3}$$

When $\beta=d\in I\!N$, then $\Gamma(\beta)(\mathcal{J}_{\beta,p_{1-\beta}}H)$ is the Williamson d-transform which plays a crucial role in the analysis of L_1-norm Dirichlet distributions and Archimedean copula, Fang et al. (1990), and McNeil and Nešlehová (2009a,b).

For the derivation of the lower tail asymptotics of W we impose an assumption on R motivated by univariate extreme value theory. Specifically, we assume that $R\sim H$ is regularly varying at 0 with some index $\gamma\in(0,\infty)$, i.e.,

$$\lim_{x\downarrow 0}\frac{H(tx)}{H(x)} \quad = \quad t^\gamma, \quad \forall t\in(0,\infty). \tag{2.4}$$

We write alternatively $H\in RV_\gamma$ or $R\in RV_\gamma$. (2.4) is equivalent with the assertion $1/R$ is regularly varying at infinity with index $-\gamma$, or the df of $1/R$ is in the max-domain of attraction of the Fréchet df $\Phi_\gamma(x)=\exp(-x^{-\gamma}), x>0$. See Resnick (1987), Bingham et al. (1987), Reiss (1989), Embrechts et al. (1997), Falk et al. (2004), De Haan and Ferreira (2006), Jessen and Mikosch (2006), or Omey and Segers (2009) for more details on regularly varying functions and max-domain of attractions.

3 Distributional Properties of Beta-Product Convolutions

In this section we discuss the model (1.1) with S being beta distributed with parameters α and β. Since we assume that $H(0)=0$, then $H_{\alpha,\beta}(0)=0$, and further as shown in Hashorva et al. (2007) $H_{\alpha,\beta}$ possesses a positive density function $h_{\alpha,\beta}$ given in terms of the Weyl-Stieltjes fractional-order integral by

$$h_{\alpha,\beta}(x) \quad = \quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}x^{\alpha-1}(\mathcal{J}_{\beta,p_{-\alpha-\beta+1}}H)(x), \quad x\in(0,\omega). \tag{3.1}$$

Clearly, if H possesses a density function h, then

$$(\mathcal{J}_{\beta,p_{-\alpha-\beta+1}}H)(x)=(I_{\beta,p_{-\alpha-\beta+1}}h)(x), \quad x\in(0,\omega). \tag{3.2}$$

In particular we have for some $\delta\in[0,1), n\in I\!N$

$$h_{1,n-\delta}(x) \quad = \quad \Gamma(n+1-\delta)(I_{n-\delta,p_{\delta-n}}h)(x), \quad x\in(0,\omega). \tag{3.3}$$

Let $D^{(n)}$ denote the n-fold derivative operator (we write alternatively $f^{(n)}$ instead of $D^{(n)}f$). Now, if $h^{(n)}_{1,n-\delta}$ exist almost everywhere, by the properties of the Weyl fractional-order integral (see Lemma 7.1) it follows easily that utilising (3.3) we can recover h as

$$x^{\delta-n}h(x) \quad = \quad \frac{(-1)^n}{\Gamma(n+1-\delta)}(I_\delta h^{(n)}_{1,n-\delta})(x), \quad x\in(0,\omega) \tag{3.4}$$

as already shown in Corollary 2.1 of Pakes and Navarro (2007). When $\delta=0$ by (7.1) in Lemma 7.1

$$h(x) \quad = \quad \frac{(-x)^n}{\Gamma(n+1)}h^{(n)}h_{1,n}(x), \quad x\in(0,\omega). \tag{3.5}$$

A more general result is stated in Theorem 2.1 of the aforementioned paper. Namely, if h exists and $h \in \mathcal{I}_{1+\alpha-\delta}$ ($h \in \mathcal{I}_{\alpha-\delta}$ is instead assumed therein, which is a misprint), then

$$h(x) \quad = \quad (-1)^n \frac{\Gamma(\alpha)}{\Gamma(\alpha+n-\delta)} x^{\alpha+n-\delta-1} (I_\delta D^{(n)} p_{1-\alpha} h_{\alpha,n-\delta})(x), \quad x \in (0,\omega) \tag{3.6}$$

provided that $h^{(n)}_{\alpha,n-\delta}$ exists almost everywhere. When $\alpha \geq \delta$ formalising we arrive at:

Theorem 3.1. *Let $H, H_{\alpha,\beta}$ be two distributions as above where $\beta = n - \delta, \delta \in [0, \alpha], n \in I\!N$. Let $h, h_{\alpha,\beta}$ denote the corresponding density functions of H and $H_{\alpha,\beta}$, respectively. If $h^{(n)}_{\alpha,\beta}(x), x > 0$ exists almost everywhere, then* (3.6) *holds.*

We present next two examples.

Example 1. a) Consider the case

$$\alpha \in (0,\infty), \quad \beta = d \in I\!N.$$

If $h^{(d)}_{\alpha,d}$ exists almost everywhere, then Theorem 3.1 implies

$$h(x) \quad = \quad (-1)^d \frac{\Gamma(\alpha)}{\Gamma(\alpha+d)} x^{\alpha+d-1} D^{(d)} (p_{1-\alpha} h_{\alpha,d})(x), \quad x \in (0,\omega). \tag{3.7}$$

b) Suppose that

$$\alpha = 1/2, \quad \beta = d - 1/2, \quad d \in I\!N.$$

Again, if $h^{(d)}_{1/2,d-1/2}$ exists almost everywhere

$$h(x) \quad = \quad (-1)^d \frac{\Gamma(1/2)}{\Gamma(d)} x^{d-1} (I_{1/2} D^{(d)} p_{1/2} h_{1/2,d-1/2})(x), \quad x \in (0,\omega) \tag{3.8}$$

holds, which reduces for $\alpha = 1/2$ to

$$h(x) \quad = \quad -\Gamma(1/2)(I_{1/2} D^{(1)} p_{1/2} h_{1/2,1/2})(x), \quad x \in (0,\omega). \tag{3.9}$$

Example 2. Let $H_{\alpha,\beta}$ be the df of $\Gamma_{\alpha+\beta,\lambda}$, a Gamma random variable with positive parameters α, λ and density function given by

$$h_{\alpha,\beta}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x), \quad x \in (0,\infty).$$

Since $p_{1-\alpha}(x) H_{\alpha,\beta}(x) = h_\lambda(x) \lambda^\alpha / \Gamma(\alpha), x > 0$, with h_λ as in the Introduction, then (3.6) implies that h is the density function of $\Gamma_{\alpha+\beta,\lambda}$, a Gamma random variable with parameters $\alpha+\beta, \lambda$. If $\Gamma_{\alpha+\beta,\lambda}$ is independent of $B_{\alpha,\beta}$, this means

$$\Gamma_{\alpha,\lambda} \stackrel{d}{=} \Gamma_{\alpha+\beta,\lambda} B_{\alpha,\beta},$$

which is a well-known property of gamma and beta random variables, see e.g., Galambos and Simonelli (2004).

A key fact when dealing with independent beta products is that if $B_{\lambda,\gamma}$ is beta distributed with positive parameters $\lambda = \alpha + \beta, \gamma$ being further independent of $B_{\alpha,\beta}$, then we have the stochastic representation

$$B_{\alpha,\beta} B_{\lambda,\gamma} \stackrel{d}{=} B_{\alpha,\beta+\gamma}. \tag{3.10}$$

In the case $\beta = d \in I\!N$ we might further write

$$B_{\alpha,d} \stackrel{d}{=} \prod_{i=1}^{d} B_{\alpha+d-i,1},$$

with $B_{\alpha+d,1},\dots,B_{\alpha,1}$ independent beta distributed random variables. The above stochastic representation is crucial for the recursive calculation of $h_{\alpha,\beta}$.

Since h need not always exist, it is of some important to recover the df H when $H_{\alpha,\beta}$ is known. Utilising (3.10) this can be achieved iteratively as shown in the next result.

Theorem 3.2. *Let $H, H_{\alpha,\beta}$ be two distribution functions of the random scaling model* (1.1) *with $H(0)=0$. If $\beta_0 := \beta > \beta_1 > \dots > \beta_k > \beta_{k+1} := 0, k \in \{0, I\!N\}$ are such that $\beta_{i-1}-\beta_i \in (0,1], i=1,\dots,k+1$, then there exist distribution functions $H_0 := H, H_1,\dots,H_{k+1} = H_{\alpha,\beta}$ determined iteratively by*

$$H_{i-1}(x) = \frac{\Gamma(\alpha+\beta_i)}{\Gamma(\alpha+\beta_{i-1})} x^{\alpha+\beta_{i-1}} \Big[(\alpha+\beta_i)(I_{\delta_i} p_{-\alpha-\beta_{i-1}} H_i)(x) - (\mathcal{J}_{\delta_i, p_{-\alpha-\beta_i}} H_i)(x) \Big], \quad x \in (0,\omega), \tag{3.11}$$

with $\delta_i := 1+\beta_i-\beta_{i-1}$. Furthermore, $H_i, i=1,\dots,k+1$ possesses a density function h_i.

We illustrate next (3.11) by two examples.

Example 3. Consider $H, H_{\alpha,\beta}$ with

$$\alpha \in (0,\infty), \quad \beta = d \in I\!N.$$

With $\beta_i = d-i, i=0,\dots,d$ there exist H_i with density function $h_i, i=1,\dots,d$ such that $H_0 = H, H_d = H_{\alpha,\beta}$ and

$$\begin{aligned} H_{i-1}(x) &= \frac{1}{\alpha+\beta_i} x^{\alpha+\beta_{i-1}} \Big[(\alpha+\beta_i) x^{-\alpha-\beta_{i-1}} H_i(x) - x^{-\alpha-\beta_i} h_i(x) \Big] \\ &= H_i(x) - \frac{x h_i(x)}{\alpha+\beta_i}, \quad x \in (0,\omega), \quad i \in \{1,\dots,d\}. \end{aligned} \tag{3.12}$$

This example shows that when $h^{(d)}_{\alpha,d}$ exists, then we can calculate h recursively by

$$h_{i-1}(x) = \frac{\alpha+\beta_i-1}{\alpha+\beta_i} h_i(x) - \frac{x h_i^{(1)}(x)}{\alpha+\beta_i}, \quad x \in (0,\infty),$$

which is an alternative calculation to (3.7). Note that if $\alpha = 1$, then H can be determined explicitly by the inverse of the Williamson d-transform (see Proposition 3.1 in McNeil and Nešlehová (2009a)).

Example 4. In an financial context assume that an investment $R \sim H$ (positive) is being subjected to some deflation effect so that the return after a period of time (say a year), is $W \stackrel{d}{=} RS$ with deflator S being uniformly distributed in $(0,1)$. The fact that $S \stackrel{d}{=} B_{1,1}$ and Example 3 imply that H and the df $H_{1,1}$ of W are related by

$$H(x) = H_{1,1}(x) - x h_{1,1}(x), \quad \forall x \in (0,\omega), \tag{3.13}$$

with ω the upper endpoint of H. Furthermore, if $h^{(1)}_{1,1}$ exists, then almost surely in $(0,\omega)$

$$h(x) = x h^{(1)}_{1,1}(x).$$

This implies

$$\lim_{x \downarrow 0} x h_{1,1}(x) = \lim_{x \to \omega} x h_{1,1}(x) = 0.$$

Next, if for some constant γ we have

$$\lim_{x \downarrow 0} x h_{1,1}(x) / H_{1,1}(x) = \gamma \in [0,1],$$

then by Proposition 2.5 in Resnick (2007) $H_{1,1} \in RV_\gamma$. Further, (3.13) implies

$$\lim_{x \to \infty} \frac{H(1/x)}{H_{1,1}(1/x)} = 1 - \gamma,$$

which can be also written alternatively in terms of survival functions as (set $R^* := 1/R$)

$$\lim_{x\to\infty} \frac{\boldsymbol{P}\{R^*\xi > x\}}{\boldsymbol{P}\{R^* > x\}} = \frac{1}{1-\gamma},$$

where $\xi = 1/B_{1,1}$ is a Pareto random variable with parameter 1. Karamata's Theorem (see e.g., De Haan and Ferreira (2006), Resnick (2007)) implies thus if $\gamma \in [0,1)$, then also $H \in RV_\gamma$. Another proof of this fact is given in Proposition 5.2 of Maulik and Resnick (2004). By (3.13) a converse result can be easily established. Note in passing that since $h(x) = xh^{(1)}_{1,1}(x)$ we have $h \in RV_{\gamma-1}$ if and only if $h^{(1)}_{1,1} \in RV_{\gamma-2}, \gamma \in I\!R$.

4 Lower Tail Asymptotics

In this section $H_{\alpha,\beta}$ is the product convolution of H with the beta distribution with parameter α, β. Hashorva and Pakes (2009) and Hashorva et al. (2010) discuss the asymptotic behaviour of the survival function $\overline{H}_{\alpha,\beta} := 1 - H_{\alpha,\beta}$, when H belongs to some max-domain of attraction of a univariate extreme value df. As shown in Hashorva and Pakes (2009) H and $H_{\alpha,\beta}$ belong (if so) to the same max-domain of attraction. Such results are important for insurance models, since the random scaling does not effect the max-domain of attraction. Another application concerns the tail asymptotics of the Archimedean copula, since if β is an integer, then $\overline{H}_{1,\beta}$ equals ψ, the density generator of an Archimedean copula, see McNeil and Nešlehová (2009a) for details. The asymptotic behaviour at 0 of the inverse of ψ is investigated in Charpentier and Segers (2007, 2008, 2009), see Remark (4.2) below.

Complementing the findings of Hashorva and Pakes (2009) we focus next on the lower tail asymptotics of $H_{\alpha,\beta}$, which boils down to determination of the min-domain of attractions of $H_{\alpha,\beta}$.

When dealing with positive random variables the min-domain of attraction, for say the df H, is determined by the max-domain of attraction of the df H_* of $1/R$. Since H is a df with lower endpoint 0, only the Fréchet or the Gumbel max-domain of attraction for H_* is possible. The first assumption is equivalent with H satisfying (2.4) with some positive index γ. In Theorem 3.2 we show that this implies that $1/W$ has also df in the Fréchet max-domain of attraction, which turns out to be the case if H_* is in the Gumbel max-domain of attraction. Further, we prove a converse result for the Fréchet case.

We state next the main result of this section.

Theorem 4.1. *Let $R \sim H$ be a positive random variable being independent of $S = B_{\alpha,\beta}$, and define the beta-product convolution df $H_{\alpha,\beta}$ with lower endpoint 0 via the random scaling model* (1.1)*. We have:*

a) If H satisfied (2.4) *with some $\gamma \in [0,\infty)$, then $H_{\alpha,\beta} \in RV_{\gamma^*}$ with $\gamma^* := \min(\gamma, \alpha)$. Furthermore, if $\alpha \neq \gamma$*

$$\lim_{s\downarrow 0} \frac{H_{\alpha,\beta}(s)}{H(s)} = K_{\alpha,\gamma} \in (0,\infty), \tag{4.1}$$

where $K_{\alpha,\gamma} = \boldsymbol{E}\{R^{-\alpha}\}$ if $\alpha > \gamma$ and

$$K_{\alpha,\gamma} = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha-\gamma)}{\Gamma(\alpha)\Gamma(\alpha+\beta-\gamma)}, \quad \alpha \in (0,\gamma).$$

b) If $1/R$ has df in the Gumbel max-domain of attraction, then $H_{\alpha,\beta} \in RV_\alpha$.

c) If $H_{\alpha,\beta}$ satisfies (2.4) *with some $\gamma \in (0,\alpha)$, then $H \in RV_\gamma$ and $h_{\alpha,\beta} \in RV_{\gamma-1}$.*

Remark 4.2.

1. In the setup of Archimedean copula Charpentier and Segers (2009) consider the asymptotics at 0 of ψ^{-1} with ψ the generator of some Archimedean copula (in the notation of McNeil and Nešlehová (2009a)). In the light of findings of the aforementioned paper $\psi^{-1} = (\overline{H}_{1,1})^{-1}$, i.e., it is the inverse of the survival function of a beta-product convolution ($\alpha = 1, \beta = 1$). By Proposition 2.6 (v) in Resnick (2007) $\psi^{-1} \in RV_\alpha, \alpha \in [0,\infty]$ implies that $\overline{H}_{1,1}$ is regularly varying at infinity with index $1/\alpha$. For the general k-dimensional Archimedean copula $\psi^{-1} = (\overline{H}_{1,k-1})^{-1}, k \in I\!N$. Consequently, the asymptotic findings of Charpentier and Segers (2009) concern the asymptotics of the survival function $\overline{H}$ and $\overline{H}_{1,k}$ and that of $h_{1,1}(x)$ as $x \to \infty$. See Hashorva and Pakes (2009), Hashorva et al. (2010) for related asymptotic results. Finally, note that the identity (3.13) of Example 4 can also be utilised to deal with these functions.

2. Statement a) of Theorem 4.1 can be formulated by dropping the distributional assumption on the random scaling variable $S \sim G$ assuming simply that $1/S$ is regularly varying at infinity with index $\alpha \in (0,\infty)$. This implies that (4.1) holds and

$$K_{\alpha,\gamma} = \boldsymbol{E}\{R^{-\alpha}\}, \quad \gamma < \alpha, \quad K_{\alpha,\gamma} = \boldsymbol{E}\{S^{-\gamma}\}, \qquad \text{if } \alpha < \gamma.$$

Note that when $\alpha = \gamma$, and $\boldsymbol{E}\{S^{-\alpha}\} = \boldsymbol{E}\{R^{-\alpha}\} = \infty$, then again (4.1) holds and further $K_{\alpha,\alpha} = \infty$, see Lemma 4.1 in Jessen and Mikosch (2006), and Problem 7.8 in Resnick (2007). In the special case that

$$F(x) = (1 + o(1))cx^\alpha, \quad G(x) = (1 + o(1))cx^\alpha \quad x \downarrow 0$$

for $c, \alpha \in (0,\infty)$ then we have

$$K_{\alpha,\alpha} = -(1 + o(1))\alpha c^{2\alpha} x^\alpha \ln x, \quad x \downarrow 0,$$

which follows from the aforementioned lemma.

3. It is well-known (see e.g., Resncik (1987), Embrechts et al. (1997)) that if a df F is in the Gumbel max-domain of attraction, then if $F(x) < 1, \forall x > 0$ we have $1 - F$ is rapidly varying at infinity, i.e.,

$$\lim_{x\to\infty} \frac{1 - F(cx)}{1 - F(x)} = 0, \quad \forall c > 1.$$

The Gumbel max-domain of attraction assumption in statement b) of Theorem 4.1 can be weakened to: The random variable $1/R$ has a rapidly varying survival function. By Theorem 5.4.1 of de Haan and Ferreira (2006) it follows that $\boldsymbol{E}\{1/R\} < \infty$. Hence if $\alpha \in (0,1]$ applying Lemma 2.1 in Gomes et al. (2004) (which generalises Breiman's Lemma) we obtain $H_{\alpha,\beta} \in RV_\alpha$.

4. If $\alpha \in (0,\infty), \beta = d$, then by (3.7) regular variation of h holds if the same is true for $D^{(d)}(p_{1-\alpha} h_{\alpha,d})$. When $\alpha = 1$ the latter reduces to $h^{(d)}_{\alpha,d}$.

5. Statement c) in Theorem 4.1 does not hold in general when $\gamma = \alpha$. For $\alpha = 1$ and a special df $H_{1,1}$, this is shown in Example 5.3 of Maulik and Resnick (2004).

We give next two simple examples.

Example 5. In the random scaling model (1.1) assume that the scaling variable S is such that $S^2 = B_{1/2,(k-1)/2}$, and the df H of R satisfies (2.4) with $\gamma \in (0,\infty)$. Applying Theorem 4.1 W^2 has df belonging to $RV_{\gamma^*/2}$ with $\gamma^* := \max(\gamma, 1)$. Consequently,

$$H_{\alpha,\beta} \in RV_{\gamma^*}, \qquad \text{and } h_{\alpha,\beta} \in RV_{\gamma^*-1}.$$

Example 6. Let H be the df of $\Gamma_{\alpha+\beta,\lambda}$ as in Example 2. Then H is regularly varying at 0 with index $\alpha + \beta$. Theorem 4.1 implies that $H_{\alpha,\beta}$ is regularly varying with index α. This can be shown directly, since in view of Example 2 $H_{\alpha,\beta}$ is Gamma distributed with parameters α, λ.

5 Applications

In this section we provide three applications concerning multivariate Dirichlet distributions. A canonical example of such distributions is the Gaussian one, and a well-known subclass of Dirichlet distributions is the class of spherically symmetric (for short spherical) distributions. A spherical random vector $\boldsymbol{X}$ in $I\!R^k, k \geq 2$ with almost surely positive random radius R has stochastic representation $R\boldsymbol{U}$ with $\boldsymbol{U}$ being uniformly distributed on the unit sphere of $I\!R^k$ being further independent of R. All the components of $\boldsymbol{X}$ have the same marginal distribution which can be a drawback for various statistical applications. Recently McNeil and Nešlehová (2009a) show the strong relation of L_1-norm Dirichlet distributions with the Archimedean copula, which is widely used in insurance and financial applications.
In general the multivariate dependence of Dirichlet distributions does not reduce to that implied by Archimedean copula, which is proved by the fact that Dirichlet distributions need not be exchangeable. Roughly speaking a multivariate Dirichlet random vector has stochastic representation given in terms of products of beta random variables, thus being closely related to the beta random scaling model of this paper.
A natural question that arises is how tractable are such random vectors? In our first application we show that even when only one marginal distribution of a weighted Dirichlet random vector (see below for the definition) is known, it is still possible to calculate the whole multivariate distribution of the weighted Dirichlet random vector. By definition this is also the case of Archimedean copulas with multivariate dependence being defined solely by a single univariate distribution function.

Our second application concerns elliptical random vectors; we derive the joint asymptotic independence of sample minima.

The last application is motivated by the dependence function of bivariate elliptical random vectors. Our asymptotical result is of some relevance when dealing with lower tail asymptotics of aggregated risk. As mention in the Introduction aggregation is a central topic in various applications; for insurance and financial applications refer to Embrechts et al. (2001) and Denuit et al. (2005).

5.1 Are Dirichlet Distributions Simple?

We give first the definition of a weighted Dirichlet random vector in $I\!R^k, k \geq 2$. Let $R \sim H$ be a positive random variable being independent of some k-dimensional random vector $\boldsymbol{U}$. We call $\boldsymbol{X}$ with stochastic representation $\boldsymbol{X} \stackrel{d}{=} R\boldsymbol{U}$ a weighted Dirichlet random vector if further

$$\boldsymbol{U} \quad \stackrel{d}{=} \quad \Big(T_1(1-V_1^{p_1})^{1/r_1}, \ldots, T_{k-1}(1-V_{k-1}^{p_{k-1}})^{1/r_{k-1}}) \prod_{i=1}^{k-2} V_i, T_k \prod_{i=1}^{k-1} V_i\Big) \tag{5.1}$$

holds with

$$T_i \stackrel{d}{=} Be(q_i), \quad 1 \leq i \leq k, \quad V_i > 0,\ V_i^{p_i} \stackrel{d}{=} B_{\alpha_i,\beta_i}, \quad \alpha_i, \beta_i, p_i, r_i \in (0,\infty), \quad 1 \leq i \leq k-1,$$

where $Be(q_i)$ is a Bernulli random variable taking only two values $\{-1, 1\}$ with $\boldsymbol{P}\{T_i = 1\} = q_i \in (0,1], i \leq k$. Further assume that $T_1, \ldots, T_k, V_1, \ldots, V_{k-1}$ are mutually independent. When $q_i = 1/2, i \leq k$ and $p_i = r_i = p \in (0,\infty)$, then $\boldsymbol{X}$ is a L_p-Dirichlet random vector, see Hashorva at al. (2007).
If we know the df H, then the df of $\boldsymbol{X}$, and in particular that of each component X_i can be easily derived.
Next, suppose that R is unobservable, and we have incomplete information about $\boldsymbol{X}$, say only the df of Q_i of

$|X_i|$ for some i with $1 < i \le k$ is known. Assume further that $q_i = q \in (-1,1), i \le k$ with q some known constant, or all q_is are also known. This is the case for $\boldsymbol{X}$ a L_p-Dirichlet distributions with $q_i = 1/2, i \le k$. By the definition we have the following stochastic representation

$$|X_i|^{r_i} \stackrel{d}{=} B_{\beta_i,\alpha_i} \prod_{j=1}^{i-1} V_j^{r_i} R \stackrel{d}{=} B_{\beta_i,\alpha_i} R_i^*.$$

Next, in the light of Theorem 3.2 the df of R_i^* can be calculated iteratively. Proceeding similarly we see that the df H of R can be calculated iteratively. Thus the whole df of $\boldsymbol{X}$ can be recovered if we know only the df Q_i, which explains why these multivariate distributions are simple. Indeed, they are multivariate distributions, however they can be recovered/determined by a single univariate df. When $H \in RV_\gamma, \gamma \in [0,\infty)$ by Theorem 4.1 we have that $Q_i \in RV_{\gamma_i}$ with

$$\gamma_i := \min\Big(\gamma, \beta_i r_i, \min_{1 \le j \le i-1} \alpha_j p_j\Big).$$

In the special case that $\boldsymbol{X}$ is a L_p-Dirichlet random vector, then

$$\gamma_i = p \min(\gamma/p, \min_{1 \le j \le i-1} \alpha_j, \beta_i), \quad i \le k.$$

Clearly, if $\alpha_i = \beta_i, i \le k$, then γ_i does not depend on i. This is for instance satisfied with $\alpha_i = \beta_i = 1/p = 1/2, i \le k$ and $\boldsymbol{U}$ being uniformly distributed on the unit sphere of $I\!R^k$ yielding

$$\gamma_i \quad = \quad \max(1, \gamma), \quad 1 \le i \le k. \tag{5.2}$$

We remark that $R\boldsymbol{U}$ is in this case a spherical random vector if additionally $q_i = 1/2, i \le k$.

5.2 Elliptical Distributions and Asymptotics of Sample Minima

When $\boldsymbol{U}$ is uniformly distributed on the unit sphere of $I\!R^k, k \ge 2$, and A is a k-dimensional nonsingular real matrix, then the random vector $\boldsymbol{X} = RA\boldsymbol{U}$ is elliptically distributed. It is well-known that the distribution function of $\boldsymbol{X}$ depends on $\Sigma := AA^\top$ but not on the matrix A itself. By the properties of $\boldsymbol{U}$ (cf. Cambanis et al. (1981)), if further the main diagonal of Σ consists of 1s, i.e., Σ is a correlation matrix, then by Lemma 6.1 in Berman (1983) (see also Berman (1992))

$$X_i \stackrel{d}{=} X_1 \stackrel{d}{=} -X_1 \stackrel{d}{=} RU_1, \quad i \le k. \tag{5.3}$$

Further we have

$$X_1^2 \stackrel{d}{=} R^2 B_{1/2, l-j/2}, \quad k := 2l + j - 1, j = 0, 1,$$

where R is independent of $B_{1/2,l-j/2}$. Clearly, the random variable $|X_1|$ is a deflation of R by $S = \sqrt{B_{1/2,l-j/2}}$. As in the previous application, if we know the df of say X_1, then the df of the associated random radius R can be calculated iteratively. In view of Theorem 3.1 we have further that R possesses a density function, which can again be calculated iteratively (or directly applying (3.7) and (3.8)), provided that X_1 possesses a density function which has an almost surely finite lth derivative.

Next, in view of (5.2) $|X_{11}|$ has df $Q \in RV_\gamma, \gamma \in (0,1]$ which is the case if for instance $H \in RV_\gamma, \gamma \in (0,\infty)$. For such Q we define a sequence of constant $a_n, n \ge 1$ asymptotically by

$$\boldsymbol{P}\{a_n^{-1} \ge X_{11} > 0\} \quad = \quad 1/(2n).$$

It follows that $a_n^{-1} = L(1/n)n^{-\gamma}$, with $L \in RV_0$ a positive slowly varying function at 0, i.e., $\lim_{n\to\infty} L(c/n)/L(1/n) = 1, \forall c \in (0,\infty)$. For such constants we have the convergence in distribution

$$a_n M_{ni} \stackrel{d}{\to} \mathcal{M}_i \sim \mathcal{G}_\gamma, \quad i \le k, \quad n \to \infty,$$

where $M_{ni} := \min_{1\le j\le n}|X_{ji}|, i=1,2$ and the df $\mathcal{G}_\gamma$ is given by

$$\mathcal{G}_\gamma(x) \quad = \quad 1-\exp(-x^\gamma), \quad x>0. \tag{5.4}$$

Since for $u\in(0,\infty)$ small enough

$$\boldsymbol{P}\{|X_{1i}|<u, |X_{1j}|<u\} = 0, \quad 1\le i\ne j\le k$$

we have

$$\lim_{u\downarrow 0} \frac{\boldsymbol{P}\{|X_{1i}|<u, |X_{1j}|<u\}}{\boldsymbol{P}\{|X_{11}|\le u\}} = 0, \quad 1\le i\ne j\le k$$

it follows that $\boldsymbol{M}_n := (M_{n1},\ldots,M_{nk})$ has asymptotic independent components meaning that the joint convergence in distribution

$$\Big(a_nM_{n1},\ldots,a_nM_{nk}\Big) \quad \stackrel{d}{\to} \quad \Big(\mathcal{M}_1,\ldots,\mathcal{M}_k\Big), \quad n\to\infty$$

holds with $\mathcal{M}_1,\ldots,\mathcal{M}_k$ as above being further independent. To this end, we note that the convergence in distribution above can be reformulated for the more general class of asymptotically elliptical random vectors (see Hashorva (2005) for asymptotic properties).

5.3 Aggregation of Two Risks: Lower Tail Asymptotics

If $\boldsymbol{X}$ is a k-dimensional elliptical random vector as above, then for given constants $\mu_i, i\le k$ we have

$$\sum_{1\le j\le k}\mu_jX_j \stackrel{d}{=} \sqrt{\sum_{1\le j\le k}\mu_j^2}X_1,$$

which is a well-known property for the Gaussian random vectors. Moreover we have for any pair $X_i,X_j, i\ne j$

$$(X_i,X_j)\stackrel{d}{=}\Big(S_1,\rho_{ij}S_1+\sqrt{1-\rho_{ij}^2}S_2\Big), \quad (S_1,S_2)\stackrel{d}{=}(R_*T_1S, R_*T_2\sqrt{1-S^2}), \quad S\sim\sqrt{B_{1/2,1/2}}, \tag{5.5}$$

with $\rho_{ij}\in(-1,1)$ the ijth entry of the correlation matrix Σ and (S_1,S_2) a bivariate spherical random vector with associated random radius R_* such that $R_*^2\stackrel{d}{=}R^2B_{1,k/2-1}$ and $T_i\stackrel{d}{=}Be(1/2), i=1,2$. Furthermore T_1,T_2,R_*,S are mutually independent.

If $\boldsymbol{X}$ is a L_p-Dirichlet random vector, the above stochastic representation does not hold in general. In this application we show that we still can determine the lower tail asymptotics of linear combinations by dropping distribution assumptions. Motivated by (5.5) we consider the lower tail asymptotics of a bivariate random vector $(X,Y_\rho), \rho\in(-1,1)$ with stochastic representation

$$(X,Y_\rho)\stackrel{d}{=}\Big(T_1RS, \rho T_1RS+\tilde{\rho}T_2R\sqrt{1-S^2}\Big), \quad \widetilde{\rho}:=\sqrt{1-\rho^2}, \tag{5.6}$$

where $T_i\stackrel{d}{=}Be(q_i), q_i\in(-1,1)$, $R\sim H$ and $S\sim G$ such that $G(0)=H(0)=0, G(1)=1$. As in the elliptical setup here again T_1,T_2,R,S are assumed to be mutually independent. Since $|X|\stackrel{d}{=}RS$, then the lower tail asymptotics of $|X|$ can be established by Theorem 4.1 under asymptotic assumptions on R and S. We note in passing that asymptotic copula properties of (X,Y_ρ) are discussed in the recent contribution Manner and Segers (2009). Further, remark that as mentioned in Remark 4.2, we do not need to specify G, apart from the asymptotic condition $G\in RV_\alpha$.

We derive next the lower tail asymptotics of $|Y_\rho|$ under the following additional assumption on G: For all positive t small enough

$$G(\rho+t)-G(\rho-t)=L_\rho(t)t^{\alpha_\rho}, \quad G(\widetilde{\rho}+t)-G(\widetilde{\rho}-t)=L_{\widetilde{\rho}}(t)t^{\alpha_{\tilde{\rho}}}, \quad \alpha_\rho,\alpha_{\widetilde{\rho}}\in[0,\infty), \tag{5.7}$$

with $L_\rho, L_{\widetilde{\rho}} \in RV_0$. Condition (5.6) can be easily checked. In the special case that the df G possesses a positive density function g continuous at ρ and $\widetilde{\rho}$, then (5.7) is satisfied with

$$\alpha_\rho = \alpha_{\widetilde{\rho}} = 1, \quad \text{and } L_\rho(t) = (2+o(1))g(\rho), \quad L_{\widetilde{\rho}}(t) = (2+o(1))g(\widetilde{\rho}), \quad t \downarrow 0. \tag{5.8}$$

We have now the following result.

Proposition 5.1. *Let $(X, Y_\rho), \rho \in (0,1)$ be a bivariate random vector with stochastic representation* (5.6). *Suppose that*

$$R \sim H \in RV_\gamma, \quad S \sim G \in RV_\alpha, \quad \alpha, \gamma \in (0,\infty)$$

and (5.7) *holds with some $\alpha_\rho, \alpha_{\widetilde{\rho}}$ and $L_\rho, L_{\widetilde{\rho}}$. Assume further that if $\gamma_\rho = \alpha_{\widetilde{\rho}}$, then $L_\rho(x) = cL_{\widetilde{\rho}}(x), \forall x > 0$ with some positive constant c. Then we have*

$$|X| \in RV_{\gamma_1}, \quad |Y_\rho| \in RV_{\gamma_2}, \tag{5.9}$$

where $\gamma_1 := \min(\alpha, \gamma)$ and $\gamma_2 = \min(\gamma, \alpha_\rho, \alpha_{\widetilde{\rho}})$.

A simple instance when we can apply Proposition 5.1 is when G possesses the density function g. In view of (5.8) the index γ_2 equals $\min(\gamma, 1)$. The following corollary is an immediate consequence of the above discussion.

Corollary 5.2. *Let $G, H, (X, Y_\rho), \rho \in (-1,1)$ be as in Proposition 5.1, and let $(X_n, Y_n), n \geq 1$ be independent bivariate random vectors with the same df as (X, Y_ρ). If $G \in RV_1$, and $a_n, b_n, n \geq 1$ are constants such that $\boldsymbol{P}\{|X| < 1/a_n\} = \boldsymbol{P}\{|Y_\rho| < 1/b_n\} = 1/n$, for all n large, then we have the joint convergence in distribution*

$$\Big(a_n \min_{1\leq j\leq n} |X_j|, b_n \min_{1\leq j \leq n} |Y_j|\Big) \stackrel{d}{\to} \Big(\mathcal{M}_1, \mathcal{M}_2\Big) \quad n \to \infty, \tag{5.10}$$

with $\mathcal{M}_1, \mathcal{M}_2$ independent with common df $\mathcal{G}_{\min(\gamma,1)}$ defined in (5.4).

6 Further Results and Proofs

Lemma 6.1. *Let T_1, T_2 be two random variables taking values $-1, 1$ with $\boldsymbol{P}\{T_1T_2 = -1\} \in (0,1]$ being independent of the scaling random variable $S \sim G$. For given $\rho \in (0,1)$ set $S_\rho := |\rho T_1 S + \widetilde{\rho} T_2 \sqrt{1-S^2}|$. If G satisfies* (5.7), *then we have*

$$\boldsymbol{P}\{S_\rho \leq u\} \quad = \quad (1+o(1))q_{1,-1}(\rho u)^{\alpha_{\widetilde{\rho}}} L_{\widetilde{\rho}}(u) + (1+o(1))q_{-1,1}(\widetilde{\rho}u)^{\alpha_\rho} L_\rho(u), \quad u \downarrow 0, \tag{6.1}$$

where $q_{i,j} := \boldsymbol{P}\{T_1 = i, T_2 = j\}$.

Note in passing that if G possesses a positive density function g continuous at ρ and $\widetilde{\rho}$, then (6.1) reduces to

$$\boldsymbol{P}\{S_\rho \leq u\} \quad = \quad (1+o(1))2\boldsymbol{P}\{T_1T_2 = -1\}[g(\rho)\widetilde{\rho} + g(\widetilde{\rho})\rho]u, \quad u \downarrow 0.$$

Proof of Lemma 6.1 By the assumptions $S \in (0,1)$ almost surely, and $T_j, j = 1,2$ assumes only two values $\{-1, 1\}$. Hence we may write for any $u \in (0,1)$ small enough

$$\begin{aligned}\boldsymbol{P}\{S_\rho \leq u\} \quad &= \quad \boldsymbol{P}\{T_1 = 1, T_2 = -1\}\boldsymbol{P}\{|\rho S - \widetilde{\rho}\sqrt{1-S^2}| \leq u\} \\ &\quad + \boldsymbol{P}\{T_1 = -1, T_2 = 1\}\boldsymbol{P}\{|\widetilde{\rho}\sqrt{1-S^2} - \rho S| \leq u\}.\end{aligned}$$

Using further the fact that S is independent of T_1, T_2 we obtain

$$\boldsymbol{P}\{|\rho S - \widetilde{\rho}\sqrt{1-S^2}| \leq u\} \quad = \quad \boldsymbol{P}\{-u \leq \rho S - \widetilde{\rho}\sqrt{1-S^2} \leq u\}$$

$$
\begin{aligned}
&= (1+o(1))\int_{\widetilde{\rho}-(1+o(1))\rho u}^{\widetilde{\rho}+(1+o(1))\rho u} dG(s)\\
&= (1+o(1))(\rho u)^{\alpha_{\widetilde{\rho}}} L_{\widetilde{\rho}}(u), \quad u \downarrow 0.
\end{aligned}
$$

As above we have further

$$
\boldsymbol{P}\{|\widetilde{\rho}\sqrt{1-S^2}-\rho S| \le u\} = (1+o(1))(\widetilde{\rho}u)^{\gamma_\rho} L_\rho(u), \quad u \downarrow 0,
$$

thus the result follows. □

Proof of Theorem 3.2 Theorem 3.3 in Hashorva and Pakes (2009) shows an iterative formula for calculating the survival function $\overline{H}$, when the survival function $\overline{H}_{\alpha,\beta}$ is known. The proof of the theorem is established with the same arguments of the aforementioned theorem utilising further (3.10). □

Proof of Theorem 4.1 a) Let $W \stackrel{d}{=} RB_{\alpha,\beta}$ with df $H_{\alpha,\beta}$. First note that

$$
\boldsymbol{P}\{B_{\alpha,\beta} < s\} = (1+o(1))\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} s^\alpha, \quad s \downarrow 0 \tag{6.2}
$$

implying thus $H_{\alpha,\beta} \in RV_\alpha$. Since for any $s \in (0,\infty)$

$$
\boldsymbol{P}\{W < s\} = \boldsymbol{P}\{R^* B^*_{\alpha,\beta} > 1/s\}, \quad R^* := \frac{1}{R}, \quad B^*_{\alpha,\beta} := \frac{1}{B_{\alpha,\beta}}
$$

the lower tail asymptotics of W is determined by the upper tail asymptotics of the random product $R^* B^*_{\alpha,\beta}$. When $\gamma \in (0,\alpha)$ we have

$$
\boldsymbol{E}\{(B^*_{\alpha,\beta})^{\gamma'}\} < \infty, \quad \forall \gamma' \in (\gamma,\alpha),
$$

thus applying Breiman's Lemma (see Breiman (1965), Cline and Samorodnitsky (2004), Gomes et al. (2004), Denisov and Zwart (2005), Jessen and Mikosch (2006), de Haan and Ferreira (2006), or Resnick (2007)) we have $H_{\alpha,\beta} \in RV_\gamma$, and further

$$
\boldsymbol{E}\{B^{-\gamma}_{\alpha,\beta}\} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha+\beta-\gamma)}.
$$

Next, if $\alpha < \gamma$, since

$$
\boldsymbol{E}\{(R^*)^{\alpha'}\} < \infty, \quad \forall \alpha' \in (0,\gamma)
$$

applying again Breiman's Lemma we find $H_{\alpha,\beta} \in RV_\alpha$. When $\alpha = \gamma$, the claim follows from the well-known result of Embrecht and Goldie (1980) on asymptotics of the product of two regularly varying random variables with equal index of regular variation, see also Lemma 4.1 in Jessen and Mikosch (2006).

b) It is well-known (see e.g., Resnick (1987)) that if $R^* = 1/R$ has df in the Gumbel max-domain of attraction, then $\boldsymbol{E}\{(R^*)^\delta\} < \infty$ for any $\delta > 0$, hence since $B_{\alpha,\beta}$ is regularly varying at 0 with index α the claim follows by Breiman's Lemma.

c) We show the proof utilising the result of Theorem 3.2. With the notation of the aforementioned theorem we have that there exist distribution functions $H_0 := H, H_1, \ldots, H_{k+1} = H_{\alpha,\beta}$ determined iteratively by

$$
H_{i-1}(x) = \frac{\Gamma(\alpha+\beta_i)}{\Gamma(\alpha+\beta_{i-1})} x^{\alpha+\beta_{i-1}}\Big[(\alpha+\beta_i)(I_{\delta_i} p_{-\alpha-\beta_i-1} H_i)(x) - (\mathcal{J}_{\delta_i, p_{-\alpha-\beta_i}} H_i)(x)\Big], \quad \forall x \in (0,\infty),
$$

with $\delta_i := 1+\beta_i-\beta_{i-1} \in [0,1)$ and $\beta_0 := \beta > \beta_1 > \cdots > \beta_k > \beta_{k+1} := 0$. By the assumption on $H_{\alpha,\beta}$ for any $x > 0$ we have (set $\lambda_{k+1} := \alpha+\beta_{k+1}+1$)

$$
(I_{\delta_{k+1}} p_{-\lambda_{k+1}} H_{k+1})(x) = \int_x^\infty (y-x)^{\delta_{k+1}-1} y^{-\lambda_{k+1}} H_{k+1}(y)\, dy
$$

$$= x^{\alpha-\beta_k} H_{k+1}(x) \int_1^\infty (y-1)^{\delta_{k+1}-1} y^{-\lambda_{k+1}} H_{k+1}(xy)/H_{k+1}(x)\, dy.$$

Applying Karamata's Theorem (see Embrechts et al. (1997), or Resnick (2007)) we have

$$\begin{aligned}
\Gamma(\delta_{k+1})(I_{\delta_{k+1}} p_{-\lambda_{k+1}} H_{k+1})(x) &= (1+o(1)) x^{-\alpha-\beta_k} H_{k+1}(x) \int_1^\infty (y-1)^{\delta_{k+1}-1} y^{-\lambda_{k+1}+\gamma}\, dy \\
&= (1+o(1)) x^{-\alpha-\beta_k} H_{k+1}(x) \frac{\Gamma(\delta_{k+1})\Gamma(\lambda_{k+1}-\delta_{k+1}-\gamma)}{\Gamma(\lambda_{k+1}-\gamma)}, \quad x \downarrow 0.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\Gamma(\delta_{k+1})(\mathcal{J}_{\delta_{k+1}, p_{1-\lambda_{k+1}}} H_{k+1})(x) &= \int_x^\infty (y-x)^{\delta_{k+1}-1} y^{1-\lambda_{k+1}}\, dH_{k+1}(y) \\
&= x^{1-\lambda_{k+1}} x^{\delta_{k+1}-1} \int_1^\infty (y-1)^{\delta_{k+1}-1} y^{1-\lambda_{k+1}}\, dH_{k+1}(xy) \\
&= (1+o(1))\gamma (I_{\delta_{k+1}} p_{-\lambda_{k+1}} H_{k+1})(x), \quad x \downarrow 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
H_k(x) &= (1+o(1)) \frac{\Gamma(\alpha+\beta_{k+1})}{\Gamma(\alpha+\beta_k)} (\alpha+\beta_{k+1}-\gamma) \frac{\Gamma(\alpha+\beta_k-\gamma)}{\Gamma(\alpha+\beta_{k+1}+1-\gamma)} H_{k+1}(x) \\
&= (1+o(1)) \frac{\Gamma(\alpha+\beta_{k+1})\Gamma(\alpha+\beta_k-\gamma)}{\Gamma(\alpha+\beta_k)\Gamma(\alpha+\beta_{k+1}-\gamma)} H_{k+1}(x), \quad x \downarrow 0,
\end{aligned}$$

hence $H_k \in RV_\gamma$. Proceeding iteratively we find that $H_0 = H \in RV_\gamma$.
Next, in view of (6.3) and the fact that $H \in RV_\gamma \in (0,\infty)$ we obtain as above

$$\begin{aligned}
h_{\alpha,\beta}(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} \int_x^\infty (y-x)^{\beta-1} y^{-\alpha-\beta+1}\, dH(y) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} H(x) \int_1^\infty (y-1)^{\beta-1} y^{-\alpha-\beta+1}\, dH(xy)/H(x) \\
&= (1+o(1))\gamma \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{H(x)}{x} \int_1^\infty (y-1)^{\beta-1} y^{-\alpha-\beta+\gamma}\, dy \\
&= (1+o(1))\gamma \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha+\beta-\gamma)} \frac{H(x)}{x}, \quad x \downarrow 0 \\
&= \gamma \frac{H_{\alpha,\beta}}{x}, \quad x \downarrow 0
\end{aligned}$$

establishing thus the claim. □

Proof of Theorem 5.1 The proof follows with the same arguments as in the proof of Theorem 4.1 applying further the result of Lemma 6.1. □

7 Appendix

We present below some basic properties of the Weyl fractional-order integral operator I_β.

Lemma 7.1. *Let β, c be positive constants, and let $h : (0,\infty) \to R$ be a given measurable function.*
a) $I_\beta h$ is continuous at 0, i.e.,

$$I_0 h = h. \tag{7.1}$$

If h is the positive density of some df H with lower endpoint 0, then for any $\alpha \geq 0$ almost surely

$$(\mathcal{J}_0 p_{-\alpha-\beta} H)(x) = x^{-\alpha-\beta} h(x), \quad x > 0. \tag{7.2}$$

b) If $h \in \mathcal{I}_\beta$ and when $\beta \in (0,1)$

$$\int_x^\delta (y-x)^{\beta-1} h(y)\, dy < \infty$$

holds for some δ positive, then $(I_\beta h)(x)$ is finite and continuous for all $x > 0$.

c) If $h \in \mathcal{I}_{\beta+c}$, then

$$I_\beta I_c h = I_c I_\beta h = I_{\beta+c} h. \tag{7.3}$$

d) Let $D^{(n)}$ denote the n-fold derivative operator ($n \in I\!N$). If the n-fold derivative $D^{(n)}h$ exists almost everywhere and $D^{(n)}h \in \mathcal{I}_\beta$, then

$$D^{(n)} I_\beta h \quad = \quad I_\beta D^{(n)} h, \quad \text{and } D^{(k)} I_n = (-1)^k I_{n-k}, \quad k = 1, \ldots, n. \tag{7.4}$$

e) For any df H with $H(0) = 0$ and upper endpoint $\omega \in (0, \infty]$ we have

$$(\mathcal{J}_{\beta+1, p_{1-\beta}} H)(x) \quad = \quad x(I_\beta p_{-1-\beta} \overline{H})(x), \quad x \in (0, \omega). \tag{7.5}$$

Proof of Lemma 7.1 If $I_y h$ is almost surely finite for all $y > 0$ small enough, then by Lemma 2.2 of Pakes and Navarro (2007) passing in the limit ($y \downarrow 0$) to the moment generating function of I_y it follows that $I_0 h = h$. Using (3.2) establishes further (7.2). Statement b) is mentioned in the Introduction of Hashorva and Pakes (2009). Both *c)* and *d)* are shown in Lemma 8.1 of the aforementioned paper. e) follows immediately from (2.2) and (2.3), and thus the result follows. □

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