Bose-Einstein condensates with $F=1$ and $F=2$. Reductions and soliton interactions of multi-component NLS models.

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ABSTRACT

We analyze a class of multicomponent nonlinear Schrödinger equations (MNLS) related to the symmetric **BD.I**-type symmetric spaces and their reductions. We briefly outline the direct and the inverse scattering method for the relevant Lax operators and the soliton solutions. We use the Zakharov-Shabat dressing method to obtain the two-soliton solution and analyze the soliton interactions of the MNLS equations and some of their reductions.

Keywords: Bose-Einstein condensates, Multicomponent nonlinear Schrödinger equations, Soliton solutions, Soliton interactions

1. INTRODUCTION

Bose-Einstein condensate (BEC) of alkali atoms in the $F = 1$ hyperfine state, elongated in x direction and confined in the transverse directions y, z by purely optical means are described by a 3-component normalized spinor wave vector $\mathbf{\Phi}(x,t) = (\Phi_1, \Phi_0, \Phi_{-1})^T(x,t)$ satisfying the nonlinear Schrödinger (MNLS) equation[1] see also:[2–6]

$$
\begin{aligned}
&i\partial_t \Phi_1 + \partial_x^2 \Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^* \Phi_0^2 = 0, \\
&i\partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^* \Phi_1 \Phi_{-1} = 0, \\
&i\partial_t \Phi_{-1} + \partial_x^2 \Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^* \Phi_0^2 = 0.
\end{aligned}
\tag{1}
$$

spinor BEC with $F = 2$ for rather specific choices of the scattering lengths in dimensionless coordinates takes the form:[7]

$$
\begin{aligned}
&i\partial_t \Phi_{\pm 2} + \partial_{xx} \Phi_{\pm 2} + 2(\vec{\mathbf{\Phi}}, \vec{\mathbf{\Phi}}^*)\Phi_{\pm 2} - (2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2)\Phi_{\mp 2}^*, \\
&i\partial_t \Phi_{\pm 1} + \partial_{xx} \Phi_{\pm 1} + 2(\vec{\mathbf{\Phi}}, \vec{\mathbf{\Phi}}^*)\Phi_{\pm 1} - (2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2)\Phi_{\mp 1}^*, \\
&i\partial_t \Phi_0 + \partial_{xx} \Phi_0 + 2(\vec{\mathbf{\Phi}}, \vec{\mathbf{\Phi}}^*)\Phi_0 - (2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2)\Phi_0^*.
\end{aligned}
\tag{2}
$$

Both models have natural Lie algebraic interpretation and are related to the symmetric spaces **BD.I** $\simeq$ SO(n + 2)/SO(n) × SO(2) with $n = 3$ and $n = 5$ respectively. They are integrable by means of inverse scattering transform method.[8,9,12] Using a modification of the Zakharov-Shabat 'dressing method' we describe the soliton solutions[1,10] and the effects of the reductions on them.

Sections 2 and 3 contain the basic details on the direct and inverse scattering problems for the Lax operator. Section 4 outlines the effects of the algebraic reductions of the MNLS. In Section 5 using the Zakharov-Shabat dressing method we derive the one- and two-soliton solutions of the MNLS and discuss their properties. Section 6 is dedicated to the analysis of the soliton interactions of the MNLS. To this end we evaluate the limits of the generic two-soliton solution for $t \to \pm\infty$. As a result we establish that the effect of the interactions on the soliton parameters is analogous to the one for the scalar NLS equation and consists in shifts of the 'center of mass' and shift in the phase.

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2. THE METHOD FOR SOLVING MNLS WITH $F = 1$ AND $F = 2$

MNLS equations for the **BD.I.** series of symmetric spaces (algebras of the type $so(n+2)$ and J dual to e_1) have the Lax representation $[L, M] = 0$ as follows[8, 11, 12]

$$L\psi(x, t, \lambda) \equiv i\partial_x\psi + U(x, t, \lambda)\psi(x, t, \lambda) = 0, \qquad M\psi(x, t, \lambda) \equiv i\partial_t\psi + V(x, t, \lambda)\psi(x, t, \lambda) = 0,$$

$$U(x, t, \lambda) = Q(x, t) - \lambda J, \qquad\qquad V(x, t, \lambda) = V_0(x, t) + \lambda V_1(x, t) - \lambda^2 J, \qquad (3)$$

$$V_1(x, t) = Q(x, t), \qquad\qquad V_0(x, t) = i\mathrm{ad}_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}_J^{-1}Q, Q(x, t)\right].$$

where $\mathrm{ad}_J X = [J, X]$ and ad_J^{-1} is well defined on the image of ad_J in $\mathfrak{g}$;

$$Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p}^* & 0 & s_0\vec{q} \\ 0 & \vec{p}^\dagger s_0 & 0 \end{pmatrix}, \qquad J = \mathrm{diag}(1, 0, \ldots 0, -1). \qquad (4)$$

The vector $\vec{q}$ for $F = 1$ (resp. $F = 2$) is 3- (resp. 5-) component and has the form

$$\vec{q} = (\Phi_1, \Phi_0, \Phi_{-1})^T, \qquad \vec{q} = (\Phi_2, \Phi_1, \Phi_0, \Phi_{-1}, \Phi_{-2})^T, \qquad (5)$$

and the corresponding matrices s_0 enter in the definition of $so(2r + 1)$ with $r = 2$ and $r = 3$:

$$X \in so(2r + 1), \qquad X + S_0X^TS_0 = 0, \qquad S_0 = \sum_{s=1}^{2r+1}(-1)^{s+1}E_{s,n+1-s}, \qquad S_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \qquad (6)$$

By E_{sp} above we mean $2r + 1 \times 2r + 1$ matrix with matrix elements $(E_{sp})_{ij} = \delta_{si}\delta_{pj}$. With the definition of orthogonality used in (6) the Cartan generators $H_k = E_{k,k} - E_{n+3-k,n+3-k}$ are represented by diagonal matrices.

If we make use of the typical reduction $Q = Q^\dagger$ (or $\vec{p}^* = \vec{q}$) the generic MNLS type equations related to **BD.I.** acquire the form:

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^\dagger, \vec{q})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{q}^* = 0. \qquad (7)$$

The Hamiltonians for the MNLS equations (7) are given by

$$H_{\mathrm{MNLS}} = \int_{-\infty}^{\infty} dx \left((\partial_x\vec{q}^\dagger, \partial_x\vec{q}) - (\vec{q}^\dagger, \vec{q})^2 + \frac{1}{2}|(\vec{q}^T, s_0\vec{q})|^2\right). \qquad (8)$$

3. THE DIRECT AND THE INVERSE SCATTERING PROBLEM

3.1 The fundamental analytic solution

We remind some basic features of the inverse scattering theory for the Lax operators L, see.[11, 12] There we have made use of the general theory developed in[13–16, 18] and the references therein. The Jost solutions of L are defined by:

$$\lim_{x\to-\infty}\phi(x, t, \lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to\infty}\psi(x, t, \lambda)e^{i\lambda Jx} = \mathbb{1} \qquad (9)$$

and the scattering matrix $T(\lambda, t) \equiv \psi^{-1}\phi(x, t, \lambda)$. The special choice of J and the fact that the Jost solutions and the scattering matrix take values in the group $SO(2r + 1)$ we can use the following block-matrix structure of $T(\lambda, t)$

$$T(\lambda, t) = \begin{pmatrix} m_1^+ & -\vec{b}^{-T} & c_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{B}^- \\ c_1^+ & \vec{B}^{+T}s_0 & m_1^- \end{pmatrix}, \qquad \hat{T}(\lambda, t) = \begin{pmatrix} m_1^- & \vec{B}^{-T} & c_1^- \\ -\vec{B}^+ & \hat{\mathbf{T}}_{22} & s_0\vec{b}^- \\ c_1^+ & -\vec{b}^{+T}s_0 & m_1^+ \end{pmatrix}, \qquad (10)$$

where $\vec{b}^{\pm}(\lambda, t)$ and $\vec{B}^{\pm}(\lambda, t)$ are $2r - 1$-component vectors, $\mathbf{T}_{22}(\lambda)$ is $2r - 1 \times 2r - 1$ block matrix, and $m_1^{\pm}(\lambda)$, and $c_1^{\pm}(\lambda)$ are scalar functions. Such parametrization is compatible with the generalized Gauss decompositions of $T(\lambda)$ which read as follows:

$$T(\lambda, t) = T_J^- D_J^+ \hat{S}_J^+, \qquad T(\lambda, t) = T_J^+ D_J^- \hat{S}_J^-,$$
$$T_J^{\mp} = e^{\pm(\vec{\rho}^{\pm}, \vec{E}_1^{\mp})}, \qquad S_J^{\pm} = e^{\pm(\vec{\tau}^{\pm}, \vec{E}_1^{\pm})}, \qquad D_J^{\pm} = \text{diag}\left((m_1^{\pm})^{\pm 1}, \mathbf{m}_2^{\pm}, (m_1^{\pm})^{\mp 1}\right),$$

(11)

where

$$\left(\vec{\rho}^+, \vec{E}_1^+\right) = \sum_{k=1}^{r-1} (\rho_k^+ E_{e_1 - e_{k+1}} + \rho_{\bar{k}}^+ E_{e_1 + e_{k+1}}) + \rho_r^+ E_{e_1},$$

$$\left(\vec{\rho}^-, \vec{E}_1^-\right) = \sum_{k=1}^{r-1} (\rho_k^- E_{-e_1 + e_{k+1}} + \rho_{\bar{k}}^- E_{-e_1 - e_{k+1}}) + \rho_r^- E_{-e_1},$$

(12)

and similar expressions for $\left(\vec{\tau}^{\pm}, \vec{E}_1^{\pm}\right)$. The functions $m_1^{\pm}$ and $n \times n$ matrix-valued functions $\mathbf{m}_2^{\pm}$ are analytic for $\lambda \in \mathbb{C}_{\pm}$. We have introduced also the notations:

$$\vec{\rho}^- = \frac{\vec{B}^-}{m_1^-}, \qquad \vec{\tau}^- = \frac{\vec{B}^+}{m_1^-}, \qquad \vec{\rho}^+ = \frac{\vec{b}^+}{m_1^+}, \qquad \vec{\tau}^+ = \frac{\vec{b}^-}{m_1^+}.$$

There are some additional relations which ensure that both $T(\lambda)$ and its inverse $\hat{T}(\lambda)$ belong to the orthogonal group $SO(2r + 1)$ and that $T(\lambda)\hat{T}(\lambda) = \mathbb{1}$.

Next we introduce the fundamental analytic solution (FAS) $\chi^{\pm}(x, t, \lambda)$ using the generalized Gauss decomposition of $T(\lambda, t)$, see:[15, 19, 20]

$$\chi^{\pm}(x, t, \lambda) = \phi(x, t, \lambda) S_J^{\pm}(t, \lambda) = \psi(x, t, \lambda) T_J^{\mp}(t, \lambda) D_J^{\pm}(\lambda),$$

(13)

This construction ensures that $\xi^{\pm}(x, \lambda) = \chi^{\pm}(x, \lambda) e^{i\lambda J x}$ are analytic functions of λ for $\lambda \in \mathbb{C}_{\pm}$. If $Q(x, t)$ is a solution of the MNLS eq. (7) then the matrix elements of $T(\lambda)$ satisfy the linear evolution equations[12]

$$i\frac{d\vec{b}^{\pm}}{dt} \pm \lambda^2 \vec{b}^{\pm}(t, \lambda) = 0, \qquad i\frac{d\vec{B}^{\pm}}{dt} \pm \lambda^2 \vec{B}^{\pm}(t, \lambda) = 0,$$
$$i\frac{dm_1^{\pm}}{dt} = 0, \qquad i\frac{d\mathbf{m}_2^{\pm}}{dt} = 0.$$

(14)

Thus the block-diagonal matrices $D^{\pm}(\lambda)$ can be considered as generating functionals of the integrals of motion. The fact that all $(2r - 1)^2$ matrix elements of $\mathbf{m}_2^{\pm}(\lambda)$ for $\lambda \in \mathbb{C}_{\pm}$ generate integrals of motion reflect the superintegrability of the model and are due to the degeneracy of the dispersion law of (7). Note that $D_J^{\pm}(\lambda)$ allow analytic extension for $\lambda \in \mathbb{C}_{\pm}$ and that their zeroes and poles determine the discrete eigenvalues[11] of L.

3.2 The Riemann-Hilbert Problem

The FAS for real λ are linearly related[12]

$$\chi^+(x, t, \lambda) = \chi^-(x, t, \lambda) G_{0,J}(\lambda, t), \qquad G_{0,J}(\lambda, t) = \hat{S}_J^-(\lambda, t) S_J^+(\lambda, t)$$

(15)

Eq. (15) can be rewriten in an equivalent form for the FAS $\xi^{\pm}(x, t, \lambda) = \chi^{\pm}(x, t, \lambda) e^{i\lambda J x}$:

$$i\frac{d\xi^{\pm}}{dx} + Q(x)\xi^{\pm}(x, \lambda) - \lambda[J, \xi^{\pm}(x, \lambda)] = 0,$$

(16)

and the relation

$$\lim_{\lambda \to \infty} \xi^{\pm}(x, t, \lambda) = \mathbb{1},$$

(17)

Then these FAS satisfy the RHP's

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G_J(x,\lambda,t), \qquad G_J(x,\lambda,t) = e^{-i\lambda J(x+\lambda t)}G_J^-(\lambda,t)e^{i\lambda J(x+\lambda t)}, \tag{18}$$

Obviously the sewing function $G_J(x,\lambda,t)$ is uniquely determined by the Gauss factors $S_J^\pm(\lambda,t)$. In addition Zakharov-Shabat's theorem[13, 14] states that $G_J(x,\lambda,t)$ depends on x and t in the way prescribed above then the corresponding FAS satisfy the linear systems (16).

If we have solved the RHP's and know the FAS $\xi^+(x,t,\lambda)$ then the formula

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda \left(J - \xi^+(x,t,\lambda)J\hat\xi^+(x,t,\lambda) \right), \tag{19}$$

allows us to recover the corresponding potential of L.

4. REDUCTIONS OF MNLS

Along with the typical reduction $Q = Q^\dagger$ mentioned above one can impose additional reductions using the reduction group proposed by Mikhailov.[21] They are automatically compatible with the Lax representation of the corresponding MNLS eq. Below we make use of two $\mathbb{Z}_2$-reductions:[22]

$$\begin{aligned} 1) && C_1 U^\dagger(x,t,\lambda^*)C_1^{-1} &= U(x,t,\lambda), & C_1 V^\dagger(x,t,\lambda^*)C_1^{-1} &= V(x,t,\lambda), \\ 2) && C_2 U^T(x,t,\lambda)C_2^{-1} &= -U(x,t,\lambda), & C_2 V^T(x,t,\lambda)C_2^{-1} &= -V(x,t,\lambda), \end{aligned} \tag{20}$$

where C_1 and C_2 are involutions of the Lie algebra $so(2r+1)$, i.e. $C_i^2 = \mathbb{1}$. They can be chosen to be either diagonal (i.e., elements of the Cartan subgroup of $SO(2r+1)$) or elements of the Weyl group.

The typical reductions of the MNLS eqs. is a class 1) reduction obtained by specifying C_1 to be the identity automorphism of $\mathfrak{g}$; below we list several choices for C_1 leading to inequivalent reductions:

$$\begin{aligned} &1a) & C_1 &= \mathbb{1}, & \vec p(x) &= \vec q^{\,*}(x), & &1b) & C_1 &= K_1, & \vec p(x) &= K_{01}\vec q^{\,*}(x), \\ &1c) & C_1 &= S_{e_2}, & \vec p(x) &= K_{02}\vec q^{\,*}(x), & &1d) & C_1 &= S_{e_2}S_{e_3}, & \vec p(x) &= K_{03}\vec q^{\,*}(x), \\ &2e) & C_2 &= K_4, & \vec q(x) &= -K_{04}s_0\vec q(x), & \vec p(x) &= -K_{04}s_0\vec p(x), \end{aligned} \tag{21}$$

where

$$K_j = \text{block-diag}\,(1, K_{0j}, 1), \qquad K_{01} = \text{diag}\,(\epsilon_1, \ldots, \epsilon_{r-1}, 1, \epsilon_{r-1}, \ldots, \epsilon_1), \quad j = 1, 2, 3, \tag{22}$$

and $\epsilon_j = \pm 1$. The matrices K_{02}, K_{03} and K_4 are not diagonal and may take the form:

$$K_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad\qquad K_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & K_{04} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$K_{02} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_{03} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

Each of the above reductions impose constraints on the FAS, on the scattering matrix $T(\lambda)$ and on its Gauss factors $S_J^\pm(\lambda)$, $T_J^\pm(\lambda)$ and $D_J^\pm(\lambda)$. These have the form:

$$\begin{aligned} (S^+(\lambda^*))^\dagger &= K_j^{-1}\hat S^-(\lambda)K_j & (T^+(\lambda^*))^\dagger &= K_j^{-1}\hat T^-(\lambda)K_j & (D^+(\lambda^*))^\dagger &= K_j^{-1}\hat D^-(\lambda)K_j \\ \vec\tau^+ &= K_{0j}\vec\tau^{-,*}, & \vec\rho^+ &= K_{0j}\vec\rho^{-,*}, \end{aligned} \tag{24}$$

where the matrices K_j are specific for each choice of the automorphisms C_1, see eqs. (21), (22).

In particular, from the last line of (24) and (22) we get:

$$(m_1^+(\lambda^*))^* = m_1^-(\lambda), \tag{25}$$

and consequently, if $m_1^+(\lambda)$ has zeroes at the points λ_k^+, then $m_1^-(\lambda)$ has zeroes at:

$$\lambda_k^- = (\lambda_k^+)^*, \qquad k = 1, \ldots, N. \tag{26}$$

5. SOLITON SOLUTIONS

Let us now make use of one of the versions of the dressing method[13, 14] which allows one to construct singular solutions of the RHP. In order to obtain N-soliton solutions one has to apply dressing procedure with a $2N$-poles dressing factor of the form

$$u(x, \lambda) = \mathbb{1} + \sum_{k=1}^{N} \left(\frac{A_k(x)}{\lambda - \lambda_k^+} + \frac{B_k(x)}{\lambda - \lambda_k^-} \right). \tag{27}$$

The N-soliton solution itself can be generated via the following formula

$$Q_{N,\mathrm{s}}(x) = \sum_{k=1}^{N} [J, A_k(x) + B_k(x)]. \tag{28}$$

The dressing factor $u(x, \lambda)$ must satisfy the equation

$$i\partial_x u + Q_{N,\mathrm{s}} u - \lambda[J, u] = 0 \tag{29}$$

and the normalization condition $\lim_{\lambda \to \infty} u(x, \lambda) = \mathbb{1}$. The construction of $u(x, \lambda) \in SO(n+2)$ is based on an appropriate anzatz specifying the form of its λ-dependence[23, 24]

The residues of u admit the following decomposition

$$A_k(x) = X_k(x) F_k^T(x), \qquad B_k(x) = Y_k(x) G_k^T(x).$$

where all matrices involved are supposed to be rectangular and of maximal rank s. By comparing the coefficients before the same powers of $\lambda - \lambda_k^{\pm}$ in (29) we convince ourselves that the factors F_k and G_k can be expressed by the fundamental analytic solutions $\chi_0^{\pm}(x, \lambda)$ as follows

$$F_k^T(x) = F_{k,0}^T [\chi_0^+(x, \lambda_k^+)]^{-1}, \qquad G_k^T(x) = G_{k,0}^T [\chi_0^-(x, \lambda_k^-)]^{-1}.$$

The constant rectangular matrices $F_{k,0}$ and $G_{k,0}$ obey the algebraic relations

$$F_{k,0}^T S_0 F_{k,0} = 0, \qquad G_{k,0}^T S_0 G_{k,0} = 0.$$

The other two types of factors X_k and Y_k are solutions to the algebraic system

$$\begin{aligned}
S_0 F_k &= X_k \alpha_k + \sum_{l \neq k} \frac{X_l F_l^T S_0 F_k}{\lambda_l^+ - \lambda_k^+} + \sum_l \frac{Y_l G_l^T S_0 F_k}{\lambda_l^- - \lambda_k^+}, \\
S_0 G_k &= \sum_l \frac{X_l F_l^T S_0 G_k}{\lambda_l^+ - \lambda_k^-} + Y_k \beta_k + \sum_{l \neq k} \frac{Y_l G_l^T S_0 G_k}{\lambda_l^- - \lambda_k^-}.
\end{aligned} \tag{30}$$

The square $s \times s$ matrices $\alpha_k(x)$ and $\beta_k(x)$ introduced above depend on χ_0^+ and χ_0^- and their derivatives by λ as follows

$$\begin{aligned}
\alpha_k(x) &= -F_{0,k}^T [\chi_0^+(x, \lambda_k^+)]^{-1} \partial_\lambda \chi_0^+(x, \lambda_k^+) S_0 F_{0,k} + \alpha_{0,k}, \\
\beta_k(x) &= -G_{0,k}^T [\chi_0^-(x, \lambda_k^-)]^{-1} \partial_\lambda \chi_0^-(x, \lambda_k^-) S_0 G_{0,k} + \beta_{0,k}.
\end{aligned} \tag{31}$$

Below for simplicity we will choose F_k and G_k to be $2r + 1$-component vectors. Then one can show that $\alpha_k = \beta_k = 0$ which simplifies the system (30). We also introduce the following more convenient parametrization for F_k and G_k, namely (see eq. (33)):

$$F_k(x, t) = S_0 |n_k(x, t)\rangle = \begin{pmatrix} e^{-z_k + i\phi_k} \\ -\sqrt{2} s_0 \vec{v}_{0k} \\ e^{z_k - i\phi_k} \end{pmatrix}, \qquad G_k(x, t) = |n_k^*(x, t)\rangle = \begin{pmatrix} e^{z_k + i\phi_k} \\ \sqrt{2} \vec{v}_{0k}{}^* \\ e^{-z_k - i\phi_k} \end{pmatrix}, \tag{32}$$

where $\vec{\nu}_{0k}$ are constant $2r - 1$-component polarization vectors and

$$z_j = \nu_j(x + 2\mu_j t) + \xi_{00}, \qquad \phi_j = \mu_j x + (\mu_j^2 - \nu_j^2)t + \delta_{00},$$

$$\langle n_j^T(x,t)|S_0|n_j(x,t)\rangle = 0, \qquad \text{or} \qquad (\vec{\nu}_{0,j} s_0 \vec{\nu}_{0,j}) = 1. \tag{33}$$

With this notations the polarization vectors automatically satisfy $\langle n_j(x,t)|S_0|n_j(x,t)\rangle = 0$.

Thus for $N = 1$ we get the system:

$$|Y_1\rangle = -\frac{(\lambda_1^+ - \lambda_1^-)|n_1\rangle}{\langle n_1^\dagger|n_1\rangle}, \qquad |X_1\rangle = \frac{(\lambda_1^+ - \lambda_1^-)S_0|n_1^*\rangle}{\langle n_1^\dagger|n_1\rangle}, \tag{34}$$

which is easily solved. As a result for the one-soliton solution we get:

$$\vec{q}_{1s} = -\frac{i\sqrt{2}(\lambda_1^+ - \lambda_1^-)e^{-i\phi_1}}{\Delta_1}\left(e^{-z_1}s_0|\vec{\nu}_{01}\rangle + e^{z_1}|\vec{\nu}_{01}^*\rangle\right), \qquad \Delta_1 = \cosh(2z_1) + \langle \vec{\nu}_{01}^\dagger|\vec{\nu}_{01}\rangle. \tag{35}$$

For $n = 3$ we put $\nu_{0k} = |\nu_{0k}|e^{\alpha_{0k}}$ get:

$$\Phi_{1s;\pm 1} = -\frac{\sqrt{2}|\nu_{01;1}\nu_{01;3}|(\lambda_1^+ - \lambda_1^-)}{\Delta_1}e^{-i\phi_1 \pm i\beta_{13}}\left(\cosh(z_1 \mp \zeta_{01})\cos(\alpha_{13}) - i\sinh(z_1 \mp \zeta_{01})\sin(\alpha_{13})\right),$$

$$\Phi_{1s;0} = -\frac{\sqrt{2}|\nu_{01;2}|(\lambda_1^+ - \lambda_1^-)}{\Delta_1}e^{-i\phi_1}\left(\sinh z_1 \cos(\alpha_{02}) + i\cosh z_1 \sin(\alpha_{02})\right), \tag{36}$$

$$\beta_{13} = \frac{1}{2}(\alpha_{03} - \alpha_{01}), \qquad \zeta_{01} = \frac{1}{2}\ln\frac{|\nu_{01;3}|}{|\nu_{01;1}|}, \qquad \alpha_{13} = \frac{1}{2}(\alpha_{03} + \alpha_{01}),$$

Note that the 'center of mass' of $\Phi_{1s;1}$ (resp. of $\Phi_{1s;-1}$) is shifted with respect to the one of $\Phi_{1s;0}$ by ζ_{01} to the right (resp to the left); besides $|\Phi_{1s;1}| = |\Phi_{1s;-1}|$, i.e. they have the same amplitudes.

For $n = 5$ we put $\nu_{0k} = |\nu_{0k}|e^{\alpha_{0k}}$ and get analogously:

$$\Phi_{1s;\pm 2} = -\frac{\sqrt{2}|\nu_{01;1}\nu_{01;5}|(\lambda_1^+ - \lambda_1^-)}{\Delta_1}e^{-i\phi_1 \pm i\beta_{15}}\left(\cosh(z_1 \mp \zeta_{01})\cos(\alpha_{15}) - i\sinh(z_1 \mp \zeta_{01})\sin(\alpha_{15})\right),$$

$$\Phi_{1s;\pm 1} = \frac{\sqrt{2}|\nu_{01;2}\nu_{01;4}|(\lambda_1^+ - \lambda_1^-)}{\Delta_1}e^{-i\phi_1 \pm i\beta_{24}}\left(\cosh(z_1 \mp \zeta_{02})\cos(\alpha_{24}) - i\sinh(z_1 \mp \zeta_{01})\sin(\alpha_{24})\right),$$

$$\Phi_{1s;0} = -\frac{\sqrt{2}|\nu_{01;3}|(\lambda_1^+ - \lambda_1^-)}{\Delta_1}e^{-i\phi_1}\left(\cosh z_1 \cos(\alpha_{03}) - i\sinh z_1 \sin(\alpha_{03})\right), \tag{37}$$

$$\beta_{15} = \frac{1}{2}(\alpha_{05} - \alpha_{01}), \qquad \zeta_{01} = \frac{1}{2}\ln\frac{|\nu_{01;5}|}{|\nu_{01;1}|}, \qquad \alpha_{15} = \frac{1}{2}(\alpha_{05} + \alpha_{01}),$$

$$\beta_{24} = \frac{1}{2}(\alpha_{04} - \alpha_{02}), \qquad \zeta_{02} = \frac{1}{2}\ln\frac{|\nu_{01;4}|}{|\nu_{01;2}|}, \qquad \alpha_{24} = \frac{1}{2}(\alpha_{04} + \alpha_{02}),$$

Similarly the 'center of mass' of $\Phi_{1s;2}$ and $\Phi_{1s;1}$ (resp. of $\Phi_{1s;-2}$ and $\Phi_{1s;-1}$) are shifted with respect to the one of $\Phi_{1s;0}$ by ζ_{01} and ζ_{02} to the right (resp to the left); besides $|\Phi_{1s;2}| = |\Phi_{1s;-2}|$ and $|\Phi_{1s;1}| = |\Phi_{1s;-1}|$.

For $N = 2$ we get:

$$|n_1(x,t)\rangle = \frac{X_2(x,t)f_{21}}{\lambda_2^+ - \lambda_1^+} + \frac{Y_1(x,t)\kappa_{11}}{\lambda_1^- - \lambda_1^+} + \frac{Y_2(x,t)\kappa_{21}}{\lambda_2^- - \lambda_1^+},$$

$$|n_2(x,t)\rangle = \frac{X_1(x,t)f_{12}}{\lambda_1^+ - \lambda_2^+} + \frac{Y_1(x,t)\kappa_{12}}{\lambda_1^- - \lambda_2^+} + \frac{Y_2(x,t)\kappa_{22}}{\lambda_2^- - \lambda_2^+},$$

$$S_0|n_1^*(x,t)\rangle = \frac{X_1(x,t)\kappa_{11}}{\lambda_2^+ - \lambda_1^+} + \frac{X_2(x,t)\kappa_{11}}{\lambda_2^+ - \lambda_1^-} + \frac{Y_2(x,t)f_{21}^*}{\lambda_2^- - \lambda_1^-}, \tag{38}$$

$$S_0|n_2^*(x,t)\rangle = \frac{X_1(x,t)\kappa_{21}}{\lambda_1^+ - \lambda_2^-} + \frac{X_2(x,t)\kappa_{22}}{\lambda_2^+ - \lambda_2^-} + \frac{Y_1(x,t)f_{12}^*}{\lambda_1^- - \lambda_2^-},$$

where

$$\kappa_{kj}(x,t) = e^{z_k+z_j+i(\phi_k-\phi_j)} + e^{-z_k-z_j-i(\phi_k-\phi_j)} + 2\left(\vec{\nu}_{0k}^\dagger, \vec{\nu}_{0j}\right),$$
$$f_{kj}(x,t) = e^{z_k-z_j-i(\phi_k-\phi_j)} + e^{z_j-z_k+i(\phi_k-\phi_j)} - 2\left(\vec{\nu}_{0k}^T s_0 \vec{\nu}_{0j}\right),$$

(39)

In other words:

$$\mathcal{M}\vec{X} \equiv \begin{pmatrix} 0 & \frac{f_{21}}{\lambda_2^+-\lambda_1^+} & \frac{\kappa_{11}}{\lambda_1^--\lambda_1^+} & \frac{\kappa_{21}}{\lambda_2^--\lambda_1^+} \\ \frac{f_{12}}{\lambda_1^+-\lambda_2^+} & 0 & \frac{\kappa_{12}}{\lambda_1^--\lambda_2^+} & \frac{\kappa_{22}}{\lambda_2^--\lambda_2^+} \\ \frac{\kappa_{11}}{\lambda_1^+-\lambda_1^-} & \frac{\kappa_{12}}{\lambda_2^+-\lambda_1^-} & 0 & \frac{f_{21}^*}{\lambda_2^--\lambda_1^-} \\ \frac{\kappa_{21}}{\lambda_1^+-\lambda_2^-} & \frac{\kappa_{22}}{\lambda_2^+-\lambda_2^-} & \frac{f_{12}^*}{\lambda_1^--\lambda_2^-} & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} |n_1\rangle \\ |n_2\rangle \\ S_0|n_1^*\rangle \\ S_0|n_2^*\rangle \end{pmatrix}.$$

(40)

We can rewrite $\mathcal{M}$ in block-matrix form:

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}, \qquad \mathcal{M}_{22} = \mathcal{M}_{11}^*, \qquad \mathcal{M}_{21} = -\mathcal{M}_{12}^T,$$

$$\mathcal{M}_{11} = \frac{f_{12}}{\lambda_2^+-\lambda_1^+}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{M}_{12} = \begin{pmatrix} \frac{\kappa_{11}}{\lambda_1^--\lambda_1^+} & \frac{\kappa_{21}}{\lambda_2^--\lambda_1^+} \\ \frac{\kappa_{12}}{\lambda_1^--\lambda_2^+} & \frac{\kappa_{22}}{\lambda_2^--\lambda_2^+} \end{pmatrix}.$$

(41)

The inverse of $\mathcal{M}$ is given by:

$$\mathcal{M}^{-1} = \begin{pmatrix} (\mathcal{M}_{11} - \mathcal{M}_{12}\hat{\mathcal{M}}_{11}^*\mathcal{M}_{21})^{-1} & -(\mathcal{M}_{11} - \mathcal{M}_{12}\hat{\mathcal{M}}_{11}^*\mathcal{M}_{21})^{-1}\mathcal{M}_{12}\hat{\mathcal{M}}_{11}^* \\ -(\mathcal{M}_{11}^* - \mathcal{M}_{21}\hat{\mathcal{M}}_{11}\mathcal{M}_{12})^{-1}\mathcal{M}_{21}\hat{\mathcal{M}}_{11} & (\mathcal{M}_{11}^* - \mathcal{M}_{21}\hat{\mathcal{M}}_{11}\mathcal{M}_{12})^{-1} \end{pmatrix},$$

(42)

One can check by direct calculation that:

$$\mathcal{M}_{11} - \mathcal{M}_{12}\hat{\mathcal{M}}_{11}^*\mathcal{M}_{21} = \frac{f_{12}^*}{\lambda_2^--\lambda_1^-}Z\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\mathcal{M}_{11}^* - \mathcal{M}_{21}\hat{\mathcal{M}}_{11}\mathcal{M}_{12} = \frac{f_{12}}{\lambda_2^+-\lambda_1^+}Z\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$Z = \left(\frac{|f_{12}|^2}{|\lambda_2^+-\lambda_1^+|^2} - \frac{\kappa_{12}\kappa_{21}}{|\lambda_2^+-\lambda_1^-|^2} + \frac{\kappa_{11}\kappa_{22}}{4\nu_1\nu_2} \right),$$

(43)

Finally we get:

$$\mathcal{M}^{-1} = \frac{1}{Z}\begin{pmatrix} 0 & \frac{f_{12}^*}{\lambda_1^--\lambda_2^-} & -\frac{\kappa_{22}}{\lambda_2^+-\lambda_2^-} & \frac{\kappa_{12}}{\lambda_2^+-\lambda_1^-} \\ -\frac{f_{12}^*}{\lambda_1^--\lambda_2^-} & 0 & \frac{\kappa_{21}}{\lambda_1^+-\lambda_2^-} & -\frac{\kappa_{11}}{\lambda_1^+-\lambda_1^-} \\ \frac{\kappa_{22}}{\lambda_2^+-\lambda_2^-} & -\frac{\kappa_{21}}{\lambda_1^+-\lambda_2^-} & 0 & -\frac{f_{12}}{\lambda_1^+-\lambda_2^+} \\ -\frac{\kappa_{12}}{\lambda_2^+-\lambda_1^-} & \frac{\kappa_{11}}{\lambda_1^+-\lambda_1^-} & \frac{f_{12}}{\lambda_2^+-\lambda_1^+} & 0 \end{pmatrix},$$

(44)

From eqs. (40) and (44) we obtain:

$$|X_1\rangle = \frac{1}{Z}\left(\frac{f_{12}^*}{\lambda_1^--\lambda_2^-}|n_2\rangle - \frac{\kappa_{22}}{\lambda_2^+-\lambda_2^-}S_0|n_1^*\rangle + \frac{\kappa_{12}}{\lambda_2^+-\lambda_1^-}S_0|n_2^*\rangle \right),$$

$$|X_2\rangle = \frac{1}{Z}\left(-\frac{f_{12}^*}{\lambda_1^--\lambda_2^-}|n_1\rangle + \frac{\kappa_{21}}{\lambda_1^+-\lambda_2^-}S_0|n_1^*\rangle - \frac{\kappa_{11}}{\lambda_1^+-\lambda_1^-}S_0|n_2^*\rangle \right),$$

$$|Y_1\rangle = \frac{1}{Z}\left(\frac{\kappa_{22}}{\lambda_2^+-\lambda_2^-}|n_1\rangle - \frac{\kappa_{21}}{\lambda_1^+-\lambda_2^-}|n_2\rangle - \frac{f_{12}}{\lambda_1^+-\lambda_2^+}S_0|n_2^*\rangle \right),$$

$$|Y_2\rangle = \frac{1}{Z}\left(-\frac{\kappa_{12}}{\lambda_2^+-\lambda_1^-}|n_1\rangle + \frac{\kappa_{11}}{\lambda_1^+-\lambda_1^-}|n_2\rangle + \frac{f_{12}}{\lambda_2^+-\lambda_1^+}S_0|n_1^*\rangle \right),$$

(45)

Inserting this result into eq. (28) we obtain the following expression for the 2-soliton solution of the MNLS:

$$Q_{2s}(x,t) = [J, A_1 + B_1 + A_2 + B_2] = \frac{1}{Z}[J, C(x,t) - S_0 C^T(x,t) S_0],$$

$$C(x,t) = \frac{\kappa_{22}}{\lambda_2^+ - \lambda_2^-}|n_1\rangle\langle n_1^\dagger| - \frac{\kappa_{12}}{\lambda_2^+ - \lambda_1^-}|n_1\rangle\langle n_2^\dagger| - \frac{\kappa_{21}}{\lambda_1^+ - \lambda_2^-}|n_2\rangle\langle n_1^\dagger| + \frac{\kappa_{11}}{\lambda_1^+ - \lambda_1^-}|n_2\rangle\langle n_2^\dagger| \tag{46}$$

$$- \frac{f_{12}^*}{\lambda_1^- - \lambda_2^-}|n_1\rangle\langle n_2|S_0 - \frac{f_{12}}{\lambda_1^+ - \lambda_2^+}S_0|n_2^*\rangle\langle n_1^\dagger|.$$

At the end of this section we note that the effect of the reductions (20)–(21) consists in constraining the polarization vectors. For the reduction 2e) we get

$$\vec{\nu}_{0k} = K_{04}\vec{\nu}_{0k} \tag{47}$$

In particular, for $n = 3$ and for $K_{04} = -\mathbb{1}$ we have $q_1 = q_3$, and q_2 arbitrary. This reduction of eq. (1) is also important for the BEC.[6] From (47) we find $\nu_{01} = \nu_{03}$. The effect of this constraint is that for the one-soliton solution we get $\Phi_{1s;1} = \Phi_{1s;3}$.

Our next remark following[25] is that this reduction applied to the $F = 1$ MNLS (1) leads to a 2-component MNLS which after the change of variables

$$q_1 = \frac{1}{2}(w_1 + iw_2), \qquad q_2 = \frac{1}{\sqrt{2}}(w_1 - iw_2), \tag{48}$$

leads to two disjoint NLS equations for w_1 and w_2 respectively.

It is only logical that applying the constraint $\nu_{01} = \nu_{03}$ the explicit expression for the one-soliton solution (36) simplifies and reduces to the standard soliton solutions of the scalar NLS.

6. TWO SOLITON INTERACTIONS

In this section we generalize the classical results of Zakharov and Shabat about soliton interactions[26] to the class of MNLS equations related to BD.I symmetric spaces. For detailed exposition see the monographs.[15, 16] These results were generalized for the vector nonlinear Schrödinger equation by Manakov,[17] see also.[27–29] The Zakharov Shabat approach consisted in calculating the asymptotics of generic N-soliton solution of NLS for $t \to \pm\infty$ and establishing the pure elastic character of the generic soliton interactions. By generic here we mean N-soliton solution whose parameters $\lambda_k^\pm = \mu_k \pm i\nu_k$ are such that $\mu_k \neq \mu_j$ for $k \neq j$. The pure elastic character of the soliton interactions is demonstrated by the fact that for $t \to \pm\infty$ the generic N-soliton solution splits into sum of N one soliton solutions each preserving its amplitude $2\nu_k$ and velocity μ_k. The only effect of the interaction consists in shifting the center of mass and the initial phase of the solitons. These shifts can be expressed in terms of $\lambda_k^\pm$ only; for detailed exposition see.[16]

We start with the simplest non-trivial case. Namely we use the 2-soliton solution derived above and calculate its asymptotics along the trajectory of the first soliton. To this end we keep $z_1(x,t)$ fixed and let $\tau = z_2 - z_1$ tend to $\pm\infty$. Therefore it will be enough to insert the asymptotic values of the matrix elements of $\mathcal{M}$ for $\tau \to \pm\infty$ and keep only the leading terms. That gives:

$$
\begin{aligned}
\kappa_{22} &= \begin{cases} e^{2\tau}\exp(\nu_2 z_1/\nu_1) + 2\mathcal{C}_1, & \text{for } \tau \to \infty, \\ e^{-2\tau}\exp(-\nu_2 z_1/\nu_1) + 2\mathcal{C}_1, & \text{for } \tau \to -\infty, \end{cases} \\
\kappa_{12} &= \begin{cases} e^{\tau}\exp((1+\nu_2/\nu_1)z_1 + i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for } \tau \to \infty, \\ e^{-\tau}\exp(-(1+\nu_2/\nu_1)z_1 - i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for } \tau \to -\infty, \end{cases} \\
\kappa_{21} &= \begin{cases} e^{\tau}\exp((1+\nu_2/\nu_1)z_1 - i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for } \tau \to \infty, \\ e^{-\tau}\exp(-(1+\nu_2/\nu_1)z_1 + i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for } \tau \to -\infty, \end{cases} \\
f_{12} &= \begin{cases} e^{\tau}\exp(-(1-\nu_2/\nu_1)z_1 + i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for } \tau \to \infty, \\ e^{-\tau}\exp((1-\nu_2/\nu_1)z_1 - i(\phi_1 - \phi_2)) + \mathcal{O}(1), & \text{for } \tau \to -\infty, \end{cases}
\end{aligned}
\tag{49}
$$

After somewhat lengthy calculations we get:

$$
\begin{aligned}
\lim_{\tau \to \infty} \vec{q}_{2s}(x,t) &= -\frac{i\sqrt{2}\nu_1 e^{-i(\phi_1 - \alpha_+)}\left(e^{-z_1 - r_+} s_0 |\vec{\nu}_{01}\rangle + e^{z_1 + r_+}|\vec{\nu}_{01}^*\rangle\right)}{\cosh(2(z_1 + r_+)) + (\vec{\nu}_{01}^\dagger, \vec{\nu}_{01})}, \\
\lim_{\tau \to -\infty} \vec{q}_{2s}(x,t) &= \frac{i\sqrt{2}\nu_1 e^{-i(\phi_1 + \alpha_+)}\left(e^{-z_1 + r_+} s_0 |\vec{\nu}_{01}\rangle + e^{z_1 - r_+}|\vec{\nu}_{01}^*\rangle\right)}{\cosh(2(z_1 - r_+)) + (\vec{\nu}_{01}^\dagger, \vec{\nu}_{01})},
\end{aligned}
\tag{50}
$$

where

$$
r_+ = \ln\left|\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}\right|, \qquad \alpha_+ = \arg\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}.
$$

In other words the 2-soliton interaction for the MNLS eqs. related to the BD.I symmetric spaces is the same as the one of the scalar NLS. Again we have that for large times the 2-soliton solution splits into sum of 1-soliton solutions with shifted center of masses and phases and the value of these shifts r_+ and α_+ are independent on the number of components of MNLS. It will be interesting to check whether the N-soliton interactions consist of sequence of elementary 2-soliton interactions and the shifts are additive.

7. CONCLUSIONS AND DISCUSSION

Using the Zakharov-Shabat dressing method we have obtained the two-soliton solution and have used it to analyze the soliton interactions of the MNLS equation. The conclusion is that after the interactions the solitons recover their polarization vectors ν_{0k}, velocities and frequency velocities. The effect of the interaction is, like in for the scalar NLS equation, shift of the center of mass $z_1 \to z_1 + r_+$ and shift of the phase $\phi_1 \to \phi_1 + \alpha_+$. Both shifts are expressed through the related eigenvalues $\lambda_j^\pm$ only.

The next step would be to analyze multi-soliton interactions. Our hypothesis is that each soliton will acquire a total shift of the center of mass that is sum of all elementary shifts from each two soliton interactions. Similar result is expected for the total phase shift of the soliton. Proofs of these facts will be published elsewhere.

REFERENCES

[1] Ieda, J., Miyakawa, T. and Wadati, M., "Exact Analysis of Soliton Dynamics in Spinor Bose-Einstein Condensates", Phys. Rev Lett. **93**, 194102 (2004).

[2] Ieda, J., Miyakawa, T. and Wadati, M., "Matter-wave solitons in an $F = 1$ spinor Bose-Einstein condensate", J. Phys. Soc. Jpn. **73**, 2996 (2004).

[3] Li, L., Li, Z., Malomed, B. A., Mihalache, D. and Liu, W. M., "Exact Soliton Solutions and Nonlinear Modulation Instability in Spinor Bose-Einstein Condensates", Phys. Rev. A **72**, 033611 (2005).

[4] Uchiyama, M., Ieda, J. and Wadati, M., "Dark solitons in $F = 1$ spinor Bose–Einstein condensate" J. Phys. Soc. Jpn. **75** 064002 (2006).

[5] Doktorov, E. V., Wang, J. and Yang, J., "Perturbation theory for bright spinor Bose-Einstein condensate solitons", Phys. Rev. A **77**, 043617 (2008).

[6] Nistazakis, H. E., Frantzeskakis, D. J., Kevrekidis, P. G., Malomed, B. A. and Carretero-Gonzalez R., "Bright-Dark Soliton Complexes in Spinor Bose-Einstein Condensates" Phys. Rev. **A 77**, 033612 (2008).

[7] Uchiyama, M., Ieda, J. and Wadati, M., "Multicomponent Bright Solitons in $F = 2$ Spinor Bose-Einstein Condensates", J. Phys. Soc. Japan,**76**, 74005 (2007).

[8] Fordy, A.P. and Kulish, P. P., "Nonlinear Schrodinger Equations and Simple Lie Algebras", Commun. Math. Phys. **89** 427–443 (1983).

[9] Tsuchida, T. and Wadati, M., "The coupled modified Korteweg-de Vries equations", J. Phys. Soc. Jpn. **67** 1175 (1998).

[10] Kostov, N. A., Atanasov, V. A., Gerdjikov, V. S. and Grahovski, G. G., "On the soliton solutions of the spinor Bose-Einstein condensate", Proceedings of SPIE **6604**, 66041T (2007).

[11] Gerdjikov, V. S., Kaup, D. J., Kostov, N. A. and Valchev, T. I., "On classification of soliton solutions of multicomponent nonlinear evolution equations" J. Phys. A: Math. Theor. **41** 315213 (2008).

[12] Gerdjikov, V. S., Kostov, N. A. and Valchev T. I., "Solutions of multi-component NLS models and Spinor Bose-Einstein condensates", Physica D **238**, 1306-1310 (2009). **arXiv:0802.4398 [nlin.SI]**.

[13] Zakharov, V. E. and A. B. Shabat A. B., "A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem". I. Functional Analysis and Its Applications, **8**, 226–235 (1974).

[14] Zakharov, V. E. and Shabat, A. B., "Integration of nonlinear equations of mathematical physics by the method of inverse scattering". II. Functional Analysis and Its Applications, **13**, 166–174 (1979).

[15] Zakharov, V. E., Manakov S V., Novikov S P. and Pitaevskii L. I., [Theory of solitons. The inverse scattering method], Plenum, N.Y. (1984).

[16] Faddeev, L. D. and Takhtadjan, L. A., [Hamiltonian Approach in the Theory of Solitons], Springer Verlag, Berlin, (1987).

[17] Manakov, S. V., "On the theory of two-dimensional stationary self-focusing of electromagnetic waves", Zh. Eksp. Teor. Fiz [Sov.Phys. JETP], **65** [**38**], 505–516 [248–253], 1973 [1974].

[18] Gerdjikov, V. S., "The Zakharov-Shabat dressing method and the representation theory of the semisimple Lie algebras", Phys. Lett. A, **126A,** 184–188, (1987).

[19] Gerdjikov, V. S., "Generalised Fourier transforms for the soliton equations. Gauge covariant formulation", Inverse Problems, **2**, 51–74, (1986).

[20] Gerdjikov V. S., "The Generalized Zakharov–Shabat System and the Soliton Perturbations" Theor. Math. Phys. **99**, 292–299 (1994).

[21] Mikhailov, A. V., "The reduction problem and the inverse scattering method", Physica D: Nonlinear Phenomena, **3**, 73–117 (1981).

[22] Gerdjikov, V. S., "On Reductions of Soliton Solutions of multi-component NLS models and Spinor Bose-Einstein condensates", In AIP Conference proceedings of First Conference on Application of Mathematics in Technical and Natural Sciences, Sozopol, Bulgaria, June 22-27, 2009 (In press).

[23] Zakharov, V. E. and Mikhailov, A. V., "On The Integrability of Classical Spinor Models in Two-dimensional Space-time", Comm. Math. Phys. **74**, 21–40 (1980).

[24] Gerdjikov, V. S., Grahovski, G. G. and Kostov, N. A., "On the multi-component NLS type equations on symmetric spaces and their reductions", Theor. Math. Phys. **144**, 1147–1156 (2005).

[25] Park, Q-Han and Shin, H. J., "Painlevé analysis of the coupled nonlinear Schrödinger equation for polarized optical waves in an isotropic medium", Phys. Rev. E **59**, 2373–2379 (1999).

[26] Zakharov, V. E. and Shabat, A. B., "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media", Soviet Physics-JETP, **34**, 62–69 (1972).

[27] M. Ablowitz, J., Prinari B. and Trubatch A. D., [Discrete and continuous nonlinear Schrödinger systems] Cambridge Univ. Press, Cambridge, (2004).

[28] Kanna T. and Lakshmanan, M., "Exact soliton solutions of coupled nonlinear Schrödinger equations: Shape-changing collisions, logic gates, and partially coherent solitons", Phys. Rev. E **67**, 046617 (2003).

[29] Tsuchida, T., "N-soliton collision in the Manakov model", Prog. Theor. Phys. **111** 151-182 (2004).