# The simplified stochastic Fermi-Ulam model revisited

A.K. Karlis[a,b], F.K. Diakonos[a], V. Constantoudis[d], P. Schmelcher[b,c]

[a] *Department of Physics, University of Athens, GR-15771 Athens, Greece*
[b] *Physikalisches Institut, Universität Heidelberg, Philosophenweg 12, 69120 Heidelberg, Germany*
[c] *Theoretische Chemie, Im Neuenheimer Feld 229, Universität Heidelberg, 69120 Heidelberg, Germany*
[d] *Institute of Microelectronics, NCSR Demokritos, P.O. Box 60228, Attiki, Greece*

**Abstract**

The description of Fermi acceleration developing in the phase-randomized simplified Fermi-Ulam model (SFUM) can be achieved in terms of a random walk taking place in momentum space. Within this framework the evolution of the probability density function of particle velocities is determined by the Fokker-Planck equation (FPE). However, the standard treatment in the literature leads to a result, which even though is in agreement with the numerical results, it is inconsistent with the transport coefficients used for the construction of the FPE. In this work we present a consistent scheme for the description of Fermi acceleration, resolving this contradiction.

## 1. Introduction

The Fermi-Ulam model [1], originally proposed for testing the feasibility of gaining energy through scattering off moving targets, i.e. Fermi acceleration [2], consists of one harmonically oscillating and one fixed infinitely heavy hard wall and an ensemble of non-interacting particles bouncing between them. Ever since, many different versions of the original model have been suggested and investigated, such as variants of the FUM with dissipation [3–7], different deterministic or random drivings of the moving wall [8, 9] the quantum-mechanical version [10–14] and the, so called, bouncer model [16], where a particle performs elastic [17] or inelastic [18–24] collisions with an oscillating infinitely heavy platform under the influence of a gravitational field. Recently, a hybrid version of the FUM and the bouncer model has also been investigated [25, 26].

The equations defining the dynamics of the FUM are of implicit form with respect to the collision time, which complicates numerical simulations and hinders

*Email addresses:* `akkarlis@gmail.com` (A.K. Karlis), `fdiakono@phys.uoa.gr` (F.K. Diakonos), `vconst@imel.demokritos.gr` (V. Constantoudis), `Peter.Schmelcher@pci.uni-heidelberg.de` (P. Schmelcher)

an analytical treatment. A simplification —known as the *static wall approximation* (SWA) [8, 27]– which treats the oscillating wall as fixed in space, yet transfer of momentum is allowed as if the wall were oscillating, has become over the time the standard approximation for studying the FUM. The SWA speeds-up numerical simulations and facilitates the analytical treatment of the problem, while it has been generalized to higher-dimensional billiards with time-dependent boundaries, such as the time-dependent Lorentz Gas [8, 28].

However, the application of the SWA in the FUM suffers from two drawbacks. The first is that it leads to a considerable underestimation of the particle acceleration. It has been shown [8, 27] that the underestimation is caused by small additional fluctuations of the time of free flight due to the displacement of the oscillating wall occurring in the exact model, which are neglected within the SWA. The second drawback has to do with the occurrence of multiple consecutive collisions between the oscillating wall and the particles which are absent in the SWA. Specifically, collisions which do not lead to a reversal of the particle velocity —corresponding to multiple consecutive collisions in the exact model— within the SWA are treated by factitiously reversing the particle velocity, in order to prevent the particle from escaping the area between the walls. However, taking into account this set of collisions in the analytical treatment of the acceleration problem poses great difficulties. Moreover, due to the acceleration of the particles, the probability measure of such events decreases with increasing number of collisions $n$, irrespective of the initial conditions imposed on the ensemble of particles, since these collision events can occur only when the particle velocity is comparable to the maximum velocity of the wall. Due to the rarity of these collisions and the complications they add to the analytical calculations they are neglected in the analytical treatment of the acceleration problem within the SWA. However, this further simplification gives rise to a fundamental inconsistency: the ensemble mean of the absolute velocity obtained analytically does not change through collisions with the "*oscillating*" wall, despite the well-established numerical result that Fermi acceleration does take place in the phase-randomized FUM. Recently, a consistent semi-analytical scheme was presented in [29], which takes into account the rare set of collision events associated with the artificial reversal of the particle velocity, and agreement between analytical and numerical results for the ensemble average magnitude of particle velocities was achieved.

A complete statistical description of the acceleration problem in the FUM, i.e. the evolution of the magnitude of particle velocities, can be obtained by viewing Fermi acceleration as a random walk taking place in the momentum space. From this perspective, Fermi acceleration can be treated as a diffusion process, and therefore can be described via a kinetic equation, such as the Fokker-Planck equation (FPE) [8, 27, 32–34]. Within this framework a yet another striking contradiction arises: namely, the diffusion transport coefficient of the FPE —coinciding with the ensemble mean increment of the magnitude of particle velocities $\langle\delta|V|\rangle$— is found equal to zero [34], yet the solution of the FPE predicts the increase of the mean magnitude of particle velocities with increasing $n$ —$\langle\delta|V|\rangle > 0$.

In this work, it is shown that the FPE reported in the literature [32–34] describing the evolution of the probability density function (PDF) of the magnitude of particle velocities is not valid, and that the observed agreement between the analytical and numerical results in this case should be regarded as accidental, i.e. the agreement breaks down when all collision events are included. Moreover, a consistent description of Fermi acceleration developing in the phase-randomized FUM by means of the FPE, which additionally takes into account all collision events, is presented. Specifically, it is proven that the FPE cannot describe the evolution of the PDF of the magnitude of particle velocities. However, a consistent FPE can be constructed describing the PDF of the algebraic value of particle velocities, resolving an apparent inconsistency that has been persisting in the literature relating to the simplified FUM for over three decades.

## 2. The simplified stochastic FUM

We investigate a phase-randomized FUM, consisting of an ensemble of non-interacting particles confined between one oscillating and one fixed infinitely heavy wall. Randomization is induced in the system by shifting the phase of the oscillating wall through the addition of a uniformly distributed random number $\eta \in [0, 2\pi)$ after each collision of a particle with the fixed wall to the phase of oscillation. The particles perform elastic collisions with the walls and move ballistically between impacts.

The simplification usually employed, known as the SWA, consists in treating the oscillating wall as immobile located at its equilibrium position, yet allowing the transfer of momentum upon impact with a particle as if the wall were harmonically oscillating. Let as consider, without loss of generality, a FUM consisting of a fixed wall on the left and a moving wall on the right, oscillating with frequency $\omega$. If we further assume that the positive direction of particle velocities is towards the right, then the dynamics of the billiard within the framework of the SWA is defined by the following set of dimensionless difference equations:

$$t_n = t_{n-1} + \frac{2}{|V_{n-1}|} \tag{1a}$$

$$V_n = -|V_{n-1} + 2u_n| \tag{1b}$$

$$u_n = \epsilon \cos(t_n + \eta_n), \tag{1c}$$

where $u_n$ is the velocity of the "oscillating" wall, $V_n$ is the algebraic value of the particle velocity immediately *after* the $n$th collision with the "oscillating" wall measured in units of $\omega w$ ($w$ denoting the spacing between the walls), $t_n$ the time when the $n$th collision occurs measured in units of $1/\omega$, $\eta_n$ a random variable uniformly distributed in the interval $[0, 2\pi)$ updated immediately after each collision between a particle and the fixed wall and $\epsilon$ the dimensionless ratio of the amplitude of oscillation to the spacing between the "oscillating" and the fixed wall. It is noted that in all numerical simulations $\epsilon$ was fixed at $1/15$

The absolute value in (1b) is introduced in order to avoid the occurrence of positive particle velocities after a collision with the “oscillating” wall, which would lead to the escape of the particle from the area between the walls. It should be stressed that such a collision, within the framework of the exact model, corresponds to a particle experiencing at least one second consecutive collision with the “oscillating” wall. Therefore, if $|V_{n-1}| < 2u_n$ and $u_n > 0$, in order to prevent the particle from escaping the region between the walls the velocity is reversed artificially. The presence of the absolute value function in (1b), nevertheless, complicates the analytical treatment of the acceleration problem. For this reason, it has become a standard practice in the treatment of the FUM to remove the absolute value function, thereby neglecting the rare collision events upon which the particle direction is not reversed after its collision with the “oscillating” wall.

## 3. Fokker-Planck equation

Fermi acceleration of particles evolving in a stochastic FUM can be described in terms of a random walk taking place in velocity space. Therefore, the evolution of the distribution of particle speeds can be determined using the Fokker-Planck equation (FPE).

$$\frac{\partial}{\partial n}\rho\left(|V|,n\right) = -\frac{\partial}{\partial|V|}\left[B\rho\left(|V|,n\right)\right] + \frac{1}{2}\frac{\partial^2}{\partial|V|^2}\left[D\rho\left(|V|,n\right)\right], \tag{2}$$

where the transport coefficients $B$, $D$ are [34]

$$B = \left(\frac{1}{\Delta n}\right)\int \Delta|V|P\ d(\Delta|V|) \tag{3a}$$

$$D = \left(\frac{1}{\Delta n}\right)\int (\Delta|V|)^2 P\ d(\Delta|V|) \tag{3b}$$

and can in general depend on $|V|$ and $n$. In (3a)-(3b) $P$ is the probability of a particle possessing the velocity $|V|$ after $n$ collisions, if it had the velocity $|V_n| - \Delta|V_n|$, $\Delta n$ collisions earlier. Assuming that $\Delta n = 1$, Eqs. (3) become

$$B \equiv \langle\delta|V_n|\rangle \tag{4a}$$

$$D \equiv \langle(\delta|V_n|)^2\rangle, \tag{4b}$$

where $\langle\delta|V|\rangle$ is the ensemble mean increment of the magnitude of the particle velocity during one mapping period, i.e. in the course of one collision. If the collisions leading to an artificial reversal of the particle velocity are neglected, i.e. the absolute value in (1b) is dropped, and further assuming that the phase upon collision follows a uniform distribution, then the FPE coefficients are [32–34]

$$B = 0 \tag{5a}$$

$$D = 2\epsilon^2. \tag{5b}$$

Omitting the absolute value function in (1b) corresponds to neglecting the collision events implicating an artificial reversal of the particle velocity. In order to take also these events into account, one has to divide collision events into two sets. The first set $C_1$ contains collisions upon which the velocity of a particle is reversed without requiring the application of the absolute value function in (1b) and its complement $C_2$ containing the collisions upon which an artificial velocity reversal is required to prevent the particles escaping the area between the walls. Then, the transport coefficients is the sum of the corresponding quantities calculated within each set of collision events $C_i$ $(i = 1, 2)$ weighted by the probability measures $w_i(n)$ $(i = 1, 2)$ of each set of events [29].

However, as shown in [29], if this class of collisions is taken into account, then the value of the transport coefficient $B \equiv \delta\langle|V|\rangle \neq 0$. On the contrary, the drift coefficient vanishes only asymptotically $(n \to \infty)$. Specifically, the inclusion of the set $C_2$ in the calculation of the mean increase and the mean square increase of the magnitude of velocity for $n \gg 1$ yields [29]

$$\langle\delta|V|\rangle = w_2(n)\frac{\pi}{2}\epsilon \propto \frac{1}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right) \tag{6a}$$

$$\langle(\delta|V|)^2\rangle = 2\epsilon^2. \tag{6b}$$

In [29] the mean increase of the absolute velocity is expressed as a function of the number of collisions $n$. Given that (1b) does not explicitly depend on $n$, the drift coefficient should formally also be independent of $n$. Therefore, it is appropriate to reformulate (6a) as a function of $|V|$.

Obviously, the magnitude of the particle velocity $|V|$ for any $n$ can be expressed as $|V| = \langle|V|\rangle + \delta|V|$, where $\delta|V|$ are the fluctuations of $|V|$. Consequently, if the fluctuations of $|V|$ are not large in comparison with $\langle|V|\rangle$, then one can make the approximation

$$|V| \simeq \langle|V|\rangle. \tag{7}$$

In order to determine the mean absolute velocity of the particles as well as the fluctuations of $|V|$ we need to determine the PDF of particle absolute velocities $\rho(|V|, n)$. This can be achieved through the application of the central limit theorem (CLT) [29], to find

$$\rho(|V|, n) = \frac{1}{\epsilon\sqrt{\pi n}} \exp\left[-|V|^2/(4\epsilon^2 n)\right]. \tag{8}$$

According to (8) the mean absolute value of particle velocities as a function of the number of collisions $n$ is

$$\langle|V_n|\rangle = \frac{2\sqrt{n}\epsilon}{\sqrt{\pi}}. \tag{9}$$

An estimation of the fluctuations of $|V|$ can be made by calculating the standard deviation of (8), to find $\sigma(|V|) = 2\sqrt{n\,(1/2 - 1/\pi)}\epsilon$. Therefore, the relative error of the approximation (7) is,

$$\frac{\sigma(|V|)}{\langle|V|\rangle} = \sqrt{\frac{\pi}{2} - 1} \simeq 0.756, \tag{10}$$

independent of the number of collisions $n$.

The statistical weight $w_2(n)$ of the set of rare events was determined in [29] semi-analytically. In this Letter, we were able to determine it fully analytically (refer to Appendix B for details), in complete agreement with the previous result. Specifically, the statistical weight $w_2(n)$ is found to the leading order of $1/n$ to be,

$$w_2(n) \simeq \frac{2}{\sqrt{n}\pi^{3/2}}. \tag{11}$$

Combining (6a) and (11) we obtain for the mean increase of the absolute velocity in the course of one collision,

$$\langle\delta|V|\rangle \simeq \frac{\epsilon}{\sqrt{\pi n}}. \tag{12}$$

Finally, to express the mean increase of the absolute particle velocity as a function of the particle absolute velocity, we solve (9) for $\sqrt{n}$, substitute the result into (12) and further assume that $\langle|V|\rangle \simeq |V|$ to obtain,

$$B \equiv \langle\delta|V|\rangle \simeq \frac{2\epsilon^2}{\pi|V|}. \tag{13}$$

The substitution of (5a) and (5b) into (2), together with reflective boundary conditions at $|V| = 0$, i.e. $\left.\frac{\partial\rho(|V|,n)}{\partial|V|}\right|_{|V|=0} = 0$, and the initial condition $\rho(|V|,0) = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma}\exp\left[-\frac{|V|^2}{2\sigma^2}\right]$, leads to the following solution of the FPE:

$$\rho\left(|V|,n\right) = \sqrt{\frac{2}{\pi(2n\epsilon^2+\sigma^2)}}\exp\left[-\frac{|V|^2}{4n\epsilon^2+2\sigma^2}\right]. \tag{14}$$

The solution of the FPE using the recalculated coefficients, that is (6b) and (13), which take into account all collision events, can be obtained utilizing a generalized Hankel transform [35] (see Appendix C for derivation). The result is,

$$\begin{aligned}\tilde{\rho}(|V|,n) \quad &= \quad \frac{1}{4^{1/\pi}\epsilon^{2/\pi}\Gamma\left(\frac{1}{2}+\frac{1}{\pi}\right)}|V|^{2/\pi}e^{-\frac{|V|^2}{4n\epsilon^2}} \\ &\quad \times\frac{{}_1F_1\left(\frac{1}{2};\frac{1}{2}+\frac{1}{\pi};\frac{|V|^2\sigma^2}{8n^2\epsilon^4+4n\sigma^2\epsilon^2}\right)}{n^{1/\pi}\sqrt{n\epsilon^2+\frac{\sigma^2}{2}}},\end{aligned} \tag{15}$$

where ${}_1F_1\left(a;b;z\right)$ is the Kummer confluent hypergeometric function. Expression (15) has the following asymptotic series expansion for $n \gg 1$,

$$\tilde{\rho}(|V|,n) \simeq \frac{1}{4^{1/\pi}\epsilon^{2/\pi}\Gamma\left(\frac{1}{2}+\frac{1}{\pi}\right)}\frac{|V|^{2/\pi}e^{-\frac{|V|^2}{4n\epsilon^2}}}{n^{1/\pi}}\left[\sqrt{\frac{1}{n}}\frac{1}{\epsilon}+\mathcal{O}\left(\frac{1}{n^{3/2}}\right)\right]. \tag{16}$$

The most fundamental difference between the two solutions (14) and (15) is that while the most probable speed of the particles evolving in the stochastic simplified FUM predicted by (14) is $|V| = 0$ for all times, if the set of rare events $C_2$ is taken into account, then the most probable speed $|V|_p$, is 0 only for $n = 0$. For $n > 0$ the maximum of the PDF (15) is shifted towards larger values of $|V|$, i.e. $|V|_p \simeq \frac{2\sqrt{n}\epsilon}{\sqrt{\pi}}$ . It can be shown that the error (10) introduced by (7) has no impact on this property of (15), but that it only changes the rate at which the maximum is shifted away from the origin.

In Fig. 1 histograms of particle speeds are shown, obtained by the simulation of $1.6 \times 10^5$ trajectories for $n = 1.5 \times 10^5$ collisions, with the initial particle velocities chosen randomly according to the PDF

$$\rho(|V|, 0) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \exp\left[-\frac{|V|^2}{2\sigma^2}\right]$$

($\sigma = 2/3$). For this particular choice of the initial distribution, $w_2(n)$ accurately described by (11) even for small $n$ [29]. In the same figure, the analytical solution given by (14) of the FPE using the coefficients (5a)-(5b) of Lieberman and Lichtenberg [32–34] is shown —black line— as well as the solution (15) of the FPE using the recalculated coefficients (6b) and (13) —red line.

Surprisingly, the use of the recalculated coefficients, which take into account all collisions —including those which implicate an artificial reversal of particle velocities— leads to a prediction of the distribution of particle speeds, which clearly does not agree with the histograms yielded by the numerical simulation. On the contrary, the use of the value of the drift coefficient $B$ published in [32–34], although inconsistent with the fact that the phase-randomized FUM exhibits Fermi acceleration, i.e. $B \equiv \langle \delta |V| \rangle \neq 0$, leads to a solution which is in agreement with the numerical results. It should be also noted that the solution (14) obtained using the coefficients (5a) and (5b) is also in agreement with the solution derived by the application of the CLT [29]. Even more, the numerical results presented in Fig. 1 seem to support the prediction of a zero drift coefficient ($B = 0$) and that only the fluctuations of particle velocities increase with time ($D \neq 0$). On the other hand, due to the restriction of the magnitude of velocity in the semi-infinite interval $(0, \infty)$, the spreading of $\rho(|V|, n)$ obviously leads to an increasing first moment of particle speeds as a function of the number of collisions $n$, even if the drift coefficient $B$ vanishes.

This apparent inconsistency can be apprehended if we examine more closely the derivation of the transport coefficients. A more rigorous calculation of the transport coefficients reveals that if the magnitude of particle velocities is chosen as an independent variable, then the drift coefficient cannot be defined (see Appendix A).

Furthermore, defining time in terms of the number of collisions with the "oscillating" wall restricts the direction of the particle velocity after a collision to be always negative for the specific convention assumed here. Thus, also the algebraic value of particle velocity is constrained in a semi-infinite interval. Consequently, since $\langle |V| \rangle = \pm \langle V \rangle$ the mean particle velocity is also non-zero,

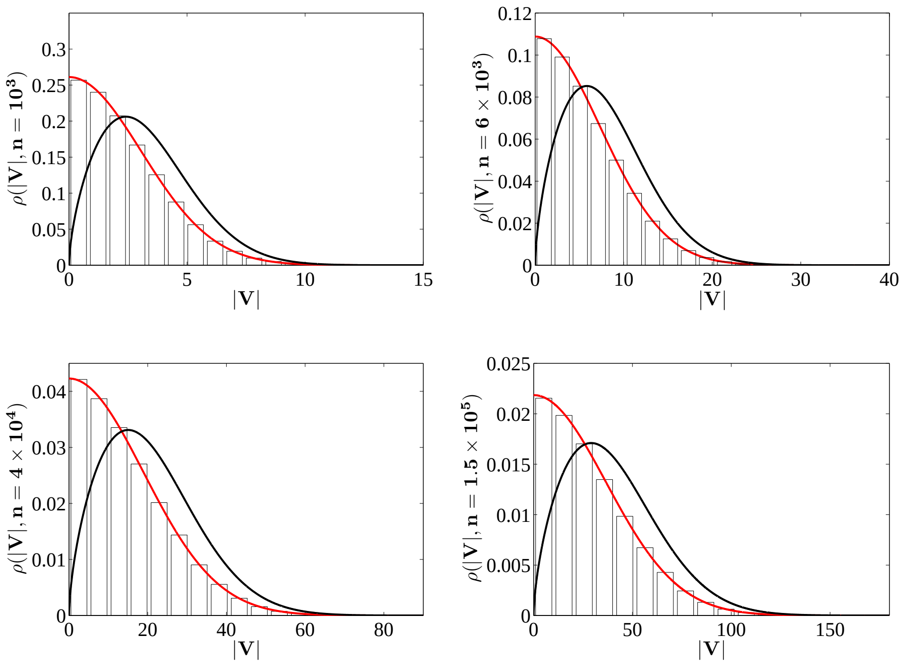


Figure 1: Histograms –bars– of the particle speeds for $n = 10^3$, $6 \times 10^3$, $4 \times 10^4$ and $1.5 \times 10^5$ collisions. Numerical results where obtained on the basis of an ensemble of $1.6 \times 10^5$ trajectories. The solution (14) obtained when the set of collision events leading to an artificial reversal of the particle velocity is neglected –red line– as well as the solution of the FPE given by (15) using the recalculated coefficients taking into account all collision events –black line– are also shown.

with the obvious physical interpretation that there exists a ratchet effect (a drift of the particle towards the direction opposite to the moving wall), even though the trajectories are bounded in space.

Therefore, to successfully treat the diffusion problem in velocity space, both independent variables of the FPE need to be redefined; the particle speed should be replaced by the algebraic value of the particle velocity, as well as, the number of collisions $n$ should be increased after *each* collision (with the "moving" and the fixed wall) so that both directions of particle velocities are taken into account. In this way, the reflection taking place on the fixed wall will be included in the treatment, which otherwise is neglected.

Let $V_{n,i}$, $(i = F, M)$ denote the particle velocity immediately *after* the n$th$ collision with the moving (M) and the fixed wall (F). Assuming that the fixed and the moving wall, comprising the FUM, are on the left and the right respectively and that the direction towards the right is taken to be positive, then the dynamics of the system are described by

$$V_{n,M} = -|V_{n-1,F} - 2u_n| \tag{17a}$$

$$V_{n,F} = -V_{n-1,M} \tag{17b}$$

$$u_n = \epsilon \cos(t_n + \eta_n) \tag{17c}$$

$$t_n = \frac{2}{V_{n-1,M}}. \tag{17d}$$

Let us further assume that the initial distribution of particle velocities is symmetrical in respect with $V = 0$ (equal flow of particles towards the fixed and the moving wall). Due to the reflection of particle velocities taking place after each collision with either of the walls, particles initially directed towards the moving wall, after the collision they will be redirected towards the fixed wall and vice-versa. Obviously, particles initially directed towards the fixed wall, on the $n$th collision will have experienced one collision less with the moving wall in comparison with a particle initially directed towards the moving wall. However, the change of the particle velocity after a single collision with the moving wall is negligible, compared to the cumulative effect induced after $n$ collisions ($n \gg 1$). For this reason, the initially symmetrical velocity distribution will remain symmetrical with respect to $V = 0$ for all number of collisions. Thus, for the specific choice of the initial distribution of particle velocities, the evolution of the ensemble of trajectories can be described by a single FPE with coefficients equal to the arithmetic mean of the corresponding coefficients, calculated separately for particles initially directed towards the fixed or the moving wall.

If the absolute value in (17a) is dropped and given that $V_{n-1,F} > 0$ (the positive direction is assumed towards the right) and further that $V_{n-1,F} > 2u_n$, $u_n > 0$ (the condition which allows us to drop the absolute value) then (17a)- (17d) can be easily uncoupled, yielding

$$\begin{aligned} V_{n,M} &= V_{n-2,M} - 2u_n && (18a) \\ V_{n,F} &= V_{n-2,F} + 2u_{n-1}. && (18b) \end{aligned}$$

Thus,

$$\begin{aligned} \delta V_{n,M} &= -2u_n && (19a) \\ \delta V_{n,F} &= 2u_{n-1}. && (19b) \end{aligned}$$

The ensemble average of $\delta V_{n,i}$, $(i = F, M)$ after one collision with the *same wall*, i.e. $\Delta n = 2$, is obtained after integration over the uniform distribution of phases upon collision. Therefore,

$$\begin{aligned} \langle\langle \delta V_{n,M} \rangle\rangle &= 0 && (20a) \\ \langle\langle \delta V_{n,F} \rangle\rangle &= 0, && (20b) \end{aligned}$$

where $\langle\langle \cdot \rangle\rangle$ denotes phase averaging. Formally, to obtain the ensemble average from (20a) and (20b) one should also integrate over the PDFs of particle velocities $\rho_i(V, n)$, $(i = F, M)$ corresponding to the particles experiencing a collision with the moving or the fixed wall, which in this case is trivial, since (20a) and (20b) are independent of the particle velocity. Therefore, the mean increment of the particle velocity during the course of two consecutive collisions is

$$\langle \delta V \rangle = \frac{\langle \delta V_{n,M} \rangle + \langle \delta V_{n,F} \rangle}{2} = 0. \tag{21}$$

Consequently,

$$B = 0. \tag{22}$$

Taking the same steps outlined above, the mean square change of particle velocity is,

$$\langle (\delta V)^2 \rangle = \frac{\langle (\delta V_{n,F})^2 \rangle + \langle (\delta V_{n,M})^2 \rangle}{2} = 2\epsilon^2. \tag{23}$$

Therefore,

$$D = \frac{1}{\Delta n} \langle (\delta V)^2 \rangle = \epsilon^2. \tag{24}$$

The solution of the FPE with coefficients given by (22) and (24) subject to natural boundary conditions ($\lim\limits_{V \to \pm\infty} \rho(V, n) = 0$) with the initial distribution of particle velocities given by

$$\begin{aligned} \rho(V, 0) &= \frac{1}{2\sqrt{2\pi}\sigma} \big( \exp\left[-(V - V_0)^2/(2\sigma^2)\right] \\ &\quad + \exp\left[-(V + V_0)^2/(2\sigma^2)\right] \big), && (25) \end{aligned}$$

is

$$\rho(V, n) = \frac{\exp\left[-\frac{(V-V_0)^2}{2(n\epsilon^2+\sigma^2)}\right] + \exp\left[-\frac{(V+V_0)^2}{2(n\epsilon^2+\sigma^2)}\right]}{2\sqrt{2\pi(n\epsilon^2 + \sigma^2)}}. \tag{26}$$

As can be seen in Fig. 2 the analytical result (26) is in agreement with the corresponding numerical results, obtained by the iteration of the "exact" set of equations (1a)-(1b) for $2.5 \times 10^5$ collisions.

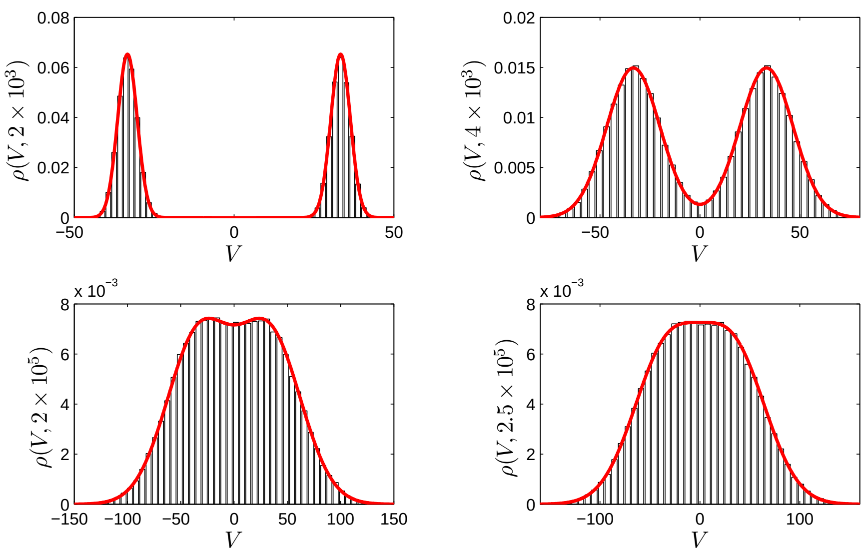


Figure 2: Histograms –bars– of the particle velocities for $n = 2000$, $4 \times 10^4$, $2 \times 10^5$ and $2.5 \times 10^5$ collisions. Numerical results where obtained on the basis of an ensemble of $4 \times 10^4$ trajectories. The velocities of the particles were initially distributed in accordance with $\rho(V,0) = \frac{1}{2\sqrt{2\pi}\sigma}\left(\exp\left[-(V-V_0)^2/(2\sigma^2)\right] + \exp\left[-(V+V_0)^2/(2\sigma^2)\right]\right)$ ($\sigma = \frac{2}{3}$, $V_0 = \frac{100}{3}$). The analytical solution given by 26 —red line— is also plotted for the sake of comparison.

In the above calculations the effect of the artificial reversal of particle velocities, which occurs on the condition that $V_{n-1,F} < 2u_n \leq 2\epsilon$, $u_n > 0$ [see (17a)], was neglected according to the previous discussion. In order to take these collisions into account we can divide the collision events into two sets, as described in [29], to find to the leading order of $(1/\sqrt{n})$

$$B' = \frac{1}{\Delta n}\langle \delta V_n \rangle = \frac{1}{\Delta n}\frac{\langle \delta V_{n,M}\rangle + \langle \delta V_{n,F}\rangle}{2} = 0 \equiv B \tag{27a}$$

$$D' = \frac{1}{\Delta n}\langle (\delta V_n)^2 \rangle = \frac{1}{\Delta n}\frac{\langle (\delta V_{n,M})^2\rangle + \langle (\delta V_{n,F})^2\rangle}{2} = \epsilon^2 \equiv D, \tag{27b}$$

where $B$, $D$ are calculated without taking into account artificial velocity reversal. As can be seen, if this specific choice of the independent variables of $\rho$ is made, the transport coefficients obtained by taking into account the class of rare collision events coincide —to the leading order of $(1/\sqrt{n})$— with the corresponding ones obtained by neglecting these collision events. Furthermore, the drift coefficient in both cases is found equal to zero, as expected, given that the particles are confined between the walls for all times. In addition, the vanishing of the drift coefficient $B$ reduces the FPE to a standard diffusion equation leading to the well known gaussian profile for the PDF [see (26)], in consistency with the prediction of the CLT and the numerical results.

## 4. Summary

The solution of the FPE for the evolution of the PDF of particle speeds using the coefficients found in the literature, which are calculated neglecting the collisions leading to the artificial reversal of the particle velocity, leads to a solution, (14), which is in agreement with the numerical results —see Fig. 1. However, the first moment of the time-evolving distribution given by (14) does not equal zero, in contradiction with the specific value of the drift coefficient used for the construction of the FPE, $B = 0$. Therefore, the differential equation for the PDF of the magnitude of particle velocities cannot be identified with the FPE, and the agreement of the obtained solution with the corresponding numerical results should be regarded as accidental.

In order to obtain a consistent solution of the FPE describing the evolution of the distribution of the particle velocities for the case of the stochastic FUM in the context of the SWA, one should use as an independent variable the algebraic value of the particle velocity rather than its magnitude and further, the time variable $n$ should be incremented after each collision, either with the "oscillating" or the fixed wall, as opposed to the convention found in the literature, where $n$ is incremented after a collision only with the "oscillating" wall. Then the solution obtained either by neglecting or by taking into account the collisions implicating the artificial reversal of the particle velocity is in agreement with the numerical results. Furthermore, the first moment of the solution of the FPE vanishes, in agreement with the drift coefficient used to obtain the evolution of the distribution of particle velocities, which also equals zero [ (22), (27a)].

## Appendix A.

As discussed in Section 3, if the FPE is defined in the semi-infinite interval $[0, \infty)$ for the description of the evolution of the *magnitude* of particle velocities, then the solution obtained is in agreement with the corresponding numerical results on the condition that the drift coefficient $B = 0$, which leads to a standard diffusion equation of the form $\frac{\partial f(x,t)}{\partial t} = D \frac{\partial^2 f(x,t)}{\partial x^2}$, giving rise to a half-gaussian solution. However, the drift coefficient $B$ defined by (3a) is identified with the first moment of the conditional probability $\rho(|V| + \Delta|V|, n + \Delta n \big| |V|, n)$, which due to Fermi acceleration is obviously non-zero. We have also shown in Sec. 3 that this apparent contradiction is resolved if the diffusion process is described instead in terms of the algebraic value of particle velocities, provided that the collisions occurring with the fixed wall are explicitly taken into account. However, in the previous discussion we took for granted that the coefficients of the FPE do exist.

However, the validity of the FPE describing a probability distribution function $p(x,t)$ requires the existence of the following limits [36]:

$$B \equiv \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-z|<\mu} dx(x-z)p(x,t+\Delta t|z,t)+O(\mu) \qquad \text{(A.1a)}$$

$$D \equiv \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|x-z|<\mu} dx(x-z)^2 p(x,t+\Delta t|z,t)+O(\mu), \qquad \text{(A.1b)}$$

with, $\mu > 0$. The difference between the definitions of the transport coefficients given by (3a), (3b) and (A.1a), (A.1b) is that in the latter, given in [34], the limit is omitted.

If we assume that the PDF of the magnitude of particle velocities $\rho(|V|, n+\Delta n|z,n)$ is the following spreading gaussian distribution

$$\rho(|V|, n+\Delta n|z,n) = \frac{1}{\sqrt{\pi\Delta n}\epsilon}\exp\left[-\frac{(|V|-z)^2}{4\Delta n\epsilon^2}\right]$$

so that $\lim_{\Delta n\to 0}\rho(|V|, n+\Delta n|z,t) = \delta(|V|-z)$ [36], then the use of (A.1a) and (A.1b) for the calculation of the transport coefficients of the FPE describing the evolution of $\rho(|V|,n)$ yields

$$B = \lim_{\Delta n\to 0}\frac{2\left(\exp\left[-\frac{z^2}{4\Delta n\epsilon^2}\right]-\exp\left[-\frac{\mu^2}{4\Delta n\epsilon^2}\right]\right)\epsilon}{\sqrt{\Delta n}\sqrt{\pi}} \qquad \text{(A.2a)}$$

$$\begin{aligned} D &= \lim_{\Delta n\to 0}\frac{2\epsilon}{\sqrt{n}\sqrt{\pi}}\Bigg\{-\left(-\frac{z^2}{4n\epsilon^2}\right)z-\exp\left(-\frac{\mu^2}{4n\epsilon^2}\right)\mu \\ &\quad +\sqrt{n}\sqrt{\pi}\epsilon\left[\mathrm{Erf}\left(\frac{z}{2\sqrt{n}\epsilon}\right)+\mathrm{Erf}\left(\frac{\mu}{2\sqrt{n}\epsilon}\right)\right]\Bigg\} = 2\epsilon^2. \end{aligned} \qquad \text{(A.2b)}$$

The limit $\Delta n \to 0$ of (A.2a) exists for $z > 0$ and equals zero. However, for $z = 0$ the limit defined by (A.2a) is not finite, and consequently the drift coefficient *cannot* be defined at the boundary $|V| = 0$.

On the other hand, if one chooses as an independent variable the algebraic value of the particle velocity and further assuming $\lim_{\Delta n\to 0}\rho(|V|, n+\Delta n|z,n) = \delta(|V|-z)$ then,

$$\rho(|V|,n) = \frac{\exp\left[-\frac{(v-z)^2}{2n\epsilon^2}\right]+\exp\left[-\frac{(v+z)^2}{2n\epsilon^2}\right]}{2\sqrt{2\pi}\sqrt{n\epsilon^2}}.$$

In this case (A.1a), (A.1b) yield for $z \geq 0$

$$
\begin{aligned}
B &= 0 \qquad\qquad (\text{A.3a}) \\
D &= \lim_{\Delta n \to 0} \frac{1}{4\Delta n^{3/2}\sqrt{2\pi}\epsilon} \exp\left[-\frac{(2z+\mu)^2}{2\Delta n\epsilon^2}\right] \\
&\times \Bigg\{ -2\Delta n \Big[ 2(-1+\exp\left[\frac{4z\mu}{\Delta n\epsilon^2}\right] z + (1+\exp\left[\frac{4z\mu}{\Delta n\epsilon^2}\right] + \\
&2\exp\left[\frac{2z(z+\mu)}{\Delta n\epsilon^2}\right]\mu\Big]\epsilon^2 \\
&-\exp\left[\frac{(2z+\mu)^2}{2\Delta n\epsilon^2}\right]\sqrt{\Delta n}\sqrt{2\pi} \\
&\times\Big[ -2\Delta n \mathrm{Erf}(\frac{\mu}{\sqrt{2}\sqrt{\Delta n}\epsilon})\epsilon^2 + (4z^2+\Delta n\epsilon^2)\mathrm{Erf}(\frac{2z-\mu}{\sqrt{2}\sqrt{\Delta n}\epsilon}) \\
&-(4z^2+\Delta n\epsilon^2)\mathrm{Erf}(\frac{2z+\mu}{\sqrt{2}\sqrt{\Delta n}\epsilon}\Big]\epsilon\Bigg\} \\
&= \epsilon^2. \qquad\qquad (\text{A.3b})
\end{aligned}
$$

Thus, if the algebraic value of the particle velocity is chosen as an independent variable then the conditions defined by (A.1a) and (A.1b) are met and the coefficients of the FPE exist and coincide with the values calculated in Sec. 3 [ (22) and (24)].

**Appendix B.**

Let us determine the statistical weight $w_2(n)$ of the set of collision events on which the particle velocity needs to be reversed "by hand", i.e. by applying the absolute value function on the right-hand side (RHS) of (1b). By mere inspection of (1b) we conclude that the particle will continue travelling towards the same direction it was prior to the collision with the "moving" wall, if and only if, $|V| < 2u$ and $u > 0$. Therefore,

$$
w_2(n) = \int_{-\pi/2}^{\pi/2} d\xi \int_0^{2u} dV \rho(V) \underbrace{\int_0^{\epsilon} dup(u) \left\{ \frac{1}{2}\delta\left[\xi - g(u)\right] + \frac{1}{2}\delta\left[\xi + g(u)\right] \right\}}_{I}, \qquad (\text{B.1})
$$

where $\rho(V)$ stands for the PDF of the algebraic value of particle velocities and $p(u)$ for the corresponding PDF of the velocity of the "moving" wall and $g(u) = \cos^{-1}(u/\epsilon)$ [see (1b)]. Given that, $u > 0$ the function $g(u)$ takes values in the interval $[0, \pi/2)$. The last term on the RHS of (B.1) compensates for the missing set of possible phases leading to $u \in (0, \epsilon]$, namely $\xi \in (3\pi/2, 2\pi] \equiv (-\pi/2, 0]$, due to the branch cut of the inverse cosine function in $g(u)$.

The PDF of particle velocities can be determined through the application of the CLT [29], leading to

$$\rho(V) = \frac{1}{2\epsilon\sqrt{\pi n}} \exp\left[-\frac{V^2}{4\epsilon^2 n}\right]. \tag{B.2}$$

The corresponding PDF of the wall velocity $p(u)$ can easily be derived, considering that the phase of oscillation of the wall upon collision follows a uniform distribution due to the random phase-shift performed before each collision, through the addition of a random number uniformally distributed in the interval $(0, 2\pi]$. Applying the fundamental transformation law of probabilities one obtains,

$$p(u) = \frac{1}{\pi\sqrt{\epsilon^2 - u^2}} \tag{B.3}$$

Using the property of the Dirac delta function $\delta\left(f(x)\right) = \sum_i \frac{\delta(f(x_i))}{|f'(x_i)|}$, where $x_i$ are the real roots of the equation $f(x) = 0$, we can rewrite the integral $I$ in (B.1) as,

$$\begin{aligned} I \quad &\equiv \quad \int_0^\epsilon du \frac{1}{\pi\sqrt{\epsilon^2 - u^2}} \left\{ \frac{1}{2}\delta\left[\xi - g(u)\right] + \frac{1}{2}\delta\left[\xi + g(u)\right] \right\} = \\ &= \int_0^\epsilon \frac{1}{2} \left[ \sum_i \frac{\delta\left(u - u_i\right)}{\left| -\frac{1}{\sqrt{\epsilon^2 - {u_i}^2}} \right|} + \sum_i \frac{\delta\left(u - {u_i}'\right)}{\left| -\frac{1}{\sqrt{\epsilon^2 - {u_i}'^2}} \right|} \right] \frac{1}{\pi\sqrt{\epsilon^2 - u^2}} du \end{aligned} \tag{B.4}$$

where, $u_i = \epsilon\cos(\xi)$ and ${u_i}' = \epsilon\cos(-\xi)$. The integration over $u$ and then $V$ of (B.1) gives,

$$w_2(n) = \int_{-\pi/2}^{\pi/2} \frac{1}{2\pi} \left\{ \frac{1}{2} \left[ \mathrm{Erf}\left(\frac{u_i}{\sqrt{n}\epsilon}\right) + \mathrm{Erf}\left(\frac{{u_i}'}{\sqrt{n}\epsilon}\right) \right] \right\} d\xi. \tag{B.5}$$

The expansion of (B.5) at $n \to \infty$ followed by integration over $\xi$ yields,

$$w_2(n) = \frac{2}{\sqrt{n}\pi^{3/2}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right). \tag{B.6}$$

It is noted that the result (B.6), to the leading order of $(1/n)$, coincides with the semi-analytical result of [29], i.e. $w_2(n) = \frac{1}{\pi}\mathrm{Erf}\left(1/\sqrt{n}\right)$.

**Appendix C.**

The substitution of (6b) and (13) into (2) together with reflective boundary conditions at $|V| = 0$ leads to the following initial-boundary value problem

(IBVP):

$$
\begin{aligned}
\frac{1}{\epsilon^2}\frac{\partial\tilde{\rho}(|V|,n)}{\partial n} &= \frac{\partial^2\tilde{\rho}(|V|,n)}{\partial|V|^2} - \frac{2}{\pi|V|}\frac{\partial\tilde{\rho}(|V|,n)}{\partial|V|} + \frac{2}{\pi|V|^2}\tilde{\rho}(|V|,n) \quad \text{(C.1a)}\\
\tilde{\rho}(|V|,0) &= \sqrt{\frac{2}{\pi}}\frac{1}{\sigma}\exp\left[-\frac{|V|^2}{2\sigma^2}\right] \quad \text{(C.1b)}\\
0 &= \left(\frac{2}{\pi|V|}\tilde{\rho}(|V|,n) - \frac{\partial\tilde{\rho}(|V|,n)}{\partial|V|}\right)\Bigg|_{|V|=0}. \quad \text{(C.1c)}
\end{aligned}
$$

Ali and Kala in [35] proposed a generalized Hankel transform, defined as

$$
\mathcal{H}_\nu\{f(x,z);s,a,c\} = \int_0^\infty z^a f(x,z)J_\nu(sz^c)dz, \tag{C.2}
$$

the inverse transform being

$$
\mathcal{H}_\nu^{-1}\left\{\hat{f}_\nu(x,s)\right\} = cz^{2c-a-1}\int_0^\infty s\hat{f}_\nu(x,s;a,c)J_\nu(sz)ds. \tag{C.3}
$$

Let us define $Q$ as the following differential operator:

$$
Q = \frac{\lambda}{z^{m-1}}\frac{\partial^2}{\partial z^2} + \frac{r}{z^m}\frac{\partial}{\partial z} - \frac{\lambda}{z^{m+1}}\left[c^2\nu^2 - a^2 + m\left(m-1+\frac{r}{\lambda}\right)\right], \tag{C.4}
$$

with

$$
\frac{r}{\lambda} = 2(a-m)+1. \tag{C.5}
$$

In [35] it is proven that the Hankel transform of $Q[f(x,z)]$ is,

$$
\begin{aligned}
\mathcal{H}_\nu\{Qf;s,a,c\} &= \underbrace{z^{a-m}\left(\lambda z\frac{\partial f}{\partial z} + rf\right)J_\nu(sz^c)}_{A}\Bigg|_0^\infty\\
&\quad -\lambda f\frac{\partial}{\partial z}\left(z^{a-m+1}J_\nu(sz^c)\right)\Bigg|_0^\infty\\
&\quad -\lambda c^2 s^2\mathcal{H}_\nu\{z^{2c-m-1}f;s,a,c\}, \quad \text{(C.6)}
\end{aligned}
$$

The differential operator on the RHS of (C.1a) acting on $\tilde{\rho}(|V|,n)$ can be identified with $Q$ letting $m=1$, $r=-2/\pi$,

$$
c^2\nu^2 - a^2 + m(m-1+r/\lambda) = -\frac{2}{\pi} \tag{C.7}
$$

and $z \equiv |V|$. From (C.5) we obtain, $a = 1/2 - 1/\pi$. The remaining parameters can be determined taking into account the boundary condition (C.1c) together with the behavior of $J_\nu(sz^c)$ for small $|V|$ and (C.7). Term $A$ of (C.6) can easily be seen that it can be decomposed into the product

$$-\underbrace{|V|^a J_\nu(s|V|^c)}_{B} \times \underbrace{\left[\frac{2}{\pi|V|}\tilde{\rho}(|V|,n) - \frac{\partial\tilde{\rho}\,[|V|,n)}{\partial|V|}\right]}_{C}\Bigg|_0^\infty . \tag{C.8}$$

For $|V| \to 0$ term $C$ of (C.8) becomes identical to the boundary condition (C.1c). Let us now set $c = 1$. Term $B$ for small $|V|$ can be expressed as,

$$B \equiv |V|^a J_\nu(s|V|^c) \overset{c=1}{=} |V|^{a+\nu}\left(\frac{2^{-\nu}s^\nu}{\Gamma(\nu+1)} + \mathcal{O}\left(|V|^2\right)\right). \tag{C.9}$$

Consequently, for $\nu = -a$,

$$\lim_{|V|\to 0} |V|^a J_\nu(s|V|^c) = \frac{2^a s^{-a}}{\Gamma(1-a)} \tag{C.10}$$

This particular choice of $\nu$ ensures that (C.8) for $|V| \to 0$ equals zero only on account of the boundary condition (C.1c), and thereby the solution obtained using (C.2) will satisfy (C.1c). Moreover, a substitution of the corresponding values of the parameters can easily be shown that satisfies (C.7).

Having the determined the parameters of (C.2) and assuming that $\tilde{\rho}(|V|,n)$ for $|V| \to \infty$ approaches 0 rapidly enough, so that the first two terms of (C.6) vanish for large $|V|$, we can now transform (C.1a) to obtain,

$$-s^2\hat{\rho}(s,n) = \frac{1}{\epsilon^2}\frac{\partial\hat{\rho}(s,n)}{\partial n} \tag{C.11}$$

Therefore, $\hat{\rho}(s,n) = A_s \exp[-s^2\epsilon^2 n]$, with

$$A_s = \int_0^\infty |V|^{1/2-1/\pi}\tilde{\rho}(|V|,0) J_{\frac{1}{\pi}-\frac{1}{2}}(s|V|)d|V| \tag{C.12}$$

The solution of (C.1) can be obtained by inverting the transform. Thus,

$$\begin{aligned}
\tilde{\rho}(|V|,n) &= \int_0^\infty |V|^{1/2+1/\pi} A_s s\hat{\rho}(s,n) J_{\frac{1}{\pi}-\frac{1}{2}}(s|V|)ds \\
&= |V|^{1/2+1/\pi}\int_0^\infty ds \exp\left[(\epsilon s)^2 n\right] \\
&\quad \times \left\{\int_0^\infty dl\ l^{1/2-1/\pi}\sqrt{\frac{2}{\pi}}\frac{1}{\sigma}\exp\left[-\frac{l^2}{2\sigma^2}\right] J_{\frac{1}{\pi}-\frac{1}{2}}(sl) J_{\frac{1}{\pi}-\frac{1}{2}}(s|V|)\right\} \\
&= \frac{1}{4^{1/\pi}\epsilon^{2/\pi}\Gamma\left(\frac{1}{2}+\frac{1}{\pi}\right)}|V|^{2/\pi} e^{-\frac{|V|^2}{4n\epsilon^2}} \frac{{}_1F_1\left(\frac{1}{2};\frac{1}{2}+\frac{1}{\pi};\frac{|V|^2\sigma^2}{8n^2\epsilon^4+4n\sigma^2\epsilon^2}\right)}{n^{1/\pi}\sqrt{n\epsilon^2+\frac{\sigma^2}{2}}},
\end{aligned} \tag{C.13}$$

where ${}_1F_1\,(a;b;z)$ is the Kummer confluent hypergeometric function.

**References**


[1] S. Ulam, Proceedings of the Fourth Berkeley Symposium on Mathematics, Statistics, and Probability, California University Press, Berkeley vol 3, 1961, p. 315.

[2] E. Fermi, Phys. Rev. 75 (1949) 1169.

[3] G.A. Luna-Acosta, Phys. Rev. A 42 (1990) 7155.

[4] E.D. Leonel, R.E. de Carvalho, Phys. Lett. A 364 (2007) 475.

[5] E.D. Leonel, P.V.E. McClintock, Phys. Rev. E 73 (2006) 066223.

[6] E.D. Leonel, J. Phys. A: Math. Theor. 40 (2007) F1077.

[7] D.F.M. Oliveira, E.D. Leonel, Braz. J. Phys. 38 (2008) 62.

[8] A.K. Karlis, P. K. Papachristou, F.K. Diakonos, V. Constantoudis, P. Schmelcher, Phys. Rev. E 76 (2007) 016214.

[9] E.D. Leonel, E.P. Marinho, Physica A 388 (2009) 4927.

[10] J.V. José, R. Cordery 1986 Phys. Rev. Lett. 56 (1986) 290.

[11] W.M. Visscher, Phys. Rev. A 36 (1987) 5031.

[12] A.J. Makowski, S.T. Dembiński , Phys. Lett. A 154 (1991) 217.

[13] M. Razavy, Phys. Rev. A 44 (1991) 2384.

[14] S.R. Jain, Phys. Rev. Lett. 70 (1993) 3553.

[15] M.L. Glasser, J. Mateo, J. Negro, L.M. Nieto, Chaos, Solitons and Fractals 41 (2009) 2067.

[16] L.D. Pustylnikov, Trans. Moscow Math. Soc. 2 (1978) 1.

[17] D.G. Ladeira, J.K. da Silva, J. Phys. A: Math. Theor. 40 (2007) 11467.

[18] P.J. Holmes, Journal of Sound and Vibration 84 (1982) 173.

[19] R.M. Everson, Physica D 19 (1986) 355.

[20] S. Celaschi, R.L. Zimmerman, Phys. Lett. A 120 (1987) 447.

[21] Z.J. Kowalik, M. Franaszek, P. Pierański, Phys. Rev. A 37 (1988) 4016.

[22] A. Mehta, J.M. Luck, Phys. Rev. Lett. 65 (1990) 393.

[23] E.D. Leonel, A.L.P. Livorati, Physica A 387 (2008) 1155.

[24] A.L.P. Livorati, D.G. Ladeira, E.D. Leonel, Phys. Rev. E 78 (2008) 056205.

[25] E.D. Leonel, P.V.E. McClintock, J. Phys. A: Math. Gen. 38 (2005) 823.

[26] D.G. Ladeira, E.D. Leonel, Chaos 17 (2007) 013119.

[27] A.K. Karlis, P.K. Papachristou, F.K. Diakonos, V. Constantoudis, P. Schmelcher, Phys. Rev. Lett. 97 (2006) 194102.

[28] A.Yu. Loskutov, A.B. Ryabov, L.G. Akinshin, J. Exp. Theor. Phys. 89 (1999) 966; J. Phys. A: Math. Gen. 33 (2000) 7973.

[29] A.K. Karlis, F.K. Diakonos, V. Constantoudis, P. Schmelcher, Phys. Rev. E 78 (2008) 046213.

[30] E.D. Leonel, P.V.E. McClintock, J.K. da Silva, Phys. Rev. Lett. 93 (2004) 014101.

[31] J.M. Hammersley, Proceedings of the Fourth Berkley Symposium on Mathematics, Statistics, and Probability, California U.P., Berkeley, Vol. 3 (1961) 79.

[32] M.A. Lieberman, A.J. Lichtenberg, Phys. Rev. A 5 (1972) 1852.

[33] A.J. Lichtenberg, M.A. Lieberman, R.H, Cohen, Physica D 1 (1980) 291.

[34] A.J. Lichtenberg, M.A. Lieberman, Regular and Chaotic Dynamics, Springer Verlag, New York, 1992.

[35] I. Ali, S. Kalla, J. Austral. Math. Soc. Ser. B 41 (1999) 105.

[36] G.W. Gardiner, Handbook of stochastic Methods, for Physics, Chemistry and the Natural Sciences, Springer Verlag, New York, 1985.