

KERNEL FORMULA APPROACH TO THE UNIVERSAL WHITHAM HIERARCHY

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Abstract

We derive the dispersionless Hirota equations of the universal Whitham hierarchy from the kernel formula approach proposed by Carroll and Kodama. Besides, we also verify the associativity equations in this hierarchy from the dispersionless Hirota equations and give a realization of the associative algebra with structure constants expressed in terms of the residue formulas.

Keywords: kernel formula, Lax formulation, dispersionless Hirota equations, Whitham hierarchy, WDVV equations, residue formula.

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1 Introduction

Dispersionless integrable hierarchies have been an active subject of research in recent years. The dispersionless KP (dKP) hierarchy[12, 13, 23, 25] and dispersionless Toda (dToda) hierarchy[22, 24] are now the prototype systems in several branches of physics and mathematics, such as topological field theories, matrix models, string theory, Laplacian growth problem and conformal maps (see, e.g., [1, 5, 4, 2, 15, 16, 21, 29, 31, 32], and references therein). The solution of the dispersionless integrable hierarchies can be characterized by a τ -function whose logarithm, $F = \log \tau$, called free energy satisfies a set of dispersionless Hirota (dHirota) equations and gives the solutions to the associativity equations or Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations introduced in the context of topological field theories[10, 8, 30].

It was shown[3] that the associativity equations are encoded in the dHirota equations for the dKP and the dToda hierarchies. There are several works devoted to derive the corresponding dHirota equations. For the dKP hierarchy, Takasaki and Takebe[25] derived them by taking the dispersionless limit of the differential Fay identity. Later, Carroll and Kodama[6] studied it from the approach of kernel formulas. As for the dToda hierarchy, Wiegmann and Zabrodin et al.[29] investigated the dHirota equations in the language of conformal mappings. Teo[28] derived the dHirota equations for both the dKP and dToda hierarchies in the content of complex analysis using the notion of Faber polynomials and Grunsky coefficients.

In this work we shall focus on the universal Whitham hierarchy which was introduced by Krichever[16] as a universal framework for both dispersionless integrable systems and Whitham modulation equations. In particular, the genus-zero case is the main target of recent researches (see e.g., [19, 20, 11, 17, 18, 26, 27]) since the universal Whitham hierarchy is the master equation of many other dispersionless integrable systems. Based on the work of Carroll and Kodama[6] (see also [7]) we like to generalize the kernel formula approach for the dKP hierarchy to derive the dHirota equations of the universal Whitham hierarchy. Moreover, motivated by the work of Boyarsky et al.[3], we intend to provide a direct verification of associative equations in the universal Whitham hierarchy from the dHirota equations point of view.

This paper is organized as follows. In the next section, we briefly recall the Lax formulation for the universal Whitham hierarchy of zero genus case. In section 3, the kernel formula approach is employed to derive the dHirota equations of the universal Whitham hierarchy. In Section 4, following the work of [3], we will prove the associativity equation of the universal Whitham hierarchy from these dHirota equations and give a realization of the associative algebra. The situations of the finite-dimensional reduction are also briefly discussed. Section 5 is devoted to the concluding remarks.

2 Lax Formulation of the Universal Whitham Hierarchy

The universal Whitham hierarchy of genus zero with $N + 1$ marked points at $q_\alpha (\alpha = 1, \dots, N)$ and ∞ on Riemann sphere is defined by the Lax functions[16]

$$\begin{aligned}\lambda_0(p) &= p + \sum_{j=2} u_j p^{-j+1} \\ \lambda_\alpha(p) &= \frac{r_\alpha}{p - q_\alpha} + \sum_{j=1} u_{\alpha j} (p - q_\alpha)^{j-1}, \quad \alpha = 1, \dots, N\end{aligned}$$

where u_j and $u_{\alpha j}$ are functions of the time variables $t_{0n} (n = 1, 2, \dots)$ and $t_{\alpha n} (\alpha = 1, \dots, N; n = 0, 1, \dots)$. One can regard λ_0 as a map defined in domain Ω_0 containing $p = \infty$, while λ_α in Ω_α containing $p = q_\alpha$. λ_0 and λ_α satisfy the Lax equations

$$\partial_{\alpha n} \lambda_\beta(p) = \{B_{\alpha n}(p), \lambda_\beta(p)\}, \quad \partial_{\alpha n} = \partial / \partial t_{\alpha n} \quad (1)$$

where the Poisson bracket is defined by $\{f, g\} = \partial_p f \partial_{01} g - \partial_{01} f \partial_p g$ and

$$\begin{aligned}B_{0n}(p) &= (\lambda_0^n)_{(0, \geq 0)} \\ B_{\alpha n}(p) &= (\lambda_\alpha^n)_{(\alpha, < 0)}, \quad n = 1, 2, \dots \\ B_{\alpha 0}(p) &= -\log(p - q_\alpha).\end{aligned}$$

Following the notation used in [19, 26] we shall denote $()_{(0, \geq 0)}$ and $()_{(\alpha, < 0)}$ as the projections to nonnegative powers in p and negative powers in $(p - q_\alpha)$, respectively. Therefore

$$\begin{aligned}\lambda_0^n(p) &= B_{0n}(p) + O(p^{-1}), \quad p \rightarrow \infty, \\ \lambda_\alpha^n(p) &= B_{\alpha n}(p) + O(1), \quad p \rightarrow q_\alpha.\end{aligned}$$

In particular, $B_{01}(p) = p$ and $B_{\alpha 1}(p) = r_\alpha / (p - q_\alpha)$. The Lax equations can be written as the Zakharov-Shabat equations

$$\partial_{\beta n} B_{\alpha m}(p) - \partial_{\alpha m} B_{\beta n}(p) + \{B_{\alpha m}(p), B_{\beta n}(p)\} = 0$$

which is equivalent to $\omega \wedge \omega = 0$ with

$$\omega = \sum_{n=1}^{\infty} dB_{0n} \wedge dt_{0n} + \sum_{\alpha=1}^N \sum_{n=0}^{\infty} dB_{\alpha n} \wedge dt_{\alpha n}.$$

Introducing Darboux coordinate (λ_0, M_0) in Ω_0 and $(\lambda_\alpha, M_\alpha)$ in Ω_α , we have

$$\begin{aligned}\omega &= d\lambda_0 \wedge dM_0 && \text{in } \Omega_0 \\ &= d\lambda_\alpha \wedge dM_\alpha && \text{in } \Omega_\alpha\end{aligned}$$

where the Orlov-Schulman functions M_0 and M_α are defined by[19, 26]

$$\begin{aligned} M_0 &= \sum_{n=1}^{\infty} n t_{0n} \lambda_0^{n-1} + \frac{t_{00}}{\lambda_0} + \sum_{n=1}^{\infty} \lambda_0^{-n-1} v_{0n}, & t_{00} &= - \sum_{\alpha=1}^N t_{\alpha 0} \\ M_\alpha &= \sum_{n=1}^{\infty} n t_{\alpha n} \lambda_\alpha^{n-1} + \frac{t_{\alpha 0}}{\lambda_\alpha} + \sum_{n=1}^{\infty} \lambda_\alpha^{-n-1} v_{\alpha n}, \end{aligned}$$

satisfies

$$\partial_{\alpha n} M_\beta = \{B_{\alpha n}, M_\beta\}$$

and the canonical relation $\{\lambda_0, M_0\} = \{\lambda_\alpha, M_\alpha\} = 1$.

From the expressions of the symplectic two-form ω we have, in Ω_0 ,

$$d \left(M_0 d\lambda_0 + \sum_{n=1}^{\infty} B_{0n} dt_{0n} + \sum_{\alpha=1}^N \sum_{n=0}^{\infty} B_{\alpha n} dt_{\alpha n} \right) = 0.$$

Therefore, there exists a S -function, S_0 , such that

$$dS_0 = M_0 d\lambda_0 + \sum_{n=1}^{\infty} B_{0n} dt_{0n} + \sum_{\alpha=1}^N \sum_{n=0}^{\infty} B_{\alpha n} dt_{\alpha n}.$$

On the other hand, in Ω_α , we have

$$dS_\alpha = M_\alpha d\lambda_\alpha + \sum_{n=1}^{\infty} B_{0n} dt_{0n} + \sum_{\alpha=1}^N \sum_{n=0}^{\infty} B_{\alpha n} dt_{\alpha n}.$$

It turns out that the Orlov-Schulman functions M_0 and M_α have the expressions

$$M_0 = \frac{\partial S_0(t)}{\partial \lambda_0}, \quad M_\alpha = \frac{\partial S_\alpha(t)}{\partial \lambda_\alpha}$$

then the S -functions admit expansions of the form

$$\begin{aligned} S_0 &= \sum_{n=1}^{\infty} t_{0n} \lambda_0^n + t_{00} \log \lambda_0 - \sum_{n=1}^{\infty} \frac{\lambda_0^{-n}}{n} v_{0n}, \\ S_\alpha &= \sum_{n=1}^{\infty} t_{\alpha n} \lambda_\alpha^n + t_{\alpha 0} \log \lambda_\alpha + \phi_\alpha - \sum_{n=1}^{\infty} \frac{\lambda_\alpha^{-n}}{n} v_{\alpha n}. \end{aligned}$$

The Hamilton-Jacobi equations are defined by

$$\partial_{\alpha n} S_\beta = B_{\alpha n}(p_\beta) = B_{\alpha n}(\partial_{01} S_\beta)$$

hence

$$\begin{aligned}
B_{0n} &= \partial_{0n} S_0(t) = \lambda_0^n - \sum_{m=1}^n \frac{\lambda_0^{-m}}{m} \partial_{0n} v_{0m} \\
&= \partial_{0n} S_\alpha(t) = \partial_{0n} \phi_\alpha - \sum_{m=1}^n \frac{\lambda_\alpha^{-m}}{m} \partial_{0n} v_{\alpha m} \\
B_{\alpha n} &= \partial_{\alpha n} S_0 = - \sum_{m=1}^n \frac{\lambda_0^{-m}}{m} \partial_{\alpha n} v_{0m} \\
&= \partial_{\alpha n} S_\beta = \delta_{\alpha\beta} \lambda_\beta^n + \partial_{\alpha n} \phi_\beta - \sum_{m=1}^n \frac{\lambda_\beta^{-m}}{m} \partial_{\alpha n} v_{\beta m} \\
B_{\alpha 0} &= \partial_{\alpha 0} S_0 = - \log \lambda_0 - \sum_{m=1}^n \frac{\lambda_0^{-m}}{m} \partial_{\alpha 0} v_{0m} \\
&= \partial_{\alpha 0} S_\beta = \delta_{\alpha\beta} \log \lambda_\beta + \partial_{\alpha 0} \phi_\beta - \sum_{m=1}^n \frac{\lambda_\beta^{-m}}{m} \partial_{\alpha 0} v_{\beta m}.
\end{aligned}$$

The above equations provide some relations between the coefficients $(u_j, u_{j\alpha}, r_\alpha, q_\alpha)$ and $(\phi_\alpha, v_{\alpha n})$.

For instance, from $B_{01} = p$ we have

$$\begin{aligned}
p &= \partial_{01} S_0(\lambda) = \lambda_0 - \sum_{m=1}^n \frac{\lambda_0^{-m}}{m} \partial_{01} v_{0m}, \quad \text{in } \Omega_0 \\
p &= \partial_{01} S_\alpha(\lambda) = \partial_{01} \phi_\alpha - \sum_{m=1}^n \frac{\lambda_\alpha^{-m}}{m} \partial_{01} v_{\alpha m}, \quad \text{in } \Omega_\alpha
\end{aligned}$$

which implies $u_2 = \partial_{01} v_{01}$, $q_\alpha = \partial_{01} \phi_\alpha$ and $r_\alpha = -\partial_{01} v_{\alpha 1}$. On the other hand, $B_{\alpha 0}(p_\alpha) = \partial_{\alpha 0} S_\alpha = -\log(p_\alpha - q_\alpha)$ which yields $r_\alpha = e^{-\partial_{\alpha 0} \phi_\alpha}$. In fact, there exist a F function [16, 19, 26] such that

$$v_{0n} = F_{0n}, \quad v_{\alpha n} = F_{\alpha n}, \quad \phi_\alpha = -F_{\alpha 0} + \sum_{\beta=1}^{\alpha} t_{\beta 0} \log(-1), \quad (2)$$

where $F_{\alpha n} = \partial_{\alpha n} F$. Then we see that $u_2 = \partial_{01}^2 F$, the dKP potential. And from $r_\alpha = e^{-\partial_{\alpha 0} \phi_\alpha}$, $\partial_{01} \partial_{\alpha 1} F - e^{\partial_{\alpha 0}^2 F} = 0$, which is just the dToda field equation. We remark here that the Lax equation for the $t_{\alpha 0}$ flow is $\partial_{\alpha 0} \lambda = \{B_{\alpha 0}, \lambda\} = \frac{-1}{p} \frac{\partial \lambda}{\partial t_{\alpha 0}}$ if setting q_α to be zero. The Poisson bracket may be written as, not so strictly, $\{f, g\} = -(p \frac{\partial f}{\partial p} \frac{\partial g}{\partial t_{\alpha 0}} - p \frac{\partial f}{\partial t_{\alpha 0}} \frac{\partial g}{\partial p})$. The time variable $-t_{\alpha 0}$ seems to play the role of the parameter s in the usual Poisson bracket of the dToda hierarchy [22, 24, 25]. Therefore the universal Whitham hierarchy can be viewed as an assembly of one dKP hierarchy and N copies of dToda-like hierarchies.

3 Carroll-Kodama Approach

3.1 Kernel Formulas

In contrast to the dKP and dToda hierarchy, the Whitham hierarchy contains $N + 1$ Lax functions λ_0 and λ_α that are defined in domains Ω_0 and $\Omega_\alpha (\alpha = 1, \dots, N)$, respectively. To derive the kernel formulas of the universal Whitham hierarchy we shall generalize the method of kernel formula to the domains $\Omega_0, \Omega_\alpha, \Omega_0 \cap \Omega_\alpha$ and $\Omega_\alpha \cap \Omega_\beta (\alpha \neq \beta)$.

In Ω_0 , from the expression of $B_{01} = p = \partial_{01} S_0$, we have

$$\lambda_0 - p - \sum_{m=1}^{\infty} \frac{\lambda_0^{-m}}{m} F_{01,0m} = 0.$$

Multiplying above by $\lambda_0^{n-1} \partial_p \lambda_0$ and taking the projection $(\cdot)_{(0, \geq 0)}$ then we obtain

$$\partial_p Q_{0,n+1}(\lambda) - p \partial_p Q_{0,n}(\lambda) - \sum_{m=1}^{n-1} \frac{F_{01,0m}}{m} \partial_p Q_{0,n-m}(\lambda) = 0, \quad n \geq 1 \quad (3)$$

where $Q_{0,n}(\lambda) = B_{0n}(\lambda)/n$. Taking the summation $\sum_{n=0}^{\infty} \mu^{-n}$ and noting that $\partial_p Q_{0,1} = 1$ we have

$$\left(\mu - p_0(\lambda) - \sum_{m=1}^{\infty} \frac{F_{01,0m}}{m} \mu^{-m} \right) \sum_{n=1}^{\infty} \partial_p Q_{0,n}(\lambda) \mu^{-n} = 1$$

or

$$\frac{1}{p_0(\mu) - p_0(\lambda)} = \sum_{n=1}^{\infty} \partial_p Q_{0,n}(\lambda) \mu^{-n}. \quad (4)$$

Integrating above with respect to $p_0(\lambda)$ by fixing the integration constant at $\mu \rightarrow \infty$, we find

$$\log \frac{\mu}{p_0(\mu) - p_0(\lambda)} = \sum_{n=1}^{\infty} \frac{B_{0,n}(\lambda)}{n} \mu^{-n} \quad (5)$$

Rewriting the expressions of $p_0 = \partial_{01} S_0$ and $B_{0n} = \partial_{0n} S_0$ in kernel formula (5) as

$$\begin{aligned} p_0(\lambda) &= \lambda - D_0(\lambda) F_{01}, \\ B_{0n}(\lambda) &= \lambda^n - D_0(\lambda) F_{0n} \end{aligned}$$

where we have used the symbols $D_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{\alpha n}$. Then the kernel formula (5) becomes

$$e^{D_0(\mu) D_0(\lambda) F} = 1 - \frac{\partial_{01} (D_0(\mu) - D_0(\lambda)) F}{\mu - \lambda}. \quad (6)$$

In Ω_α , from $B_{\alpha n} = \partial_{\alpha n} S_\alpha$, we have

$$B_{\alpha 1} = \frac{r_\alpha}{p - q_\alpha} = \lambda_\alpha + \partial_{\alpha 1} \phi_\alpha - \sum_{m=1}^{\infty} \frac{\lambda_\alpha^{-m}}{m} F_{\alpha 1, \alpha m}$$

Multiplying above by $\lambda_\alpha^{n-1} \partial_p \lambda_\alpha$ and taking the projection $(\cdot)_{(0, \leq -2)}$ then we obtain

$$\partial_p Q_{\alpha, n+1}(\lambda) - \left(\frac{r_\alpha}{p - q_\alpha} - \partial_{\alpha 1} \phi_\alpha \right) \partial_p Q_{\alpha, n}(\lambda) - \sum_{m=1}^{n-1} \frac{F_{\alpha 1, \alpha m}}{m} \partial_p Q_{\alpha, n-m}(\lambda) = 0, \quad n \geq 1 \quad (7)$$

where $Q_{\alpha, n}(\lambda) = B_{\alpha n}(\lambda)/n$. Taking the summation $\sum_{n=0}^{\infty} \mu^{-n}$ and noting that $\partial_p Q_{\alpha, 1} = -\frac{r_\alpha}{(p - q_\alpha)^2}$ we have

$$\left(\mu - \left(\frac{r_\alpha}{p - q_\alpha} - \partial_{\alpha 1} \phi_\alpha \right) - \sum_{m=1}^{\infty} \frac{F_{\alpha 1, \alpha m}}{m} \mu^{-m} \right) \sum_{n=1}^{\infty} \partial_p Q_{\alpha, n}(\lambda) \mu^{-n} = -\frac{r_\alpha}{(p - q_\alpha)^2}$$

or

$$\frac{p_\alpha(\mu) - q_\alpha}{(p_\alpha(\mu) - p_\alpha(\lambda))(p_\alpha(\lambda) - q_\alpha)} = \sum_{n=1}^{\infty} \partial_p Q_{\alpha, n}(\lambda) \mu^{-n}. \quad (8)$$

Integrating above with respect to $p_\alpha(\lambda)$ by fixing the integration constant at $\mu \rightarrow \infty$, we find

$$\log \frac{p_\alpha(\lambda) - q_\alpha}{p_\alpha(\lambda) - p_\alpha(\mu)} = \sum_{n=1}^{\infty} \frac{B_{\alpha, n}(\lambda)}{n} \mu^{-n} \quad (9)$$

Rewriting the expressions of $p_\alpha - q_\alpha$, p_α and $B_{\alpha n}$ in kernel formula (9) as

$$\begin{aligned} p_\alpha - q_\alpha &= e^{-\partial_{\alpha 0} S_\alpha(\lambda)} = -\frac{1}{\lambda} e^{\partial_{\alpha 0}(\partial_{\alpha 0} + D_\alpha(\lambda))F} \\ p_\alpha(\lambda) &= \partial_{01} S_\alpha(\lambda) = \partial_{01} \phi_\alpha - D_\alpha(\lambda) F_{01} \\ B_{\alpha n} &= \partial_{\alpha n} S_\alpha(\lambda) = \lambda^n + \partial_{\alpha n} \phi_\alpha - D_\alpha(\lambda) F_{\alpha n} \end{aligned}$$

then the kernel formula (9) becomes

$$e^{(\partial_{\alpha 0} + D_\alpha(\mu))(\partial_{\alpha 0} + D_\alpha(\lambda))F} = -\frac{\mu \lambda}{\mu - \lambda} \partial_{01} (D_\alpha(\mu) - D_\alpha(\lambda)) F. \quad (10)$$

In $\Omega_0 \cap \Omega_\alpha$, let us rewrite (5) as

$$\frac{\mu}{\partial_{01} S_0(\mu) - \partial_{01} S_0(\lambda)} = e^{\sum_{n=1}^{\infty} \frac{\partial_{0n} S_0(\lambda)}{n} \mu^{-n}}.$$

By replacing $S_0(\lambda)$ by $S_\alpha(\lambda)$ and using the relations

$$\begin{aligned} \partial_{01} S_\alpha(\lambda) &= \partial_{01} \phi_\alpha - D_\alpha(\lambda) F_{01} \\ \partial_{0n} S_\alpha(\lambda) &= \partial_{0n} \phi_\alpha - D_\alpha(\lambda) F_{0n} \end{aligned}$$

then we have

$$\mu e^{D_0(\mu)(\partial_{\alpha 0} + D_\alpha(\lambda))F} = \mu - \partial_{01} (D_0(\mu) - \partial_{\alpha 0} - D_\alpha(\lambda)) F. \quad (11)$$

On the other hand, one can derive (11) from the kernel formula (9) by expressing it as

$$\frac{e^{-\partial_{\alpha 0} S_\alpha(\lambda)}}{\partial_{01} S_\alpha(\lambda) - \partial_{01} S_\alpha(\mu)} = e^{\sum_{n=1}^{\infty} \frac{\partial_{\alpha n} S_\alpha(\lambda)}{n} \mu^{-n}}.$$

After replacing $S_\alpha(\lambda)$ by $S_0(\lambda)$, then the above equation becomes

$$\lambda e^{D_0(\lambda)(\partial_{\alpha 0} + D_\alpha(\mu))F} = \lambda - \partial_{01}(D_0(\lambda) - \partial_{\alpha 0} - D_\alpha(\mu))F$$

which, after making an exchange $\lambda \leftrightarrow \mu$, coincides with (11).

In $\Omega_\alpha \cap \Omega_\beta$ with $\alpha \neq \beta$, we rewrite (9) as

$$\frac{e^{-\partial_{\alpha 0} S_\alpha(\lambda)}}{\partial_{01} S_\alpha(\lambda) - \partial_{01} S_\alpha(\mu)} = e^{\sum_{n=1} \frac{\partial_{\alpha n} S_\alpha(\lambda)}{n} \mu^{-n}}.$$

By replacing $S_\alpha(\lambda)$ by $S_\beta(\lambda)$, then above equation becomes

$$\frac{e^{-\partial_{\alpha 0} S_\beta(\lambda)}}{\partial_{01} S_\beta(\lambda) - \partial_{01} S_\alpha(\mu)} = e^{\sum_{n=1} \frac{\partial_{\alpha n} S_\beta(\lambda)}{n} \mu^{-n}}. \quad (12)$$

Since

$$\begin{aligned} e^{-\partial_{\alpha 0} S_\beta(\lambda)} &= e^{-\partial_{\alpha 0} \phi_\beta + \partial_{\alpha 0} D_\beta(\lambda)F} = -\epsilon_{\alpha\beta} e^{\partial_{\alpha 0}(\partial_{\beta 0} + D_\beta(\lambda))F}, \\ \partial_{01} S_\beta(\lambda) - \partial_{01} S_\alpha(\mu) &= \partial_{01}(D_\alpha(\mu) + \partial_{\alpha 0} - \partial_{\beta 0} - D_\beta(\lambda))F, \\ \partial_{\alpha n} S_\beta(\lambda) &= \partial_{\alpha n} \phi_\beta - \sum_{m=1} \frac{F_{\alpha n, \beta m}}{m} \lambda^{-m}, \end{aligned}$$

where ϕ_β is defined by (2) and

$$\epsilon_{\alpha\beta} = \begin{cases} +1, & \alpha \leq \beta \\ -1, & \alpha > \beta \end{cases}$$

Thus eq.(12) can be expressed as

$$\epsilon_{\alpha\beta} e^{(\partial_{\alpha 0} + D_\alpha(\mu))(\partial_{\beta 0} + D_\beta(\lambda))F} = -\partial_{01}(D_\alpha(\mu) + \partial_{\alpha 0} - \partial_{\beta 0} - D_\beta(\lambda))F \quad (13)$$

Eqs. (6), (10), (11) and (13) constitute the dispersionless Hirota equations of the genus-zero universal Whitham hierarchy. They describe algebraic relations between second derivatives of the free energy F . Remarkably, our results coincide with that by Takasaki and Takebe [26] who derived these equations from the Hamilton-Jacobian equations in the context of the Faber polynomials.

3.2 Dispersionless Hirota Equations

Using the parametrization[27]: $\hat{t}_{0n} = nt_{0n}$, $\hat{t}_{\alpha 0} = t_{\alpha 0}$ and $\hat{t}_{\alpha n} = nt_{\alpha n}$. the corresponding time derivative is written as $\hat{\partial}_{\alpha n} = \frac{\partial}{\partial \hat{t}_{\alpha n}}$. Then the four dispersionless Hirota equations can be

rewritten as[26] :

$$\begin{aligned}
\hat{D}_0(\mu)\hat{D}_0(\lambda)F &= \log \frac{p_0(\mu) - p_0(\lambda)}{\mu - \lambda}, \\
\hat{D}_\alpha(\mu)\hat{D}_\alpha(\lambda)F &= \log \frac{\mu\lambda(p_\alpha(\mu) - p_\alpha(\lambda))}{\mu - \lambda}, \\
\hat{D}_0(\mu)\hat{D}_\alpha(\lambda)F &= \log \frac{p_0(\mu) - p_\alpha(\lambda)}{\mu}, \\
\hat{D}_\alpha(\mu)\hat{D}_\beta(\lambda)F &= \log \frac{p_\alpha(\mu) - p_\beta(\lambda)}{\varepsilon_{\alpha\beta}},
\end{aligned} \tag{14}$$

where we have use the new symbols : $\hat{D}_0(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{0n} = \sum_{n=1}^{\infty} z^{-n} \hat{\partial}_{0n}$, and $\hat{D}_\alpha(z) = \partial_{\alpha 0} + \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{\alpha n} = \sum_{n=0}^{\infty} z^{-n} \hat{\partial}_{\alpha n}$. This set of equations will be our main interests in the following discussion. Note that they have the same form if applying the differentiation operator $\hat{D}_\delta(\nu)$ on them. The result can be summarized as a single equation:

$$\hat{D}_\alpha(\mu)\hat{D}_\beta(\lambda)\hat{D}_\delta(\nu)F = -\frac{(\hat{D}_\alpha(\mu) - \hat{D}_\beta(\lambda))\hat{D}_\delta(\nu)\hat{F}_{01}}{p_\alpha(\mu) - p_\beta(\lambda)}, \tag{15}$$

where $\alpha, \beta, \delta = 0, 1, 2, \dots, N$. In fact, we may rewrite (14) as a set of equations satisfied by the second derivatives of F . Letting $\mu \rightarrow \lambda$ in (14) we have

$$\begin{aligned}
k\hat{F}_{0k,01} &= P_{k+1}(X), \quad X_1 = 0, X_{j \geq 2} = \sum_{n+m=j} \hat{F}_{0n,0m} \\
k\hat{F}_{\alpha k,01} &= e^{\hat{F}_{0\alpha 0, \alpha 0}} P_{k-1}(X), \quad X_{j \geq 1} = \sum_{n+m=j} \hat{F}_{\alpha n, \alpha m} \\
k\hat{F}_{0k,01} &= P_{k+1}(X) - \sum_{l=1}^{k+1} Y_l P_{k+1-l}(X), \quad X_{j \geq 1} = \sum_{n+m=j} \hat{F}_{0n, \alpha m}, Y_j = \sum_{n+m=j} n\hat{F}_{0n, \alpha m} \\
k\hat{F}_{\alpha k,01} &= \mp e^{\hat{F}_{\alpha 0, \beta 0}} \sum_{l=1}^k Y_l P_{k-l}(X), \quad X_{j \geq 1} = \sum_{n+m=j} \hat{F}_{\alpha n, \beta m}, Y_j = \sum_{n+m=j, n \geq 1} n\hat{F}_{\alpha n, \beta m}
\end{aligned}$$

where $k \geq 1$ and $P_k(X)$ are Schur polynomials defined by $e^{\sum_{j=1}^{\infty} X_j s^j} = \sum_{k=0}^{\infty} P_k(X) s^k$. The first nontrivial one of each dHirota equation is given below

$$\begin{aligned}
\hat{F}_{03,01} - \hat{F}_{02,02} - \hat{F}_{01,01}^2/2 &= 0 \\
\hat{F}_{\alpha 1,01} - e^{\hat{F}_{\alpha 0, \alpha 0}} &= 0 \\
\hat{F}_{01,01} + \hat{F}_{02, \alpha 0} + \hat{F}_{01, \alpha 0}^2/2 &= 0 \\
\hat{F}_{\alpha 1,01} \pm e^{\hat{F}_{\alpha 0, \beta 0}} \hat{F}_{\alpha 1, \beta 0} &= 0
\end{aligned}$$

We remark that, in terms of $t_{\alpha n}$, the first and the second equations are just the dKP and dToda equations, respectively.

4 Associativity Equations in the Whitham Hierarchy

4.1 WDVV equations

Let F be a function of the time variables t_J where the subscribes $J = \{(0, n), (\alpha, m)\}$ with $\alpha = 1, \dots, N; n \geq 1, m \geq 0$ and denoting $\hat{F}_J = \hat{\partial}_J F$, $\hat{F}_{KL} = \hat{\partial}_J \hat{\partial}_K F$, etc. Let us define $\hat{t}_I = \hat{t}_{01}$ and introduce the nondegenerate metric $\hat{\eta}_{JK} = \hat{F}_{JKI}$ which provides a transformation between $\{\hat{t}_J\}$ and $\{\hat{F}_{KI}\}$. Thus

$$\hat{F}_{JKL} = \sum_A \frac{\partial \hat{F}_{JK}}{\partial \hat{F}_{AI}} \frac{\partial \hat{F}_{AI}}{\partial \hat{t}_L} = \sum_A \hat{C}_{JK}^A \hat{\eta}_{AL} \quad (16)$$

where the coefficients \hat{C}_{JK}^A defined above connect \hat{F}_{JKL} and $\hat{\eta}_{AL}$. Notice that setting $J = I$ one has $\hat{\eta}_{KL} = \sum_A \hat{C}_{IK}^A \hat{\eta}_{AL}$ and hence $\hat{C}_{IK}^A = \delta_K^A$. If \hat{C}_{JK}^L are structure constants of an associative algebra generated by $\{\hat{\phi}_J\}$, i.e., $\hat{\phi}_J \cdot \hat{\phi}_K = \sum_L \hat{C}_{JK}^L \hat{\phi}_L$, then the associativity condition $(\hat{\phi}_J \cdot \hat{\phi}_K) \cdot \hat{\phi}_L = \hat{\phi}_J \cdot (\hat{\phi}_K \cdot \hat{\phi}_L)$ yields $\hat{C}_{JK}^M \hat{C}_{ML}^N = \hat{C}_{JM}^N \hat{C}_{KL}^M$, or, equivalently,

$$\sum_M \hat{C}_{JK}^M \hat{F}_{MLN} = \sum_M \hat{C}_{KL}^M \hat{F}_{JMN}.$$

The above equation is known as the WDVV equations arising in the context of topological field theory[10, 8, 30]. Define a four indices quantity $\hat{X}_{JKLMN} = \sum_L \hat{C}_{JK}^L \hat{F}_{LMN}$, then the WDVV equations can be replaced by the condition that \hat{X}_{JKLMN} is symmetric with respect to permutations of any indices. Using the operator $\hat{D}_\alpha(z)$ introduced in the previous section, it is convenient to define the generating functions for \hat{F}_{JK} , \hat{F}_{JKL} , \hat{C}_{JK}^L and \hat{X}_{JKLM} as follows:

$$\hat{D}_\alpha(u) \hat{D}_\beta(v) F = \sum_{n,m=0} u^{-n} v^{-m} \hat{F}_{\alpha n, \beta m}, \quad (17)$$

$$\hat{D}_\alpha(u) \hat{D}_\beta(v) \hat{D}_\gamma(w) F = \sum_{n,m,l=0} u^{-n} v^{-m} w^{-l} \hat{F}_{\alpha n, \beta m, \gamma l}, \quad (18)$$

$$\hat{C}^J(u_\alpha, v_\beta) = \sum_{n,m=0} u^{-n} v^{-m} \hat{C}_{\alpha n, \beta m}^J, \quad (19)$$

$$\hat{X}(u_\alpha, v_\beta, w_\gamma, z_\delta) = \sum_{n,m,k,l=0} u^{-n} v^{-m} w^{-k} z^{-l} \hat{X}_{\alpha n, \beta m, \gamma k, \delta l}. \quad (20)$$

Hence the WDVV equations can be expressed in terms of the generating function as

$$\hat{X}(u_\alpha, v_\beta, w_\gamma, z_\delta) = \hat{X}(u_\alpha, w_\gamma, v_\beta, z_\delta).$$

4.2 From dHirota to WDVV

Applying $\hat{\partial}_{\alpha l}$ on the first line of dHirota equation(14), we get

$$\hat{D}_0(u) \hat{D}_0(v) \hat{\partial}_{\alpha l} F = - \frac{(\hat{D}_0(u) - \hat{D}_0(v)) \hat{\partial}_{\alpha l} \hat{F}_{01}}{p_0(u) - p_0(v)}$$

The left hand side is

$$\begin{aligned}
\hat{D}_0(u)\hat{D}_0(v)\hat{\partial}_{\alpha l}F &= \sum_{n,m=1} u^{-n}v^{-m}\hat{F}_{0n,0m,\alpha l} \\
&= \sum_J \sum_{n,m=1} u^{-n}v^{-m}\hat{C}_{0n,0m}^J\hat{F}_{J,\alpha l,I} \\
&= \sum_J \hat{C}^J(u_0, v_0)\hat{F}_{J,\alpha l,I},
\end{aligned}$$

while the right hand side is

$$-\frac{(\hat{D}_0(u) - \hat{D}_0(v))\hat{\partial}_{\alpha l}\hat{F}_{01}}{p_0(u) - p_0(v)} = -\sum_{k=1} \frac{u^{-k} - v^{-k}}{p_0(u) - p_0(v)}\hat{F}_{0k,\alpha l,I}.$$

Hence, we have

$$\hat{C}^J(u_0, v_0) = \begin{cases} -\frac{u^{-n}-v^{-n}}{p_0(u)-p_0(v)} & J = (0, n) \\ 0 & J = \{(\alpha, n); \alpha \neq 0\} \end{cases} \quad (21)$$

By the same way, one can get the other structure constant generating functions from the rest of equation(14) . The formulas are listed below:

$$\hat{C}^J(u_\alpha, v_\alpha) = \begin{cases} -\frac{u^{-n}-v^{-n}}{p_\alpha(u)-p_\alpha(v)} & J = \{(\alpha, n); \alpha \neq 0\} \\ 0 & J = (0, n), \end{cases}$$

$$\hat{C}^J(u_0, v_\alpha) = \begin{cases} \frac{-u^{-n}}{p_0(u)-p_\alpha(v)} & J = (0, n) \\ \frac{v^{-n}}{p_0(u)-p_\alpha(v)} & J = \{(\alpha, n); \alpha \neq 0\}, \end{cases}$$

and

$$\hat{C}^J(u_\alpha, v_\beta) = \begin{cases} \frac{-u^{-n}}{p_\alpha(u)-p_\beta(v)} & J = \{(\alpha, n); \alpha \neq 0\} \\ \frac{v^{-n}}{p_\alpha(u)-p_\beta(v)} & J = \{(\beta, n); \beta \neq 0\}. \end{cases}$$

We remark here that by comparing above with (19) it is easy to show that

$$\begin{aligned}
\hat{C}_{0n,0m}^{0k} &= 0, \quad k > n + m - 1; & \hat{C}_{\alpha n,\alpha m}^{\alpha k} &= 0, \quad k > n + m + 1 \\
\hat{C}_{0n,\alpha m}^{0k} &= 0, \quad k > n - 1; & \hat{C}_{0n,\alpha m}^{\alpha k} &= 0, \quad k > m \\
\hat{C}_{\alpha n,\beta m}^{\alpha k} &= 0, \quad k > n; & \hat{C}_{\alpha n,\beta m}^{\beta k} &= 0, \quad k > m
\end{aligned}$$

and hence the infinite sum in (16) involves only finite terms.

We can now show that any solution F of the universal Whitham hierarchy obey the associativity equation with the help of those structure constants. To show that the generating function (20) is totally symmetric with respect to the permutation of $u_\alpha, v_\beta, w_\gamma, z_\delta$, however, it is enough to prove the symmetry with respect to the permutation of v_β and w_γ in (20), that is, $\hat{X}(u_\alpha, v_\beta, w_\gamma, z_\delta) = \hat{X}(u_\alpha, w_\gamma, v_\beta, z_\delta)$.

For the case of $\alpha, \beta, \gamma, \delta \neq 0$, we need to prove that

$$\begin{aligned} & (p_\alpha(u) - p_\gamma(w))(\hat{D}_\alpha(u) - \hat{D}_\beta(v))\hat{D}_\gamma(w)\hat{D}_\delta(z)F \\ &= (p_\alpha(u) - p_\beta(v))(\hat{D}_\alpha(u) - \hat{D}_\gamma(w))\hat{D}_\beta(v)\hat{D}_\delta(z)F, \end{aligned} \quad (22)$$

By virtue of (15), one can easily check that it is an identity. Take as the second case : $\alpha = \beta = \gamma = \delta = 0$, we need to prove that

$$\begin{aligned} & (p_0(u) - p_0(w))(\hat{D}_0(u) - \hat{D}_0(v))\hat{D}_0(w)\hat{D}_0(z)F \\ &= (p_0(u) - p_0(v))(\hat{D}_0(u) - \hat{D}_0(w))\hat{D}_0(v)\hat{D}_0(z)F, \end{aligned}$$

Consulting (15), one can see that it actually holds. As for the case : $\alpha \neq \gamma \neq \delta, \beta = 0$, the one needed to be proven is that

$$\begin{aligned} & (p_\alpha(u) - p_\gamma(w))(\hat{D}_0(v) - \hat{D}_\alpha(u))\hat{D}_\gamma(w)\hat{D}_\delta(z)F \\ &= (p_0(v) - p_\alpha(u))(\hat{D}_\alpha(u) - \hat{D}_\gamma(w))\hat{D}_0(v)\hat{D}_\delta(z)F \end{aligned}$$

One can also verify that it is true according to (15). For the other cases , the calculation is similar to the above one, we omit that here.

4.3 Realization of the Associative algebra

To construct the generators $\{\hat{\phi}_J\}$ which realize the associative algebra with structure constants \hat{C}_{JK}^L obtained in the previous subsection we define

$$\hat{\phi}_{\alpha n}(p) = \begin{cases} \frac{dQ_{\alpha n}(p)}{dp}, & \alpha = 0, 1, \dots, N; n \geq 1. \\ \frac{dB_{\alpha 0}(p)}{dp}, & \alpha = 1, \dots, N. \end{cases} \quad (23)$$

such that $\hat{\phi}_{0n}(p)$ are polynomials in p and $\hat{\phi}_{\alpha n}(p)$ for $\alpha \neq 0$ are polynomials in $(p - q_\alpha)^{-1}$. In particular, $\hat{\phi}_{01}(p) = 1$, the unit element of the algebra. Equations (3) and (7) provide the recursion relations for $\hat{\phi}_{0n}(p)$ and $\hat{\phi}_{\alpha n}(p)$, respectively. There are four classes to be discussed.

In Ω_0 , from the kernel formula (4) we have

$$\frac{1}{p_0(z) - p} = \sum_{n=1} z^{-n} \hat{\phi}_{0n}(p). \quad (24)$$

Thus

$$\begin{aligned}
\frac{1}{(p_0(z) - p)(p_0(w) - p)} &= \sum_{n,m=1} z^{-n} w^{-m} \hat{\phi}_{0n}(p) \cdot \hat{\phi}_{0m}(p) \\
&= -\frac{1}{p_0(z) - p_0(w)} \left(\frac{1}{p_0(z) - p} - \frac{1}{p_0(w) - p} \right) \\
&= -\frac{1}{p_0(z) - p_0(w)} \left(\sum_{l=1} (z^{-l} - w^{-l}) \hat{\phi}_{0l}(p) \right) \\
&= \sum_{l=1} \hat{C}^{0l}(z_0, w_0) \hat{\phi}_{0l}(p),
\end{aligned}$$

which together with (19) implies

$$\hat{\phi}_{0n} \cdot \hat{\phi}_{0m} = \sum_{l=1} \hat{C}_{0n,0m}^{0l} \hat{\phi}_{0l}(p). \quad (25)$$

In Ω_α , note that the kernel formula (8) can be rewritten as the following form:

$$\frac{1}{p_\alpha(w) - p} = \sum_{m=0} w^{-m} \hat{\phi}_{\alpha m}(p). \quad (26)$$

By the same way as above, we have

$$\hat{\phi}_{\alpha n} \cdot \hat{\phi}_{\alpha m} = \sum_{l=0} \hat{C}_{\alpha n, \alpha m}^{\alpha l} \hat{\phi}_{\alpha l}(p).$$

While in $\Omega_0 \cap \Omega_\alpha$, it's easy to see that

$$\begin{aligned}
\frac{1}{p_0(z) - p} \frac{1}{p_\alpha(w) - p} &= \sum_{n=1, m=0} z^{-n} w^{-m} \hat{\phi}_{0n}(p) \cdot \hat{\phi}_{\alpha m}(p) \\
&= -\frac{1}{p_0(z) - p_\alpha(w)} \left(\frac{1}{p_0(z) - p} - \frac{1}{p_\alpha(w) - p} \right) \\
&= -\frac{1}{p_0(z) - p_\alpha(w)} \left(\sum_{n=1} z^{-n} \hat{\phi}_{0n} - \sum_{n=0} w^{-n} \hat{\phi}_{\alpha n} \right) \\
&= \sum_{n=1} \hat{C}^{0n}(z_0, w_\alpha) \hat{\phi}_{0n} + \sum_{n=0} \hat{C}^{\alpha n}(z_0, w_\alpha) \hat{\phi}_{\alpha n},
\end{aligned}$$

which implies that

$$\hat{\phi}_{0n} \cdot \hat{\phi}_{\alpha m} = \sum_{l=1} \hat{C}_{0n, \alpha m}^{0l} \hat{\phi}_{0l}(p) + \sum_{l=0} \hat{C}_{0n, \alpha m}^{\alpha l} \hat{\phi}_{\alpha l}(p).$$

Finally, in $\Omega_\alpha \cap \Omega_\beta$, we have

$$\hat{\phi}_{\alpha n} \cdot \hat{\phi}_{\beta m} = \sum_{l=0} \hat{C}_{\alpha n, \beta m}^{\alpha l} \hat{\phi}_{\alpha l}(p) + \sum_{l=0} \hat{C}_{\alpha n, \beta m}^{\beta l} \hat{\phi}_{\beta l}(p).$$

Therefore, the generators $\{\hat{\phi}_J\}$ indeed satisfy the algebra $\hat{\phi}_J \cdot \hat{\phi}_J = \sum_L \hat{C}_{JK}^L \hat{\phi}_L$ with structure constants defined in the previous section.

Next let us derive the residue formulas for the third derivatives of F directly from the dHirota equations. Applying $\hat{\partial}_{01}$ on the first dHirota equation(14), we see that

$$\hat{D}_0(u)\hat{D}_0(v)\hat{F}_{01} = \frac{-1}{p_0(u) - p_0(v)}(\hat{D}_0(u) - \hat{D}_0(v))\hat{F}_{01,01},$$

and thus

$$\hat{D}_0(u)\hat{D}_0(v)\hat{D}_0(w)F = \sum_{i=1}^3 \text{res}_{p_i} \frac{\hat{D}_0(\lambda(p))\hat{F}_{01,01}}{(p - p_0(u))(p - p_0(v))(p - p_0(w))} dp. \quad (27)$$

Expanding both side of (27) in the powers of u, v, w , we get that

$$\hat{F}_{0i,0j,0k} = \frac{1}{2\pi i} \oint_{C_\infty} \frac{\hat{\partial}_{01}\lambda(p)}{\lambda'(p)} \hat{\phi}_{0i}\hat{\phi}_{0j}\hat{\phi}_{0k} dp,$$

where, due to the fact that $0 = \hat{\partial}_{01}p(p) = \hat{\partial}_{01}p(\lambda) + \partial_\lambda p(\lambda)\hat{\partial}_{01}\lambda$, we have replaced $\hat{D}_0(\lambda)\hat{F}_{01,01} = -\hat{\partial}_{01}p(\lambda) = \hat{\partial}_{01}\lambda/\lambda'(p)$ in the numerator of (27). For those cases containing different domain indices, one can derive the formulas by straightforward calculations. In fact, they can be summarized as a single equation :

$$\hat{D}_\alpha(u)\hat{D}_\beta(v)\hat{D}_\gamma(w)F = \sum_{i=1}^3 \text{res}_{p_i} \left(\frac{\hat{D}(\lambda)\hat{F}_{01,01}}{(p - p_\alpha(u))(p - p_\beta(v))(p - p_\gamma(w))} dp \right),$$

where $\alpha, \beta, \gamma = 0, 1, \dots, N$ and $\hat{D}(\lambda)$ denotes as a bookkeeping notation defined by $\hat{D}(\lambda(p_\alpha)) = \hat{D}_\alpha(\lambda)$.

4.4 Finite-dimensional reductions

We now consider finite-dimensional reductions of the Whitham hierarchy. Suppose $\phi_\alpha(\lambda)$ are times-independent functions so that $E(p) = \phi_\alpha(\lambda(p))$ is a meromorphic function of p in $\Omega_\alpha(\alpha = 0, \dots, N)$ with the number of poles being unchanged under the flows $\hat{t}_{\alpha n}$. Then the sum of the residue in the third derivative of F is now the zeros of $E'(p)$:

$$\hat{F}_{JKL} = \sum_{E'(p_s)=0} \text{res}_{p_s} \left(\frac{-\hat{\partial}_{01}E(p)}{E'(p)} \hat{\phi}_J(p)\hat{\phi}_K(p)\hat{\phi}_L(p) dp \right). \quad (28)$$

Such a finite-dimensional reduction imposes a set of additional constraints on the system and the algebra becomes finite-dimensional. Since unknowns of the system are functions of the second derivative of the free energy F , one can choose $\hat{F}_{01,P} \equiv \hat{F}_{1P}$ as the independent variables called primary fields where the time parameters $\{\hat{t}_P\}$ span a finite-dimensional small phase space. All other \hat{F}_{ID} can be expressed in terms of these primary fields through a functional

relation $Q_D(F_{IP})$ where D denotes those time indices outside the set P . With respect to this separation, we rewrite the \hat{F}_{KLM} as

$$\begin{aligned}\hat{F}_{KLM} &= \sum_{J \in P} \left(\hat{C}_{KL}^J + \sum_D \hat{C}_{KL}^D \frac{\partial Q_D}{\partial F_{IJ}} \right) \hat{F}_{JMI} \\ &= \sum_{J \in P} \tilde{C}_{KL}^J \hat{F}_{JMI}\end{aligned}\tag{29}$$

The object $\tilde{C}_{KL}^J = \hat{C}_{KL}^J + \sum_D \hat{C}_{KL}^D \frac{\partial Q_D}{\partial F_{IJ}}$ define the structure constant of a finite-dimensional algebra generated by the primary fields. Consequently the finite-dimensional WDVV equation is now defined by $\tilde{X}_{JKLM} \equiv \sum_{A \in P} \tilde{C}_{JK}^A \hat{F}_{ALM}$ ($J, K, L, M \in P$). Following [3], one can show that this \tilde{X}_{JKLM} is totally symmetric with respect to all the indices in the set P by proving that $\tilde{X}_{JKLM} = \tilde{X}_{JKLM}$ for $J, K, L, M \in P$. This completes the verification for the associativity equations corresponding to finite-dimensional reductions of the dHirota equations. Define a scalar product :

$$\langle f \cdot g \rangle = \sum_{E'(p_s)=0} \text{res}_{p_s} \left(\frac{-\hat{\partial}_{01} E(p)}{E'(p)} f(p) g(p) dp \right),\tag{30}$$

then we see that $\eta_{JK} = \hat{F}_{JKI} = \langle \hat{\phi}_J \cdot \hat{\phi}_K \rangle$ because of $\hat{\phi}_I = 1$. Therefore

$$\begin{aligned}\hat{F}_{JKL} &= \langle \hat{\phi}_J \hat{\phi}_K \cdot \hat{\phi}_L \rangle = \sum_{A \in P} \tilde{C}_{JK}^A \langle \hat{\phi}_A \cdot \hat{\phi}_L \rangle \\ &= \sum_{A \in P} \tilde{C}_{JK}^A \hat{F}_{ALI}.\end{aligned}\tag{31}$$

This gives another proof of the WDVV equation since now it reads $\tilde{X}_{JKLM} = \langle \hat{\phi}_J \hat{\phi}_K \cdot \hat{\phi}_L \hat{\phi}_M \rangle$ and the symmetry with respect to the permutation is now obvious. Let us mention the well-known finite-dimensional reduction constructed by Krichever[16], in which $\phi_\alpha(\lambda) = \lambda_\alpha^{n_\alpha}$ and $E(p)$ is a rational function of the form

$$E(p) = p^{n_0} + u_{n_0-2} p^{n_0-2} + \cdots + u_0 + \sum_{\alpha=1}^N \sum_{s=1}^{n_\alpha} \frac{v_{\alpha s}}{(p - q_\alpha)^s}\tag{32}$$

which provides the algebraic orbits(or Lax function) of genus-zero Whitham hierarchy. Note that the number of zeros p_s in (28) is equal to the number of unknowns $\{u_{n_0}, \cdots, u_{n_0-2}, v_{\alpha s}, q_\alpha\}$ and the small phase space is characterized by the time-parameters \hat{t}_P with P being the set $\{(0, i), i = 1, \cdots, n_0 - 1; (\alpha, m), \alpha = 1, \cdots, N; m = 0, \cdots, n_\alpha\}$.

5 Concluding Remarks

We have demonstrated that the kernel formula approach developed by Carroll and Kodama for the dKP hierarchy can be generalized well to the universal Whitham hierarchy. The obtained

dHirota equations are exactly the same as those derived by Takasaki and Takebe using the notion of Faber polynomials. We extract the structure constants of an associate algebra from the dHirota equations and verify the associativity equations of the algebra. Finally, we give a realization of the associative algebra with the structure constants expressed in terms of the residue formula and verify the associativity equations for finite-dimensional reductions. We like to emphasize that the kernel formula approach indeed provides an efficient way to derive the dHirota equations of the genus-zero Whitham hierarchy. It would be interesting to generalize and/or reformulate the kernel formula approach to higher genus [16]. We hope to address this issue in the future work.

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