EXACT INTERNAL WAVES OF A BOUSSINESQ SYSTEM*

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We consider a Boussinesq system describing one-dimensional internal waves which develop at the boundary between two immiscible fluids, and we restrict to its traveling waves. The method which yields explicitly all the elliptic or degenerate elliptic solutions of a given nonlinear, any order algebraic ordinary differential equation is briefly recalled. We then apply it to the fluid system and, restricting in this preliminary report to the generic situation, we obtain all the solutions in that class, including several new solutions.

Keywords: Boussinesq system; internal waves; elliptic solutions; solitary waves.

1. Introduction

At the boundary between two immiscible fluids, one observes the formation of waves, called *internal waves*. These are typically described by Boussinesq

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systems such as [2]

$$\begin{cases} (1-\mu b\Delta)\partial_t\zeta + c_{\mathrm{v}} \,\mathrm{div}\, \mathbf{v} + c_{\mathrm{NL}} \,\mathrm{div}(\zeta\mathbf{v}) + \mu a \,\mathrm{div}\, \Delta\mathbf{v} = 0, \\ (1-\mu d\Delta)\partial_t\mathbf{v} + c_{\mathrm{z}} \,\mathrm{grad}\, \zeta + \dfrac{c_{\mathrm{NL}}}{2} \,\mathrm{grad}\, |\mathbf{v}|^2 + \mu c\Delta \,\mathrm{grad}\, \zeta = 0, \end{cases} \tag{1}$$

in which $\mu, a, b, c, d, c_{\mathrm{NL}}, c_{\mathrm{v}}, c_{\mathrm{z}}$ are constant, $\mu c_{\mathrm{NL}} \neq 0$.

We restrict here to one-dimensional situations, relevant for instance when the fluids are inside a channel, and our purpose is to obtain traveling waves $\zeta = u(\xi) - c_{\mathrm{v}}/c_{\mathrm{NL}}, \mathbf{v} = v(\xi) + c_0/c_{\mathrm{NL}}, \xi = x - c_0 t$ (the translation on u suppresses any dependence on c_{v} and the one on v shortens the expressions below) in closed form by a nonperturbative method. The conservative form of the equations (1) allows each equation to be integrated once, and the considered system will be

$$\begin{cases} \mu b c_0 u'' + \mu a v'' + c_{\mathrm{NL}} uv + K_1 = 0, \\ \mu d c_0 v'' + \mu c u'' + \dfrac{c_{\mathrm{NL}}}{2} v^2 + c_{\mathrm{z}} u + K_2 = 0. \end{cases} \tag{2}$$

The above system is essentially the same as those considered by Chen [4] and Nguyen and Dias [10] .

Chen [4] already found all the traveling waves in which u and v are polynomials (of degree $1, 2$ or 4) in $\tanh k\xi$ and $\mathrm{sech}\, k\xi$. In the present work, we obtain the closed form expressions of *all*[a] those solutions of (2) which are either elliptic (doubly periodic in the ξ complex plane) or degenerate elliptic, i.e. rational in one exponential $e^{k\xi}$ (simply periodic in the ξ complex plane, which includes the above mentioned solutions) or rational in ξ.

The method, based on those complex singularities of (u, v) which depend on the initial conditions ("movable" singularities [5]), implements classical results by Briot, Bouquet [3] and Poincaré [7]. First presented in [9], it was later turned into an algorithm [6].

2. Singularity analysis

In order to know whether closed-form solutions to (2) might exist, a prerequisite [5] is to investigate the singularities of (u, v) in the complex plane of ξ. One must distinguish whether the total differential order of system (2) is four or two, depending on the value of the determinant

$$\delta \equiv bdc_0^2 - ac. \tag{3}$$

We leave the nongeneric case $\delta = 0$ to a forthcoming detailed study.

[a]In this short report, only the generic case is presented.

In the generic case $\delta \neq 0$, the system is equivalent to

$$\delta \neq 0: \begin{cases} \mu\delta u'' + c_{\rm NL}\left(dc_0 uv - \frac{a}{2}v^2\right) - ac_{\rm z}u + dc_0 K_1 - aK_2 = 0, \\ \mu\delta v'' + c_{\rm NL}\left(\frac{bc_0}{2}v^2 - cuv\right) + bc_0 c_{\rm z}u + bc_0 K_2 - cK_1 = 0. \end{cases} \tag{4}$$

For an easier computation of u knowing v, it is convenient to introduce the shift

$$v_{\rm s} = v - \frac{bc_0}{c_{\rm NL}c}c_{\rm z}. \tag{5}$$

In order to find all the elliptic and degenerate elliptic solutions, one must first determine all the families of movable poles *and* movable zeros (i.e. movable poles of $1/u$ or $1/v_{\rm s}$).

Let us first determine the poles. Assume that, near a movable singularity ξ_0, the variables (u, v) behave algebraically

$$u \sim u_0\chi^{p_1},\ v \sim v_0\chi^{p_2},\ \chi = \xi - \xi_0,\ u_0 v_0 \neq 0, \tag{6}$$

with p_1, p_2 not both positive integers. Balancing the highest derivatives and the nonlinear terms, one generically (nongeneric cases will be dealt with in a forthcoming paper) obtains double pole behaviours

$$\begin{cases} p_1 = -2,\ p_2 = -2, \\ c_{\rm NL}u_0 v_0 + 6\mu(bc_0 u_0 + av_0) = 0,\ \frac{c_{\rm NL}}{2}v_0^2 + 6\mu(dc_0 v_0 + cu_0) = 0. \end{cases} \tag{7}$$

Whenever $acD \neq 0$, with

$$D^2 \equiv (b - 2d)^2 c_0^2 + 8ac, \tag{8}$$

this system (7) admits two solutions (u_0, v_0),

$$\delta acD \neq 0: \begin{cases} u_{0,\varepsilon} = \frac{3\mu}{2cc_{\rm NL}}\left(b(2d-b)c_0^2 - 4ac + \varepsilon bc_0 D\right), \\ v_{0,\varepsilon} = \frac{3\mu}{c_{\rm NL}}\left(-(b+2d)c_0 + \varepsilon D\right),\ \varepsilon^2 = 1. \end{cases} \tag{9}$$

Let us next determine the movable zeros. By elimination between (4), it is easy to establish the fourth order ODE for $1/v_{\rm s}$. Its movable poles are one simple pole of arbitrary residue plus, when $K_{11} \neq 0$, one double pole, with

$$K_{11} = K_1 - \frac{\lambda D - 2dc_0}{c}\left(K_2 + \frac{c_{\rm z}^2}{2c^2}(\lambda D - 2dc_0)^2\right). \tag{10}$$

The movable zeros of u are less easy to establish, but it is sufficient for our purpose, as explained below, to know that u has always at least two movable zeros.

One must then compute [5] the Fuchs indices i of the linearized system of (4) near all the movable singularities. Near the movable double poles (6), the resulting indicial equation (we skip the details) only depends on one adimensional parameter λ,

$$(i+1)(i-6)\left(i^2-5i+\frac{12}{1+\varepsilon\lambda}\right)=0, \tag{11}$$

$$\lambda=\frac{(b+2d)c_0}{D}. \tag{12}$$

Since some Fuchs indices i are generically noninteger, the general solution of the system (4) is multivalued. Nongenerically, for the general solution to be singlevalued, it is necessary that, for both signs ε, all roots i of (11) be integer. Denoting these roots as

$$-1,6,\frac{5+Q}{2},\frac{5-Q}{2} \text{ for } \varepsilon=+1,\ -1,6,\frac{5+R}{2},\frac{5-R}{2} \text{ for } \varepsilon=-1, \tag{13}$$

the elimination of λ between the products of the roots

$$\frac{25-Q^2}{4}=\frac{12}{1+\lambda},\ \frac{25-R^2}{4}=\frac{12}{1-\lambda}, \tag{14}$$

yields the diophantine equation

$$\frac{24}{25-Q^2}+\frac{24}{25-R^2}=1, \tag{15}$$

which admits no solution for odd positive integers (Q,R).

Despite its generically multivalued general solution, the system (4) may still admit singlevalued particular solutions. For this it is necessary that the Laurent series whose first term is (6),

$$u=\sum_{j=0}^{+\infty}u_j\chi^{j-2},\ v=\sum_{j=0}^{+\infty}v_j\chi^{j-2}, \tag{16}$$

exists, i.e. that no impossibility occurs when computing the coefficients u_j,v_j. The invariance of the system (2) under $\xi\to-\xi$ forbids the occurrence of odd powers of χ in the Laurent series of u and v.

For convenience, the three defining equations for δ,D,λ can be solved for a,bc_0,δ in terms of dc_0,λ,D,c, yielding

$$a=\frac{D^2-(4dc_0-\lambda D)^2}{8c},\ bc0=-2dc_0+\lambda D,\ \delta=(\lambda^2-1)D^2, \tag{17}$$

$$u_{0,\varepsilon}=\frac{3\mu D}{4cc_{\rm NL}}(\lambda-\varepsilon)\left[4dc_0-D(\lambda-\varepsilon)\right],\ v_{0,\varepsilon}=\frac{3\mu D}{c_{\rm NL}}(\varepsilon-\lambda). \tag{18}$$

For a generic λ, the values $i = 2$ and $i = 4$ are not roots of (11),

$$\frac{(b+2d)^2 c_0^2}{(b-2d)^2 c_0^2 + 8ac} \notin \{1, 4\}, \tag{19}$$

so no impossibility can occur when computing the next coefficients u_2, v_2, u_4, v_4. The only obstruction arises from the Fuchs index $i = 6$, which generates two necessary conditions (one for each sign ε) for the absence of movable logarithms (again we skip the details of this classical computation),

$$Q_6 \equiv c_{\mathrm{z}}(\lambda D - 3dc_0)\left(\tilde{K}_1 - \frac{2Du_0}{cv_0}\tilde{K}_2\right) = 0, \tag{20}$$

$$\tilde{K}_1 = cK_1 - a(b-2d)c_0\frac{c_{\mathrm{z}}^2}{c},\ \tilde{K}_2 = K_2 + \left((b-2d)^2 c_0^2 + 2ac\right)\frac{c_{\mathrm{z}}^2}{2c^2}. \tag{21}$$

This defines three subcases,

$$c_{\mathrm{z}} = 0, \tag{22}$$

$$(b-d)c_0 = 0, \tag{23}$$

$$\tilde{K}_1 = \frac{2Du_0}{cv_0}\tilde{K}_2. \tag{24}$$

The first two subcases are independent of the sign ε, and the third subcase can be enforced either for one sign (condition (24)) or for both signs, leading to the stronger condition

$$\tilde{K}_1 = 0,\ \tilde{K}_2 = 0. \tag{25}$$

For the first two cases, a first integral exists,

$$\begin{aligned}(b-d)c_0 c_{\mathrm{z}} = 0:\ K_6 = {} & \mu c_{\mathrm{NL}}\left(c^2 {u'}^2 + 2cdc_0 u'v' + (ac - (b-d)dc_0^2){v'}^2\right) \\ & + c_{\mathrm{NL}}^2\left(cuv^2 + \frac{(d-b)c_0}{3}v^3\right) + c_{\mathrm{NL}} c c_{\mathrm{z}} u^2 \\ & + 2cK_2 u + (2cK_1 - 2K_2(b-d)c_0)\, v,\end{aligned} \tag{26}$$

and the first integral for the third subcase $\tilde{K}_1 = \tilde{K}_2 = 0$, yet to be found, is not quartic in (u', v').

3. Method to find all the elliptic solutions

For full details on the method, we refer to [6,5].

The input is an N-th order ($N \geq 2$) any degree autonomous algebraic ordinary differential equation (ODE) admitting a Laurent series.

The output is made of all its elliptic or degenerate elliptic solutions in closed form.

Let us first recall a classical definition. The *elliptic order* of a nondegenerate elliptic (genus one) function [1] is the number of poles, counting multiplicity of course, inside a period parallelogram. It is equal to the number of zeros. This equality breaks down under degeneracy to genus zero, e.g. for the rational function $u = (\xi - a)(\xi - b)/(\xi - c)$.

The successive steps of the algorithm are [6]:

(1) Find the analytic structure of singularities (in our case two families of movable double poles for both u and v, see (7), one movable simple zero and, if $K_{11} \neq 0$, one movable double zero for v, at least two movable simple zeros for u). Deduce the total number of poles (or, if greater, of zeros) of the unknown function and its derivative, here $m = 4, n = 6$ for (v, v'), more for (u, u').
(2) Compute slightly more than $(m+1)^2$ terms in each Laurent series.
(3) Choose one of the dependent variables (u, v) (call it U) and define the first order m-th degree subequation $F(U, U') = 0$ (it contains at most $(m+1)^2$ coefficients $a_{j,k}$),

$$F(U, U') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} U^j {U'}^k = 0,\ a_{0,m} \neq 0. \tag{27}$$

(4) Require at least one Laurent series of U to obey $F(U, U') = 0$,

$$F \equiv \chi^{m(p-1)} \left(\sum_{j=0}^{J} F_j \chi^j + \mathcal{O}(\chi^{J+1}) \right),\ \forall j\ :\ F_j = 0, \tag{28}$$

and solve this **linear overdetermined** system for $a_{j,k}$.
(5) Integrate each resulting first order ODE $F(U, U') = 0$.

The key advantage of this method is that the system of equations $F_j = 0$ for the unknown coefficients $a_{j,k}$ is *linear* and infinitely overdetermined, therefore quite easy to solve.

4. Elliptic and degenerate elliptic solutions, generic case

By generic, we mean that the fixed constants $a, c, bc_0, dc_0, c_z, K_1, K_2$ obey the nonvanishing conditions $ac\delta D \neq 0$, (19) and only one of the three vanishing conditions (22), (23), (24).

When the algorithm of section 3 is applied to a system of ODEs such as (2), a key practical ingredient is to select a "good" variable U, i.e. one whose total number of poles (or, if greater, of zeros) of U and U' is the smallest possible. Since this number is always smaller for (v, v') $((4,6))$ than for (u, u'), the natural choice is $U = v$.

Moreover, the already mentioned invariance under $\xi \to -\xi$ forbids the occurrence of odd powers of $U' = v'$ in the subequation (27).

Since $U = v$ admits two distinct Laurent series, the search for elliptic or degenerate elliptic solutions splits into (step 4): either require one Laurent series to obey (the odd-parity terms have been removed)

$$F \equiv {U'}^2 + a_{3,0}U^3 + a_{2,0}U^2 + a_{1,0}U + a_{0,0} = 0, \tag{29}$$

or require both Laurent series to obey

$$F \equiv {U'}^4 + a_{3,2}U^3{U'}^2 + a_{6,0}U^6 + a_{2,2}U^2{U'}^2 + a_{5,0}U^5 + a_{1,2}U{U'}^2 + a_{4,0}U^4 \\ +a_{0,2}{U'}^2 + a_{3,0}U^3 + a_{2,0}U^2 + a_{1,0}U + a_{0,0} = 0. \tag{30}$$

4.1. *Elliptic and degenerate elliptic solutions, one series*

One of the three necessary conditions (22), (23), (24) must hold true. In this section we simply denote (u_0, v_0) anyone of the two values $(u_{0,\varepsilon}, v_{0,\varepsilon})$ (Eqs. (9) or (18)).

In step (2) of section 3, it is quicker to compute simultaneously both Laurent coefficients (u_j, v_j) from system (4). Going to $j = 8$ is sufficient to obtain all the coefficients in (29) and ensure that system (4) is indeed a differential consequence of (29).

In step (4), with the definition (5), the resulting subequation (29) is

$$v_{\rm s}'^2 - \frac{4}{v_0}v_{\rm s}^3 + b_2 v_{\rm s}^2 + b_1 v_{\rm s} + b_0 = 0, \tag{31}$$

$$b_2 = \left(D - 4\frac{3dc_0 - \lambda D}{\lambda - \varepsilon}\right)\frac{c_{\rm z}}{c\mu D}, \tag{32}$$

$$c_{\rm z}(b-d)c_0 = 0: \quad b_1 = 0,\ b_0 = 0,\ \tilde{K}_1 = 2\frac{Du_0}{cv_0}\tilde{K}_2, \tag{33}$$

$$c_{\rm z}(b-d)c_0 \ne 0: \quad b_1 = \frac{1}{c_{\rm NL}\mu}\left(\frac{8\tilde{K}_2}{D(\lambda-\varepsilon)} - 4(3dc_0 - \lambda D)\frac{c_{\rm z}^2}{c^2}\right),\ b_0 = \text{arb},$$

i.e. one additional constraint is found in the case $c_{\rm z}(b-d)c_0 = 0$.

Step (5) is immediate. Indeed, subequation (31) is nothing else, up to an affine transformation, than the canonical equation of Weierstrass [1],

$$v_{\rm s} = v_0\wp(\xi - \xi_0, g_2, g_3) + v_2 - \frac{bc_0}{c_{\rm NL}c}c_{\rm z}, \tag{34}$$

$${\wp'}^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),\ \wp'' = 6\wp^2 - \frac{g_2}{2}. \tag{35}$$

The result is

$$\begin{cases} v = v_0 \wp - \dfrac{D u_0 c_{\mathrm{z}}}{c^2 c_{\mathrm{NL}} v_0},\ g_3 = 28 \dfrac{v_6}{v_0}, \\ u = \dfrac{u_0}{v_0} \left(v_{\mathrm{s}} - (8 d c_0 - D(3\lambda - \varepsilon)) \dfrac{c_{\mathrm{z}}}{2 c c_{\mathrm{NL}}} \right) \\ \qquad + \left(\tilde{K}_1 - \dfrac{2 D u_0}{c v_0} \tilde{K}_2 \right) \dfrac{}{c c_{\mathrm{NL}} D \mu (\lambda - 2\varepsilon)} \dfrac{v_0}{v_{\mathrm{s}}}, \\ c_{\mathrm{z}}(b-d) c_0 = 0 :\ \tilde{K}_1 = ((3\varepsilon - \lambda) D - 4 d c_0) \dfrac{\tilde{K}_2}{2},\ g_2 = \dfrac{c_{\mathrm{z}}^2}{12 c^2 \mu^2},\ g_3 = \dfrac{c_{\mathrm{z}}^3}{(6 c \mu)^3}, \\ c_{\mathrm{z}}(b-d) c_0 \neq 0 :\ \tilde{K}_1 = \dfrac{2 D u_0}{c v_0} \tilde{K}_2,\ g_3 = \text{arbitrary}, \\ \qquad g_2 = \dfrac{1}{(\lambda - \varepsilon)^2 D^2 \mu^2} \left(- \dfrac{8 \tilde{K}_2}{3} + (12 d c_0 - (3\lambda + \varepsilon) D) \dfrac{c_{\mathrm{z}}^2}{12 c^2} \right). \end{cases} \tag{36}$$

The effective expression of $\wp$ depends on the values of (g_2, g_3) in (35), according to the identities [1] ,

$$\wp(z, g_2, g_3) = \begin{cases} \text{doubly periodic ("cnoidal wave")}\ , g_2^3 - 27 g_3^2 \neq 0, \\ 3q \coth^2(\sqrt{3q} z) - 4q,\ g_2 = 12 q^2, g_3 = -8 q^3, \\ \dfrac{1}{z^2},\ g_2 = g_3 = 0. \end{cases} \tag{37}$$

For each of the three subcases (22), (23), (24), one thus obtains two closed form solutions (one for each sign ε), namely:
two rational solutions for the condition (22),

$$(22):\ v_{\mathrm{s}} = v = v_0 (\xi - \xi_0)^{-2},\ u = \frac{u_0}{(\xi - \xi_0)^2} - \frac{\tilde{K}_2}{\mu c c_{\mathrm{NL}}} (\xi - \xi_0)^2,$$
$$\tilde{K}_1 = ((3\varepsilon - \lambda) D - 4 d c_0) \frac{\tilde{K}_2}{2},\ v_6 = 0,\ K_6 = 0, \tag{38}$$

two solutions rational in one exponential for the condition (23),

$$(23):\ v = v_0 \left(\tau^2 - \frac{k^2}{3} \right) - \frac{D u_0 c_{\mathrm{z}}}{c^2 c_{\mathrm{NL}} v_0},\ \tau = \frac{k}{2} \tanh \frac{k(\xi - \xi_0)}{2},$$
$$u = \frac{u_0}{v_0} \left(v_{\mathrm{s}} + (\lambda - 3\varepsilon)) \frac{D c_{\mathrm{z}}}{6 c c_{\mathrm{NL}}} \right) - \frac{\tilde{K}_2}{\mu c c_{\mathrm{NL}}} \frac{v_0}{v_{\mathrm{s}}},$$
$$k^2 = - \frac{c_{\mathrm{z}}}{12 c \mu},\ \tilde{K}_1 = (9\varepsilon - 7\lambda) \frac{D \tilde{K}_2}{6},\ v_6 = \frac{v_0}{28} \frac{c_{\mathrm{z}}^3}{(6 c \mu)^3}, \tag{39}$$

and two doubly periodic solutions for the condition (24),

$$(24):\ v = v_0 \wp + \frac{c_{\mathrm{z}}}{4 c c_{\mathrm{NL}}} (12 d c_0 - D(5\lambda - \varepsilon)),$$

$$u = \frac{u_0}{v_0}\left(v_{\mathrm{s}} - (8dc_0 - D(3\lambda - \varepsilon))\frac{c_{\mathrm{z}}}{2cc_{\mathrm{NL}}}\right),$$

$$g_2 = \frac{-32c^2\tilde{K}_2 + c_{\mathrm{z}}^2(12dc_0 - D(3\lambda + \varepsilon))^2}{12(\lambda - \varepsilon)^2\mu^2 D^2 c^2},$$

$$g_3 = \frac{28v_6}{v_0},\ \tilde{K}_1 = \frac{2Du_0}{cv_0}\tilde{K}_2,\ v_6 = \text{arbitrary}. \tag{40}$$

When only one Laurent series for v is enforced, the method therefore yields two solutions of each possible kind (elliptic, rational in one exponential, rational).

Remarks.

(1) If one is interested in finding only the nondegenerate elliptic (genus one) solutions, such as (40), a quicker method to find them all is to combine the present method with the conditions on the residues as explained in [8].

(2) For the rational and trigonometric solutions, the ODE obeyed by u has the type (30) (degree four), e.g. for the rational solution $u = \alpha(\xi - \xi_0)^{-2} + \beta(\xi - \xi_0)^2$,

$$\left(\alpha {u'}^2 - 2u^3 + 8\alpha\beta u\right)^2 - 4\left(u^2 - 4\alpha\beta\right)^3 = 0. \tag{41}$$

This is why u should not be chosen to apply the algorithm.

4.2. *Elliptic and degenerate elliptic solutions, two series*

One of the three necessary conditions (22), (23), (25) must hold true.

The subequation (30) is assumed to be nonfactorizable (nonzero value for the discriminant in ${U'}^2$). In order to determine all the $a_{j,k}$ in subequation (30) and ensure that system (4) is indeed a differential consequence of (30), it is necessary and sufficient to compute the series up to $j = 14$ included, i.e. 8 terms in the series for u and 8 terms in the series for v if one uses the system (4) to perform the computation.

For the condition (25), the found subequation (30) is the product of two factors like (29), the solution would only represent (40), so we discard it.

For either condition (22), (23), one finds the unique subequation

$$c_{\mathrm{z}}(d-b)c_0 = 0 :\ F \equiv \left[{v_{\mathrm{s}}'}^2 + \Gamma(v_{\mathrm{s}} - A)(v_{\mathrm{s}}^2 - B)\right]^2 - \Delta(v_{\mathrm{s}}^2 - B)^3 = 0, \tag{42}$$

in which the constants (A, B, Γ, Δ) take the values

$$A = -\frac{3(\lambda^2 - 1)D}{4cc_{\mathrm{NL}}\lambda}c_{\mathrm{z}},\ B = -6\frac{\lambda^2 - 1}{c_{\mathrm{NL}}^2(\lambda^2 - 2)}\tilde{K}_2,\ \Gamma = \frac{4c_{\mathrm{NL}}\lambda}{3\mu D(\lambda^2 - 1)}, \tag{43}$$

$$\Delta = \left(\frac{4c_{\rm NL}}{3\mu D(\lambda^2-1)}\right)^2,\ \tilde{K}_1 = \left(-2dc_0 + \frac{D\lambda(\lambda^2-3)}{2(\lambda^2-2)}\right)\tilde{K}_2. \tag{44}$$

Because of the parity invariance, the general solution of (42) cannot involve $\wp'$ and is the quotient of two second degree polynomials of $\wp$. It can be obtained by brute force with the Maple command `with(algcurves);Weierstrassform(...)` [7], but the structure of singularities allows a straightforward integration. Indeed, the *a priori* solution

$$v_{\rm s} = \frac{a_2\wp^2 + a_1\wp + a_0}{(\wp - e_1)(\wp - e_2)}, \tag{45}$$

must have two double poles with principal parts $v_{0,\varepsilon}(\xi - \xi_\varepsilon)^{-2}$, therefore the poles e_1, e_2 must be [1] two of the three zeros e_j of $\wp'$, Eq. (35),

$$\wp(\xi_j) = e_j,\ j = 1,2;\ \xi \to \xi_j:\ \wp(\xi) - e_j \sim \frac{1}{2}\left(6e_j^2 - \frac{g_2}{2}\right)(\xi - \xi_j)^2. \tag{46}$$

The result,

$$v_{\rm s} = a_2 + \frac{v_{0,1}}{2}\left(6e_1^2 - \frac{g_2}{2}\right)\frac{1}{\wp - e_1} + \frac{v_{0,2}}{2}\left(6e_2^2 - \frac{g_2}{2}\right)\frac{1}{\wp - e_2}, \tag{47}$$

in which $v_{0,1}, v_{0,2}$ are the two values of $v_{0,\varepsilon}$, Eq. (9), can be written as

$$\begin{aligned}
&\frac{3\mu D}{c_{\rm NL}} v_{\rm s} = 3\lambda e_3 + e_1 - e_2 - (e_1 - e_2)\frac{(2\wp + e_3 + \lambda(e_1 - e_2))(\wp - e_3)}{(\wp - e_1)(\wp - e_2)},\\
&e_1 + e_2 = -e_3 = \frac{c_{\rm z}}{12\mu c},\ (e_1 - e_2)^2 = \frac{2}{3\lambda^2\mu^2(\lambda^2-2)}\tilde{K}_2 + \left(\frac{c_{\rm z}}{4\mu c}\right)^2,\\
&g_2 = \frac{c_{\rm z}}{6\mu c}\frac{\tilde{K}_2}{3\lambda^2\mu^2(\lambda^2-2)} + \left(\frac{c_{\rm z}}{6\mu c}\right)^3,\ g_3 = \frac{2\tilde{K}_2}{3\lambda^2\mu^2(\lambda^2-2)} + 3\left(\frac{c_{\rm z}}{6\mu c}\right)^2.
\end{aligned} \tag{48}$$

This represents two solutions since $e_1 - e_2$ can take two opposite values.

This solution is truely elliptic, except when two of the three roots e_1, e_2, e_3 are equal. Finally, the value of u is easily obtained from the second equation of system (4).

5. Conclusion and perspectives

When the coefficients of the Boussinesq system (1) take generic values as defined at the beginning of section 4, we have obtained in closed form *all* the singlevalued solutions which are either elliptic ("cnoidal") or rational in one exponential (this includes polynomials of tanh and sech) or rational. Naturally, one then must select those solutions which make the internal waves real and bounded.

When the quadratic nonlinearities in the Boussinesq system (1) are insufficient and cubic nonlinearities must be incorporated, the resulting system [10] can be handled similarly. The various Boussinesq systems displayed in [2] could similarly be processed.

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