Bihamiltonian Structure of the Two-component BKP Hierarchy

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Abstract

We employ a Lax pair representation of the two-component BKP hierarchy and construct its bihamiltonian structure with R-matrix techniques.

Key words: BKP hierarchy, hamiltonian structure, R-matrix

1 Introduction

The Kadomtsev-Petviashvili (KP) hierarchy of type B (BKP for short) was introduced in [6, 7], and generalized to multi-component cases by Date, Jimbo, Kashiwara, Miwa [4] in the form of bilinear equations. Among these multi-component integrable systems, the two-component BKP hierarchy is of special interest.

As the two-component BKP hierarchy was proposed, it was shown that the solution space of tau functions of this hierarchy can be regarded as the vacuum orbit in the two-component neutral free fermionic Fock representation of the infinite dimensional Lie algebra D_∞ [5, 14], which corresponds to the infinite Dynkin diagram of type D [15]. The Lie algebra D_∞ can be reduced to the affine Lie algebra $D_n^{(1)}$ under the $(2n-2,2)$-reduction in [5], see also [14, 17]. This reduction leads the two-component BKP hierarchy to a hierarchy that is equivalent with the Kac-Wakimoto hierarchy corresponding to the principal vertex operator realization of the basic representation of $D_n^{(1)}$, the Drinfeld-Sokolov hierarchy associated to the Lie algebra $D_n^{(1)}$ and the zeroth vertex c_0 of its Dynkin diagram, as well as the Givental-Milanov hierarchy satisfied by the total descendant for the D_n

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singularity, see [9, 12, 13, 16, 19, 26] and references therein. Such a reduction is analogous with the one that reduces the KP hierarchy to the nth Gelfand-Dickey hierarchy (see e.g. [8]) that corresponds to the reduction of Lie algebras: $A_\infty \mapsto A_n^{(1)}$. So in this sense to compare the two-component BKP hierarchy with the KP hierarchy would deepen our understanding of integrable hierarchies and relevant theories, such as Jacobi/Prym varieties in algebraic geometry and Landau-Ginzburg Models of topological strings, see e.g. [22, 23].

In this article our aim is to study the two-component BKP hierarchy from the view point of Hamiltonian structures. To our best knowledge, this topic has not been considered in the literature, possibly for the reason that the KP-analogue Lax pair representation of the two-component BKP hierarchy was unknown. Recall that the two-component BKP hierarchy was defined to be the bilinear equation of a single tau function:

$$\begin{aligned} &\mathrm{res}_z z^{-1} X(\mathbf{t};z)\tau(\mathbf{t},\hat{\mathbf{t}})X(\mathbf{t}';-z)\tau(\mathbf{t}',\hat{\mathbf{t}}') \\ &= \mathrm{res}_z z^{-1} X(\hat{\mathbf{t}};z)\tau(\mathbf{t},\hat{\mathbf{t}})X(\hat{\mathbf{t}}';-z)\tau(\mathbf{t}',\hat{\mathbf{t}}'), \end{aligned} \tag{1.1}$$

where $\mathbf{t} = (t_1, t_3, t_5, \cdots)$, $\hat{\mathbf{t}} = (\hat{t}_1, \hat{t}_3, \hat{t}_5, \cdots)$, and X is a vertex operator given by

$$X(\mathbf{t};z) = \exp\left(\sum_{k\in\mathbb{Z}_+^{\mathrm{odd}}} t_k z^k\right) \exp\left(-\sum_{k\in\mathbb{Z}_+^{\mathrm{odd}}} \frac{2}{k\, z^k}\frac{\partial}{\partial t_k}\right).$$

Here the residue of a Laurent series is taken as $\mathrm{res}_z(\sum_{i\in\mathbb{Z}} f_i z^i) = f_{-1}$. In [22] Shiota proposed a scalar Lax representation of the hierarchy (1.1), though this did not draw much attention for it contains pseudo-differential operators with derivations of two spatial variables. Recently, a Lax pair representation of the two-component BKP hierarchy was found by Liu, Zhang and one of the authors [19]. It was shown that the hierarchy (1.1) can be redefined by certain extension of the following Lax equations (see Section 3 below):

$$\frac{\partial P}{\partial t_k} = [(P^k)_+, P], \quad \frac{\partial \hat{P}}{\partial t_k} = [(P^k)_+, \hat{P}], \tag{1.2}$$

$$\frac{\partial P}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, P], \quad \frac{\partial \hat{P}}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, \hat{P}] \tag{1.3}$$

with $k \in \mathbb{Z}_+^{\mathrm{odd}}$, where

$$P = D + \sum_{i\geq 1} u_i D^{-i}, \quad \hat{P} = D^{-1}\hat{u}_{-1} + \sum_{i\geq 1} \hat{u}_i D^i \text{ with } D = \frac{\mathrm{d}}{\mathrm{d}x}$$

are pseudo-differential operators such that $P^* = -DPD^{-1}$, $\hat{P}^* = -D\hat{P}D^{-1}$. Note that the first equation in (1.2) is just the Lax representation of the BKP hierarchy appearing in [6]. Our arguments will be based on the Lax pair representation (1.2), (1.3) of the two-component BKP hierarchy.

One can see that the expression (1.2), (1.3) is very similar with the Lax pair representation of the two-dimensional Toda hierarchy [25]. Since the latter hierarchy carries a tri-Hamiltonian structure [1], following the idea of [1] we want to use the R-matrix theory to construct a bihamiltonian structure of the hierarchy (1.2), (1.3).

We are also motivated by the recent work [2], in which Carlet, Dubrovin and Mertens constructed an infinite-dimensional Frobenius manifold underlying the two-dimensional Toda hierarchy. Due to the similarity of the Lax representations mentioned above, we expect that there also exists an infinite dimensional Frobenius manifold that underlies the two-component BKP hierarchy. A hint is that the potential F of the dispersionless two-component BKP hierarchy (in the notion of [23], namely the dispersionless limit of the logarithm of the tau function, see Section 3 below) was discovered to satisfy certain infinite-dimensional WDVV-type associativity equations [3]. While in the finite-dimensional case, the concept of Frobenius manifolds [10] can be regarded as a geometric description of the WDVV equations, and associated to certain nondegenerate Frobenius manifold there lives a Poisson pencil so that a bihamiltonian hierarchy can be constructed [11]. We hope that this article and follow-up work might help to understand the theory of infinite-dimensional manifolds.

This article is arranged as follows. In next section we recall the definition and some properties of pseudo-differential operators introduced in [19], and in Section 3 we recall the Lax pair representation of the two-component hierarchy. In Sections 4 and 5, an R-matrix will be used to construct Poisson brackets on an algebra of pseudo-differential operators, and after appropriate reductions of the Poisson brackets we obtain a bihamiltonian structure of the two-component BKP hierarchy. Finally we compute the dispersionless limit of this bihamiltonian structure in Section 6, and give some remarks in Section 7.

2 Pseudo-differential operators

For preparation we first recall the notion of pseudo-differential operators over a differential algebra with a gradation as introduced in [19].

Let $\mathcal{A}$ be a commutative ring with unity, and $D : \mathcal{A} \to \mathcal{A}$ be a derivation. The algebra of usual pseudo-differential operators is

$$\mathcal{D}^- = \left\{ \sum_{i<\infty} f_i D^i \mid f_i \in \mathcal{A} \right\}. \tag{2.1}$$

This algebra is topologically complete with a topological basis given by the following filtration:

$$\cdots \subset \mathcal{D}^-_{(d-1)} \subset \mathcal{D}^-_{(d)} \subset \mathcal{D}^-_{(d+1)} \subset \cdots, \quad \mathcal{D}^-_{(d)} = \left\{ \sum_{i\le d} f_i D^i \mid f_i \in \mathcal{A} \right\},$$

and in this algebra two elements are multiplied as series of the following product of monomials:

$$f D^i \cdot g D^j = \sum_{r\ge 0} \binom{i}{r} f\, D^r(g)\, D^{i+j-r}, \quad f, g \in \mathcal{A}. \tag{2.2}$$

Now assume there is a gradation on $\mathcal{A}$ such that

$$\mathcal{A} = \prod_{i\ge 0} \mathcal{A}_i, \quad D : \mathcal{A}_i \to \mathcal{A}_{i+1}, \quad \mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j},$$

and consider the linear space

$$\mathcal{D} = \left\{ \sum_{i\in\mathbb{Z}} f_i D^i \mid f_i \in \mathcal{A} \right\}.$$

Obviously $\mathcal{D}^- \subset \mathcal{D}$.

For any $k \in \mathbb{Z}$, denote by $\mathcal{D}_k$ the set of homogeneous operators with degree k in $\mathcal{D}^-$, i.e.,

$$\mathcal{D}_k = \left\{ \sum_{i\le k} f_i D^i \mid f_i \in \mathcal{A}_{k-i} \right\}.$$

Let $\mathcal{D}^+$ be a subspace of $\mathcal{D}$ that reads

$$\mathcal{D}^+ = \bigcup_{d\in\mathbb{Z}} \mathcal{D}^+_{(d)}, \quad \mathcal{D}^+_{(d)} = \prod_{k\ge d} \mathcal{D}_k, \tag{2.3}$$

and $\mathcal{D}^+$ have a topological basis given by the filtration

$$\cdots \supset \mathcal{D}^+_{(d-1)} \supset \mathcal{D}^+_{(d)} \supset \mathcal{D}^+_{(d+1)} \supset \cdots.$$

In fact, every element $A \in \mathcal{D}^+$ has the following normal expansion [19]

$$A = \sum_{i\in\mathbb{Z}} \left(\sum_{j\ge \max\{0, m-i\}} a_{i,j} \right) D^i, \quad a_{i,j} \in \mathcal{A}_j$$

with some integer m. Note that $\mathcal{D}_k \cdot \mathcal{D}_l \subset \mathcal{D}_{k+l}$ according to the multiplication defined by (2.2), then this multiplication can be naturally extended to $\mathcal{D}^+$ such that $\mathcal{D}^+$ becomes an associative algebra.

Definition 2.1 ([19]) *Elements of $\mathcal{D}^-$ (resp. $\mathcal{D}^+$) are called pseudo-differential operators of the first type (resp. the second type) over $\mathcal{A}$. The intersection of $\mathcal{D}^-$ and $\mathcal{D}^+$ in $\mathcal{D}$ is denoted by*

$$\mathcal{D}^b = \mathcal{D}^- \cap \mathcal{D}^+,$$

and its elements are called bounded pseudo-differential operators.

Sometimes to indicate the algebra $\mathcal{A}$ and the derivation D, we will use the notations $\mathcal{D}^\pm(\mathcal{A}, D)$ instead of $\mathcal{D}^\pm$.

Pseudo-differential operators of the second type have similar properties with the operators in $\mathcal{D}^-$. For any operator

$$A = \sum_{i\in\mathbb{Z}} f_i D^i \in \mathcal{D}^\pm, \tag{2.4}$$

its positive part, negative part, residue and adjoint operator are defined to be respectively

$$A_+ = \sum_{i\geq 0} f_i D^i, \quad A_- = \sum_{i<0} f_i D^i, \tag{2.5}$$

$$\operatorname{res} A = f_{-1}, \quad A^* = \sum_{i\in\mathbb{Z}} (-D)^i \cdot f_i. \tag{2.6}$$

Note that the projections given in (2.5) induce the following decompositions of spaces

$$\mathcal{D}^\pm = (\mathcal{D}^\pm)_+ \oplus (\mathcal{D}^\pm)_-, \tag{2.7}$$

particularly one easily sees that

$$(\mathcal{D}^-)_+ \subset \mathcal{D}^b, \quad (\mathcal{D}^+)_- \subset \mathcal{D}^b. \tag{2.8}$$

An element A of $(\mathcal{D}^\pm)_+$ is call a *differential operator*, and it is denoted by $A(f)$ the action of a differential operator A on $f \in \mathcal{A}$.

We need more notations below. Elements of the quotient space $\mathcal{F} = \mathcal{A}/D(\mathcal{A})$ are called *local functionals*, which are denoted as

$$\int f \, \mathrm{d}x = f + D(\mathcal{A}), \quad f \in \mathcal{A}.$$

Introduce a map

$$\langle\ \rangle : \ \mathcal{D} \to \mathcal{F}, \quad A \mapsto \langle A \rangle = \int \operatorname{res} A \, \mathrm{d}x. \tag{2.9}$$

Then the pairing

$$\langle A, B \rangle = \langle AB \rangle \tag{2.10}$$

defines an inner product on each of $\mathcal{D}^\pm$.

Given any subspace $\mathcal{S} \subset \mathcal{D}^\pm$, we denote by $\mathcal{S}^*$ the dual space of $\mathcal{S}$ (c.f. the notation of adjoint operators). Via the above inner product, we have the following identification of dual spaces

$$(\mathcal{D}^\pm)^* = \mathcal{D}^\pm. \tag{2.11}$$

Consider the decompositions (2.7), it is easy to see that

$$\left((\mathcal{D}^\pm)_\pm\right)^* = (\mathcal{D}^\pm)_\mp.$$

We decompose $\mathcal{D}^\pm$ in another way as

$$\mathcal{D}^\pm = \mathcal{D}^\pm_0 \oplus \mathcal{D}^\pm_1, \tag{2.12}$$

where

$$\mathcal{D}^\pm_\nu = \left\{ A \in \mathcal{D}^\pm \mid A^* = (-1)^\nu A \right\}, \quad \nu = 0, 1.$$

Since $\langle A \rangle = -\langle A^* \rangle$ for $A \in \mathcal{D}^\pm$, then the dual subspaces of $\mathcal{D}^\pm_\nu$ read

$$(\mathcal{D}^\pm_\nu)^* = \mathcal{D}^\pm_{1-\nu}, \quad \nu = 0, 1. \tag{2.13}$$

For more details on properties of pseudo-differential operators one can refer to [8, 19].

3 The two-component BKP hierarchy

The two types of pseudo-differential operators serve in [19] to give a scalar Lax pair representation of the two-component BKP hierarchy, which is reviewed as follows.

Let $\tilde{M}$ be an infinite-dimensional manifold with local coordinates

$$(a_1, a_3, a_5, \dots, b_1, b_3, b_5, \dots),$$

and $\tilde{\mathcal{A}}$ be the algebra of differential polynomials on $\tilde{M}$:

$$\tilde{\mathcal{A}} = C^\infty(\tilde{M})[[a_k^{(s)}, b_k^{(s)} \mid k \in \mathbb{Z}_+^{\rm odd}, s \geq 1]].$$

We assign a gradation on $\tilde{\mathcal{A}}$ by

$$\deg f = 0 \text{ for } f \in C^\infty(\tilde{M}), \quad \deg a_k^{(s)} = \deg b_k^{(s)} = s$$

which make $\tilde{\mathcal{A}}$ a topologically complete algebra:

$$\tilde{\mathcal{A}} = \prod_{i \geq 0} \tilde{\mathcal{A}}_i, \quad \tilde{\mathcal{A}}_i \cdot \tilde{\mathcal{A}}_j \subset \tilde{\mathcal{A}}_{i+j}.$$

Note that this gradation is induced from the derivation

$$D:\tilde{\mathcal{A}}\to\tilde{\mathcal{A}},\quad D=\sum_{s\ge 0}\sum_{k\in\mathbb{Z}_+^{\rm odd}}\left(a_k^{(s+1)}\frac{\partial}{\partial a_k^{(s)}}+b_k^{(s+1)}\frac{\partial}{\partial b_k^{(s)}}\right)$$

with $a_k^{(0)}=a_k$, $b_k^{(0)}=b_k$. So one can define the algebras $\tilde{\mathcal{D}}^{\pm}=\mathcal{D}^{\pm}(\tilde{\mathcal{A}},D)$ of pseudo-differential operators as was done in last section.

Introduce two operators

$$\Phi=1+\sum_{i\ge1}a_iD^{-i}\in\tilde{\mathcal{D}}^-,\quad \Psi=1+\sum_{i\ge1}b_iD^{i}\in\tilde{\mathcal{D}}^+, \tag{3.1}$$

where $a_2,a_4,a_6,\dots,b_2,b_4,b_6,\dots\in\tilde{\mathcal{A}}$ are determined by the following conditions

$$\Phi^*=D\Phi^{-1}D^{-1},\quad \Psi^*=D\Psi^{-1}D^{-1}. \tag{3.2}$$

Then the two-component BKP hierarchy (1.1) can be redefined to be

$$\frac{\partial\Phi}{\partial t_k}=-(P^k)_-\Phi,\quad \frac{\partial\Psi}{\partial t_k}=\left((P^k)_+-\delta_{k1}\hat{P}^{-1}\right)\Psi, \tag{3.3}$$

$$\frac{\partial\Phi}{\partial\hat{t}_k}=-(\hat{P}^k)_-\Phi,\quad \frac{\partial\Psi}{\partial\hat{t}_k}=(\hat{P}^k)_+\Psi, \tag{3.4}$$

where $k\in\mathbb{Z}_+^{\rm odd}$, and the operators P, $\hat{P}$ read

$$P=\Phi D\Phi^{-1}\in\tilde{\mathcal{D}}^-,\quad \hat{P}=\Psi D^{-1}\Psi^{-1}\in\tilde{\mathcal{D}}^+. \tag{3.5}$$

It was shown that the operators P and $\hat{P}$ have the following expressions:

$$P=D+\sum_{i\ge1}u_iD^{-i},\quad \hat{P}=D^{-1}\hat{u}_{-1}+\sum_{i\ge1}\hat{u}_iD^{i}, \tag{3.6}$$

with $\hat{u}_{-1}=(\Psi^{-1})^*(1)$, and they satisfy

$$P^*=-DPD^{-1},\quad \hat{P}^*=-D\hat{P}D^{-1}, \tag{3.7}$$

which implies

$$(P^k)_+(1)=0,\quad (\hat{P}^k)_+(1)=0,\quad k\in\mathbb{Z}_+^{\rm odd}. \tag{3.8}$$

Observe that the coefficients of P and $\hat{P}$ are elements of the quotient algebra $\tilde{\mathcal{A}}/\mathcal{I}$, where $\mathcal{I}$ is an ideal of $\tilde{\mathcal{A}}$ defined by the equations (3.5), moreover, among these coefficients only the ones with odd subscript are independent, while the others are determined by the conditions (3.7).

Assume that

$$\mathbf{u}=(u_1,u_3,\dots,\hat{u}_{-1},\hat{u}_1,\hat{u}_3,\dots) \tag{3.9}$$

serves as a coordinate of some infinite-dimensional manifold M. Let $\mathcal{A}$ be the algebra of differential polynomials on M:

$$\mathcal{A} = C^\infty(M)[[\mathbf{u}^{(s)} \mid s \geq 1]],$$

namely $\mathcal{A} = \tilde{\mathcal{A}}/\mathcal{I}$. Similarly as above, one can assign a gradation to $\mathcal{A}$ that is induced from the derivation

$$D : \mathcal{A} \to \mathcal{A}, \quad D = \sum_{s\geq 0} \mathbf{u}^{(s+1)} \cdot \frac{\partial}{\partial \mathbf{u}^{(s)}}$$

with $\mathbf{u}^{(0)} = \mathbf{u}$, and define the algebras $\mathcal{D}^\pm = \mathcal{D}^\pm(\mathcal{A}, D)$ of pseudo-differential operators over $\mathcal{A}$. Obviously $P \in \mathcal{D}^-$, $\hat{P} \in \mathcal{D}^+$. Then it is easy to see that the two-component BKP hierarchy (3.3), (3.4) can be restricted to the algebra $\mathcal{A}$ as follows:

$$\frac{\partial P}{\partial t_k} = [(P^k)_+, P], \quad \frac{\partial \hat{P}}{\partial t_k} = [(P^k)_+, \hat{P}], \tag{3.10}$$

$$\frac{\partial P}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, P], \quad \frac{\partial \hat{P}}{\partial \hat{t}_k} = [-(\hat{P}^k)_-, \hat{P}] \tag{3.11}$$

with $k \in \mathbb{Z}_+^{\mathrm{odd}}$.

In the present paper we regard the two-component BKP hierarchy as the evolutionary equations (3.10), (3.11) defined on the algebra $\mathcal{A}$. In fact, the hierarchy (3.10), (3.11) possesses a tau function $\tau = \tau(\mathbf{t}, \hat{\mathbf{t}})$ that solves the bilinear equation (1.1). More exactly, this tau function is defined by [19]

$$\omega = \mathrm{d}(2\,\partial_x \log \tau) \quad \text{with} \quad x = t_1, \tag{3.12}$$

where ω is a closed 1-form:

$$\omega = \sum_{k\in\mathbb{Z}_+^{\mathrm{odd}}} (\operatorname{res} P^k\, \mathrm{d}t_k + \operatorname{res} \hat{P}^k\, \mathrm{d}\hat{t}_k).$$

Remark 3.1 The dispersionless limit of the flows (3.10), (3.11) first exists in [23], where Takasiki also considered the dispersionless limit of the logarithm of the tau function as given in (3.12). Inspired by [23], Chen and Tu [3] discovered that the leading term of $\log \tau$ solves an infinite-dimensional associativity equation of WDVV type.

4 R-matrix and pseudo-differential operators

To show that the two-component BKP hierarchy (3.10), (3.11) possesses a bihamiltonian structure, we need to construct a Poisson pencil for it.

The method is to use the standard R-matrix theory and introduce Poisson brackets on a Lie algebra (see [21, 18, 20] and references therein), then restrict the Poisson brackets to certain submanifold of the Lie algebra. Our approach is similar with that used by Carlet [1] for the two-dimensional Toda hierarchy.

We first recall the R-matrix formalism. Let $\mathfrak{g}$ be a Lie algebra, and $R : \mathfrak{g} \to \mathfrak{g}$ be a linear transformation. Then R is called an R-matrix [21] on $\mathfrak{g}$ if it defines a Lie bracket by

$$[X, Y]_R = [R(X), Y] + [X, R(Y)], \quad X, Y \in \mathfrak{g}. \tag{4.1}$$

A sufficient condition for a transformation R being an R-matrix is that R solves the modified Yang-Baxter equation [21]

$$[R(X), R(Y)] - R([X, Y]_R) = -[X, Y] \tag{4.2}$$

for all X, $Y \in \mathfrak{g}$.

Assume that $\mathfrak{g}$ is an associative algebra, with the Lie bracket defined naturally by commutators, and there is a map $\langle\ \rangle : \mathfrak{g} \to \mathbb{C}$ that defines a non-degenerate symmetric invariant bilinear form (inner product) $\langle\,,\rangle$ by

$$\langle X, Y\rangle = \langle XY\rangle = \langle YX\rangle, \quad X, Y \in \mathfrak{g}.$$

Via this inner product one can identify $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$. The tangent and the cotangent bundles of $\mathfrak{g}$ are denoted by $T\mathfrak{g}$ and $T^*\mathfrak{g}$ respectively, with fibers at any point $A \in \mathfrak{g}$ of the form $T_A\mathfrak{g} = \mathfrak{g}$, $T_A^*\mathfrak{g} = \mathfrak{g}^*$.

Let R^* be the adjoint transformation of R with respect to the above inner product. We introduce the notations of the symmetric and the anti-symmetric parts of R respectively as

$$R_s = \frac{1}{2}(R + R^*), \quad R_a = \frac{1}{2}(R - R^*).$$

The R-matrix formalism is briefly stated as follows. Given an R-matrix $R : \mathfrak{g} \to \mathfrak{g}$ that satisfies certain conditions, there define three compatible Poisson brackets on $\mathfrak{g}$, say, the linear, the quadratic and the cubic brackets in the notion of [18, 20].

In particular, let us recall the quadratic bracket, which will be used to construct a Poisson pencil for the two-component BKP hierarchy.

Lemma 4.1 ([18, 20]) *Let f, g be two arbitrary smooth functions on $\mathfrak{g}$, and $\nabla f, \nabla g \in T_A^*\mathfrak{g}$ be their gradients at any point $A \in \mathfrak{g}$. Given a linear transformation $R : \mathfrak{g} \to \mathfrak{g}$, if both R and its anti-symmetric part R_a satisfy the modified Yang-Baxter equation* (4.2)*, then the quadratic bracket*

$$\{f, g\}(A) = \frac{1}{4}\big(\langle [A, \nabla f], R(A\nabla g + \nabla g \cdot A)\rangle - \langle [A, \nabla g], R(A\nabla f + \nabla f \cdot A)\rangle\big) \tag{4.3}$$

defines a Poisson bracket on $\mathfrak{g}$.

Note that the bracket (4.3) can be rewritten as

$$\{f,g\}(A) = \langle \nabla f, \mathcal{P}_A(\nabla g)\rangle,$$

where $\mathcal{P} : T^*\mathfrak{g} \to T\mathfrak{g}$ is a Poisson tensor given by

$$\mathcal{P}_A(\nabla g) = -\frac{1}{4}[A, R(A\nabla g + \nabla g \cdot A)] - \frac{1}{4}AR^*([A,\nabla g]) - \frac{1}{4}R^*([A,\nabla g])A,$$

namely,

$$\mathcal{P}_A(\nabla g) = -\frac{1}{2}A\big(R_s(A\nabla g) + R_a(\nabla g \cdot A)\big) + \frac{1}{2}\big(R_a(A\nabla g) + R_s(\nabla g \cdot A)\big)A. \tag{4.4}$$

Henceforth we take $\mathfrak{g}$ to be the algebra

$$\mathfrak{D} = \mathcal{D}^- \times \mathcal{D}^+,$$

where $\mathcal{D}^-$ and $\mathcal{D}^+$ are the sets of pseudo-differential operators of the first type and the second type over some differential algebra $\mathcal{A}$ as defined in Section 2. In $\mathfrak{D}$ the elements read $\mathbf{X} = (X, \hat{X})$, and the operations are defined diagonally as

$$(X,\hat{X}) + (Y,\hat{Y}) = (X+Y, \hat{X}+\hat{Y}), \quad (X,\hat{X})(Y,\hat{Y}) = (XY, \hat{X}\hat{Y}).$$

So $\mathfrak{D}$ is indeed an associative algebra. It is easy to see that $\mathfrak{D}$ is equipped with an inner product define by

$$\langle (X,\hat{X}), (Y,\hat{Y})\rangle = \langle (X,\hat{X})(Y,\hat{Y})\rangle = \langle X, Y\rangle + \langle \hat{X}, \hat{Y}\rangle,$$

see (2.9), (2.10). Via this inner product we have the identification of dual spaces as above:

$$\mathfrak{D}^* = (\mathcal{D}^-)^* \times (\mathcal{D}^+)^* = \mathcal{D}^- \times \mathcal{D}^+ = \mathfrak{D}.$$

Consider an R-matrix on $\mathfrak{D}$ that has the same form with the one used in [1] to construct the tri-Hamiltonian structure of the two-dimensional Toda hierarchy, namely,

$$R : \mathfrak{D} \to \mathfrak{D}, \quad (X,\hat{X}) \mapsto (X_+ - X_- + 2\hat{X}_-, \hat{X}_- - \hat{X}_+ + 2X_+). \tag{4.5}$$

As done in [1], since $R = \Pi - \tilde{\Pi}$, where

$$\Pi(X,\hat{X}) = (X_+ + \hat{X}_-, \hat{X}_- + X_+), \quad \tilde{\Pi}(X,\hat{X}) = (X_- - \hat{X}_-, \hat{X}_+ - X_+)$$

are two projections of $\mathfrak{D}$ onto its subalgebras, more exactly,

$$\Pi\mathfrak{D} = \{(X,X) \mid X \in \mathcal{D}^b\}, \quad \tilde{\Pi}\mathfrak{D} = (\mathcal{D}^-)_- \times (\mathcal{D}^+)_+,$$

$$\Pi^2 = \Pi, \quad \tilde{\Pi}^2 = \tilde{\Pi}, \quad \tilde{\Pi}\,\Pi = 0 = \Pi\,\tilde{\Pi}, \quad \Pi + \tilde{\Pi} = \mathrm{id},$$

then transformation R satisfies the modified Yang-Baxter equation (4.2) and is indeed an R-matrix on $\mathfrak{D}$. Moreover, according to the inner product on $\mathfrak{D}$ introduced above, the adjoint transformation of R reads

$$R^* : \mathfrak{D} \to \mathfrak{D}, \quad (X, \hat{X}) \mapsto (X_- - X_+ + 2\hat{X}_-, \hat{X}_+ - \hat{X}_- + 2X_+),$$

hence the symmetric and anti-symmetric parts of R are given by

$$R_s(X, \hat{X}) = 2(\hat{X}_-, X_+), \quad R_a(X, \hat{X}) = (X_+ - X_-, \hat{X}_- - \hat{X}_+). \tag{4.6}$$

Observe that R_a can be expressed as the difference of two projections onto subalgebras of $\mathfrak{D}$, hence R_a also solves the Yang-Baxter equation (4.2). Thus the R-matrix given in (4.5) fulfills the condition of Lemma 4.1.

We regard $\mathfrak{D}$ as an infinite-dimensional manifold, whose coordinate is given by the coefficients of the general expression of its elements

$$\mathbf{A} = \left(\sum_{i\in\mathbb{Z}} w_i D^i, \sum_{i\in\mathbb{Z}} \hat{w}_i D^i \right) \in \mathfrak{D}, \tag{4.7}$$

and let the set $\mathcal{F}$ of local functionals over the differential algebra $\mathcal{A}$ (see Section 2) play the role of $C^\infty(\mathfrak{g})$. For any $F \in \mathcal{F}$, the variational gradient of F at the point $\mathbf{A}$ given in (4.7) is defined to be

$$\frac{\delta F}{\delta \mathbf{A}} = \left(\sum_{i\in\mathbb{Z}} D^{-i-1} \frac{\delta F}{\delta w_i(x)}, \sum_{i\in\mathbb{Z}} D^{-i-1} \frac{\delta F}{\delta \hat{w}_i(x)} \right),$$

where $\delta F/\delta w(x) = \sum_{j\ge 0} (-D)^j \left(\partial F/\partial w^{(j)}\right)$. Note that $\delta F/\delta \mathbf{A}$ is not contained in $\mathfrak{D}^* = \mathfrak{D}$ in general, so to go forward we need to do some restriction.

It shall be indicated that, in this paper we only consider functionals with variational gradients lying in $\mathfrak{D}$. We denote by $\mathcal{F}_0$ the set of such functionals.

Now one can use Lemma 4.1 and the formulae (4.4), (4.6) to obtain the following result.

Lemma 4.2 *Let F and G be two arbitrary functionals in $\mathcal{F}_0$. On the algebra $\mathfrak{D}$ there is a quadratic Poisson bracket*

$$\{F, G\}(\mathbf{A}) = \left\langle \frac{\delta F}{\delta \mathbf{A}}, \mathcal{P}_{\mathbf{A}} \left(\frac{\delta G}{\delta \mathbf{A}} \right) \right\rangle, \quad \mathbf{A} = (A, \hat{A}) \in \mathfrak{D}, \tag{4.8}$$

where the Poisson tensor $\mathcal{P} : T\mathfrak{D}^* \to T\mathfrak{D}$ is defined by

$$\begin{aligned} \mathcal{P}_{(A,\hat{A})}(X, \hat{X}) = \big(& A(XA)_- - (AX)_- A - A(\hat{A}\hat{X})_- + (\hat{X}\hat{A})_- A, \\ & \hat{A}(\hat{X}\hat{A})_+ - (\hat{A}\hat{X})_+ \hat{A} - \hat{A}(AX)_+ + (XA)_+ \hat{A} \big). \end{aligned} \tag{4.9}$$

Aiming at Hamiltonian structures of the two-component BKP hierarchy, we need to reduce the Poisson structure (4.9) to an appropriate submanifold of $\mathfrak{D}$. Recall the decompositions (2.12), let us decompose the space $\mathfrak{D}$ as

$$\mathfrak{D} = \mathfrak{D}_0 \oplus \mathfrak{D}_1, \tag{4.10}$$

where $\mathfrak{D}_\nu = \mathcal{D}_\nu^- \times \mathcal{D}_\nu^+$ for $\nu = 0, 1$. Then the subspaces $\mathfrak{D}_0$ and $\mathfrak{D}_1$ are dual to each other with respect to the inner product on $\mathfrak{D}$ defined above. So for any $\mathbf{A} \in \mathfrak{D}_\nu$ we have $T^*_{\mathbf{A}}\mathfrak{D}_\nu = (\mathfrak{D}_\nu)^* = \mathfrak{D}_{1-\nu}$ for $\nu = 0, 1$. It is straightforward to verify the following lemma.

Lemma 4.3 *The Poisson structure* (4.9) *on* $\mathfrak{D}$ *can be properly restricted to each of its subspaces* $\mathfrak{D}_0$ *and* $\mathfrak{D}_1$.

5 Bihamilonian representation of the two-component BKP hierarchy

In this section, we are to find a submanifold of $\mathfrak{D}$ where lives the Poisson pencil for the two-component BKP hierarchy, then after a further reduction of the Poisson structure constructed in last section we express the hierarchy (3.10), (3.11) to the form of Hamiltonian equations.

Recall the operators $P \in \mathcal{D}^-$, $\hat{P} \in \mathcal{D}^+$ given in (3.5), we let

$$\mathbf{A} = (P^2 D^{-1}, D\hat{P}^2). \tag{5.1}$$

It is easy to see that $\mathbf{A} \in \mathfrak{D}_1$ (see (4.10)), and $\mathbf{A} = (A, \hat{A})$ has the following expression:

$$A = P^2 D^{-1} = D + \sum_{i\geq 0}(v_{-i}D^{-2i-1} + f_{-i}D^{-2i-2}), \tag{5.2}$$

$$\hat{A} = D\hat{P}^2 = \rho D^{-1}\rho + \sum_{i\geq 1}(\hat{v}_i D^{2i-1} + \hat{f}_i D^{2i-2}), \quad \rho = \hat{u}_{-1}. \tag{5.3}$$

Denote $\hat{v}_0 = \rho^2$, and $\mathbf{v} = (v_0, v_{-1}, \dots, \hat{v}_0, \hat{v}_1, \dots)$. It is easy to see that the coordinate $\mathbf{v}$ is related to $\mathbf{u}$ given in (3.9) by a Miura-type transformation, and that f_{-i}, $\hat{f}_i$ are linear functions of derivatives of $\mathbf{v}$ determined by the symmetry property $(A^*, \hat{A}^*) = -(A, \hat{A})$. Hence the flows of the hierarchy (3.10), (3.11) can be described in the coordinate $\mathbf{v}$.

Given any local functional $F \in \mathcal{F}_0$ (remind the notation $\mathcal{F}_0$ in last section), its variational gradient with respect to $\mathbf{A}$, say $\delta F/\delta \mathbf{A}$, is defined to be $\mathbf{X} = (X, \hat{X}) \in \mathfrak{D}$ with

$$X = \frac{1}{2}\sum_{i\geq 0}\left(\frac{\delta F}{\delta v_{-i}(x)}D^{2i} + D^{2i}\frac{\delta F}{\delta v_{-i}(x)}\right), \tag{5.4}$$

$$\hat{X} = \frac{1}{2}\sum_{i\geq 0}\left(\frac{\delta F}{\delta \hat{v}_i(x)}D^{-2i} + D^{-2i}\frac{\delta F}{\delta \hat{v}_i(x)}\right). \tag{5.5}$$

Or in a coordinate-free way, $\delta F/\delta \mathbf{A} = \mathbf{X}$ can be defined by

$$\delta F = \langle \mathbf{X}, \delta \mathbf{A}\rangle, \quad \mathbf{X} \in \mathfrak{D}_0. \tag{5.6}$$

Note that in the latter definition, the variational gradient is determined up to a kernel part $\mathbf{Z} = (Z, \hat{Z}) \in \mathfrak{D}_0$ such that

$$Z_+ = 0, \quad \hat{Z}_- = 0, \quad \hat{Z}_+(\rho) = 0. \tag{5.7}$$

Let us consider the coset $(D, 0) + \mathcal{U}$ consisting of operators of the form (5.1), where

$$\mathcal{U} = (\mathcal{D}_1^-)_- \times \big((\mathcal{D}_1^+)_+ \times \mathcal{M}\big), \quad \mathcal{M} = \{\rho D^{-1}\rho \mid \rho \in \mathcal{A}\}. \tag{5.8}$$

Here $\mathcal{M}$ is regarded as a 1-dimensional manifold with coordinate ρ, and this manifold has tangent spaces of the form

$$T_\rho\mathcal{M} = \{\rho D^{-1} f + f D^{-1}\rho \mid f \in \mathcal{A}\}.$$

So the tangent bundle, denoted by $T\mathcal{U}$, of the coset $(D, 0) + \mathcal{U}$ has fibers

$$T_{\mathbf{A}}\mathcal{U} = (\mathcal{D}_1^-)_- \times \big((\mathcal{D}_1^+)_+ \times T_\rho\mathcal{M}\big), \quad \mathbf{A} \in (D, 0) + \mathcal{U}, \tag{5.9}$$

while the cotangent bundle $T^*\mathcal{U}$ of $(D, 0) + \mathcal{U}$ is composed of

$$T^*_{\mathbf{A}}\mathcal{U} = (\mathcal{D}_0^-)_+ \times \big((\mathcal{D}_0^+)_- \times T^*_\rho\mathcal{M}\big), \quad T^*_\rho\mathcal{M} = \mathcal{A}. \tag{5.10}$$

From (5.4), (5.5) one sees that $\delta F/\delta\mathbf{A} \in T^*_{\mathbf{A}}\mathcal{U}$ for any $F \in \mathcal{F}_0$. Now we are ready to do the desired reduction of the Poisson structure.

Lemma 5.1 *The map*

$$\mathcal{P} : T^*\mathcal{U} \to T\mathcal{U} \tag{5.11}$$

defined by the formula (4.9) *is a Poisson tensor on the coset* $(D, 0) + \mathcal{U}$ *that consists of operators of the form* (5.1).

Proof. We only need to show that the map defined by (4.9) admits the restriction to the coset $(D, 0) + \mathcal{U}$, i.e., the following map is well defined:

$$\mathcal{P}_{\mathbf{A}} : T^*_{\mathbf{A}}\mathcal{U} \to T_{\mathbf{A}}\mathcal{U}, \quad \mathbf{A} \in (D, 0) + \mathcal{U}. \tag{5.12}$$

Assume $\mathbf{X} = (X, \hat{X}) \in T^*_{\mathbf{A}}\mathcal{U} \subset \mathfrak{D}_0$. From Lemma 4.3 one has $\mathcal{P}_{\mathbf{A}}(X) \in \mathfrak{D}_1$. Moreover, it is easy to see that the first component of $\mathcal{P}_{\mathbf{A}}(X)$ belongs to $(\mathcal{D}_1^-)_-$. On the other hand, for any $\hat{Y} \in (\mathcal{D}^+)_+$ we have

$$(\hat{A}\hat{Y} + \hat{Y}^*\hat{A})_- = (\rho D^{-1}\rho\hat{Y} + \hat{Y}^*\rho D^{-1}\rho)_-$$

$$
\begin{aligned}
&= -(\hat{Y}^*\rho D^{-1}\rho)^*_- + \hat{Y}^*(\rho)D^{-1}\rho \\
&= \rho D^{-1}\hat{Y}^*(\rho) + \hat{Y}^*(\rho)D^{-1}\rho \in T_\rho\mathcal{M},
\end{aligned}
$$

then by taking $\hat{Y} = (\hat{X}\hat{A})_+, (AX)_+$ it follows that the second component of $\mathcal{P}_{\mathbf{A}}(\mathbf{X})$ lies in $(\mathcal{D}_1^+)_+ \times T_\rho\mathcal{M}$. Thus $\mathcal{P}_{\mathbf{A}}(\mathbf{X}) \in T_{\mathbf{A}}\mathcal{U}$, i.e., the map (5.12) is well defined. The lemma is proved. □

Remark 5.2 The proof of this lemma is the simplest case of the Dirac reduction procedure for Poisson tensors, see e.g. [20]. In fact, one can express the manifolds $\mathfrak{D}_1$ and $\mathfrak{D}_1^*$ as

$$
\mathfrak{D}_1 = \mathcal{U} \times \mathcal{V} = T_{\mathbf{A}}\mathcal{U} \times \mathcal{V}_{\mathbf{A}}, \quad \mathfrak{D}_1^* = \mathfrak{D}_0 = T^*_{\mathbf{A}}\mathcal{U} \times \mathcal{V}^*_{\mathbf{A}}, \tag{5.13}
$$

where

$$
\begin{aligned}
&\mathcal{V} = \mathcal{V}_{\mathbf{A}} = (\mathcal{D}_1^-)_+ \times \mathcal{N}, \quad \mathcal{N} = \{X \in (\mathcal{D}_1^+)_- \mid \operatorname{res} X = 0\}, \\
&\mathcal{V}^*_{\mathbf{A}} = (\mathcal{D}_0^-)_- \times (T^*_\rho)^\perp\mathcal{M}, \quad (T^*_\rho)^\perp\mathcal{M} = \{\hat{Y} \in (\mathcal{D}_0^+)_+ \mid \hat{Y}(\rho) = 0\}.
\end{aligned}
$$

Similar as the proof of Lemma 5.1, one can show that the map

$$
\mathcal{P}_{\mathbf{A}} = \begin{pmatrix} \mathcal{P}^{\mathcal{U}\mathcal{U}}_{\mathbf{A}} & \mathcal{P}^{\mathcal{U}\mathcal{V}}_{\mathbf{A}} \\ \mathcal{P}^{\mathcal{V}\mathcal{U}}_{\mathbf{A}} & \mathcal{P}^{\mathcal{V}\mathcal{V}}_{\mathbf{A}} \end{pmatrix} : T^*_{\mathbf{A}}\mathcal{U} \times \mathcal{V}^*_{\mathbf{A}} \to T_{\mathbf{A}}\mathcal{U} \times \mathcal{V}_{\mathbf{A}}
$$

defined by (4.9) is diagonal. Hence from Lemma 4.3 it follows that the map (4.9) gives a Poisson tensor on the coset $(D, 0) + \mathcal{U} \subset \mathcal{D}_1$.

Lemma 5.3 *On the coset $(D, 0) + \mathcal{U}$ there are two compatible Poisson tensors defined by the following formulae:*

$$
\begin{aligned}
\mathcal{P}_1(X, \hat{X}) = \big(&A(XD^{-1})_- + D^{-1}(XA)_- - (D^{-1}X)_-A - (AX)_-D^{-1} \\
&- A(D\hat{X})_- - D^{-1}(\hat{A}\hat{X})_- + (\hat{X}D)_-A + (\hat{X}\hat{A})_-D^{-1}, \\
&\hat{A}(\hat{X}D)_+ + D(\hat{X}\hat{A})_+ - (D\hat{X})_+\hat{A} - (\hat{A}\hat{X})_+D \\
&- \hat{A}(D^{-1}X)_+ - D(AX)_+ + (XD^{-1})_+\hat{A} + (XA)_+D\big),
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
\mathcal{P}_2(X, \hat{X}) = \big(&A(XA)_- - (AX)_-A - A(\hat{A}\hat{X})_- + (\hat{X}\hat{A})_-A, \\
&\hat{A}(\hat{X}\hat{A})_+ - (\hat{A}\hat{X})_+\hat{A} - \hat{A}(AX)_+ + (XA)_+\hat{A}\big)
\end{aligned} \tag{5.15}
$$

with $(X, \hat{X}) \in T^*_{\mathbf{A}}\mathcal{U}$ at any point $\mathbf{A} = (A, \hat{A}) \in (D, 0) + \mathcal{U}$.

Proof. It follows from Lemma 5.1 that $\mathcal{P}_2$ is a Poisson tensor on the coset $(D, 0) + \mathcal{U}$.

Introduce a shift transformation on $(D,0)+\mathcal{U}$ as

$$\mathscr{S}:(A,\hat{A})\mapsto(A+sD^{-1},\hat{A}+sD)$$

with s being a parameter. Then the push-forward of the Poisson tensor $\mathcal{P}_2$ reads

$$(\mathscr{S}_*\mathcal{P}_2)(X,\hat{X})=\mathcal{P}_2(X,\hat{X})+s\mathcal{P}_1(X,\hat{X})+s^2\mathcal{P}_0(X,\hat{X}),\tag{5.16}$$

where

$$\begin{aligned}\mathcal{P}_0(X,\hat{X})=&\big(D^{-1}(XD^{-1})_- -(D^{-1}X)_-D^{-1}-D^{-1}(D\hat{X})_- +(\hat{X}D)_-D^{-1},\\ &D(\hat{X}D)_+ -(D\hat{X})_+D-D(D^{-1}X)_+ +(XD^{-1})_+D\big).\end{aligned}$$

By virtue of the symmetry property $(X^*,\hat{X}^*)=(X,\hat{X})$ that yields the formulae

$$\begin{aligned}&(XD^{-1})_\pm=X_\pm D^{-1}\mp X_+(1)D^{-1},\\ &(D^{-1}X)_\pm=D^{-1}X_\pm\mp D^{-1}\cdot X_+(1),\\ &(D\hat{X})_\pm=D\hat{X}_\pm,\quad(\hat{X}D)_\pm=\hat{X}_\pm D,\end{aligned}$$

one can check $\mathcal{P}_0(X,\hat{X})=0$. Hence the expansion (5.16) implies that $\mathcal{P}_1$ is a Poisson tensor, moreover, compatible with $\mathcal{P}_2$. The lemma is proved. □

Denote by $\{\cdot,\cdot\}_{1,2}$ the Poisson brackets given in (4.8) with Poisson tensors being $\mathcal{P}_{1,2}$ respectively. Now we arrive at the main result of this article.

Theorem 5.4 *The two-component BKP hierarchy* (3.10), (3.11) *can be expressed in the following bihamiltonian recursion form*

$$\frac{\partial F}{\partial t_k}=\{F,H_{k+2}\}_1(\mathbf{A})=\{F,H_k\}_2(\mathbf{A}),\tag{5.17}$$

$$\frac{\partial F}{\partial\hat{t}_k}=\{F,\hat{H}_{k+2}\}_1(\mathbf{A})=\{F,\hat{H}_k\}_2(\mathbf{A})\tag{5.18}$$

with $k\in\mathbb{Z}_+^{\mathrm{odd}}$, *where* $F\in\mathcal{F}_0$, $\mathbf{A}=(P^2D^{-1},D\hat{P}^2)$ *as given in* (5.1), *and the Hamiltonians are*

$$H_k=\frac{2}{k}\langle P^k\rangle,\quad\hat{H}_k=-\frac{2}{k}\langle\hat{P}^k\rangle,\quad k\in\mathbb{Z}_+^{\mathrm{odd}}.\tag{5.19}$$

Proof. First let us compute the variational gradients of the Hamiltonian functionals. Since

$$\delta H_k=\langle P^{k-2},\delta P^2\rangle=\langle DP^{k-2},\delta(P^2D^{-1})\rangle=\langle(DP^{k-2},0),\delta\mathbf{A}\rangle$$

and similarly

$$\delta\hat{H}_k = \langle(0, -\hat{P}^{k-2}D^{-1}), \delta\mathbf{A}\rangle,$$

then up to kernel parts given in (5.7) we have the variational gradients of the Hamiltonians:

$$\frac{\delta H_k}{\delta\mathbf{A}} = (DP^{k-2}, 0), \quad \frac{\delta\hat{H}_k}{\delta\mathbf{A}} = (0, -\hat{P}^{k-2}D^{-1}) \tag{5.20}$$

One can easily see that different choices of the kernel parts do not change the definition of the Poisson tensors $\mathcal{P}_{1,2}$.

According to the flows (3.10), (3.11) one has

$$\frac{\partial\mathbf{A}}{\partial t_k} = \left([(P^k)_+, P^2]D^{-1}, D[(P^k)_+, \hat{P}^2]\right).$$

Note that

$$\frac{\partial F}{\partial t_k} = \left\langle \frac{\delta F}{\delta\mathbf{A}}, \frac{\partial\mathbf{A}}{\partial t_k} \right\rangle,$$

then to show (5.17) we only need to verify the equations

$$\frac{\partial\mathbf{A}}{\partial t_k} = \mathcal{P}_1\left(\frac{\delta H_{k+2}}{\delta\mathbf{A}}\right) = \mathcal{P}_2\left(\frac{\delta H_k}{\delta\mathbf{A}}\right). \tag{5.21}$$

The verification is straightforward by substituting (5.20) into (5.14), (5.15) and by using the formulae induced from (3.8):

$$(DP^kD^{-1})_\pm = D(P^k)_\pm D^{-1}, \quad (D\hat{P}^kD^{-1})_\pm = D(\hat{P}^k)_\pm D^{-1}, \quad k \in \mathbb{Z}_+^{\mathrm{odd}}.$$

In the same way one can check the equations (5.18). The theorem is proved. $\square$

This theorem implies that the tau function (3.12) of the two-component BKP hierarchy is defined from the tau-symmetry of Hamiltonian densities [11] up to the signs of $\hat{H}_k$.

Remark 5.5 One can also construct Hamiltonian structures of the two-component BKP hierarchy by reducing the linear and the cubic Poisson brackets induced from the R-matrix mentioned in last section. However, from these brackets we could not find any bihamiltonian recursion relations like (5.17), (5.18).

6 Dispersionless limit of the bihamiltonian structure

Before studying the infinite-dimensional Frobenius manifolds, let us compute the leading term of the bihamiltonian structure of the two-component BKP hierarchy.

The method is to replace pseudo-differential operators by Laurent series of symbols. In the dispersionless case, the operator $\mathbf{A} = (P^2 D^{-1}, D\hat{P}^2)$ becomes

$$(a(z), \hat{a}(z)) = \left(z + \sum_{i\geq 0} v_{-i} z^{-2i-1}, \sum_{i\geq 0} \hat{v}_i z^{2i-1} \right), \tag{6.1}$$

and the coordinate-type local functionals $v_{-i}(y)$, $\hat{v}_j(y)$ have variational gradients $(z^{2i}\delta(x-y), 0)$, $(0, z^{-2j}\delta(x-y))$ respectively. By substituting them into the Poisson brackets defined by the formulae (4.8), (5.14), (5.15), we obtain the following result. Here for the convenience of expression we set $v_1 = 1$, $v_i = 0$ when $i \geq 2$, and $\hat{v}_j = 0$ when $j \leq -1$.

i) The first bracket: for $i, j \geq 0$,

$$\begin{aligned} \{v_{-i}(x), v_{-j}(y)\}_1^{[0]} &= (1-\delta_{i0}-\delta_{j0})\big(2(i+j-1)v_{-i-j+1}(x)\,\delta'(x-y) \\ &\quad + (2j-1)v'_{-i-j+1}(x)\,\delta(x-y)\big), \end{aligned} \tag{6.2}$$

$$\begin{aligned} \{\hat{v}_i(x), \hat{v}_j(y)\}_1^{[0]} &= -(1-\delta_{i0}-\delta_{j0})\big(2(i+j-1)\hat{v}_{i+j}(x)\,\delta'(x-y) \\ &\quad + (2j-1)\hat{v}'_{i+j}(x)\,\delta(x-y)\big), \end{aligned} \tag{6.3}$$

$$\begin{aligned} &\{v_{-i}(x), \hat{v}_j(y)\}_1^{[0]} \\ &\quad = 2(i-j)\big((1-\delta_{j0})v_{j-i}(x) + (1-\delta_{i0})\hat{v}_{j-i+1}(x)\big)\delta'(x-y) \\ &\quad\quad - (2j-1)\big((1-\delta_{j0})v'_{j-i}(x) + (1-\delta_{i0})\hat{v}'_{j-i+1}(x)\big)\delta(x-y). \end{aligned} \tag{6.4}$$

ii) The second bracket: for $i, j \geq 0$,

$$\begin{aligned} &\{v_{-i}(x), v_{-j}(y)\}_2^{[0]} \\ &\quad = \sum_{r=-1}^{i-1} \Big(2(i+j-2r-1)v_{-r}(x)\,v_{-i-j+r+1}(x)\,\delta'(x-y) \\ &\quad\quad + (2j-2r-1)v_{-r}(x)\,v'_{-i-j+r+1}(x)\,\delta(x-y) \\ &\quad\quad + (2i-2r-1)v'_{-r}(x)\,v_{-i-j+r+1}(x)\,\delta(x-y)\Big), \end{aligned} \tag{6.5}$$

$$\begin{aligned} \{\hat{v}_i(x), \hat{v}_j(y)\}_2^{[0]} &= -\sum_{r=0}^{i} \Big(2(i+j-2r+1)\hat{v}_r(x)\,\hat{v}_{i+j-r+1}(x)\,\delta'(x-y) \\ &\quad + (2j-2r+1)\hat{v}_r(x)\,\hat{v}'_{i+j-r+1}(x)\,\delta(x-y) \\ &\quad + (2i-2r+1)\hat{v}'_r(x)\,\hat{v}_{i+j-r+1}(x)\,\delta(x-y)\Big), \end{aligned} \tag{6.6}$$

$$\begin{aligned} &\{v_{-i}(x), \hat{v}_j(y)\}_2^{[0]} \\ &\quad = \sum_{r=\max\{-1, i-j-1\}}^{i-1} \Big(2(i-j)v_{-r}(x)\,\hat{v}_{-i+j+r+1}(x)\,\delta'(x-y) \end{aligned}$$

$$
\begin{aligned}
&+ (2r-2j+1)v_{-r}(x)\,\hat{v}'_{-i+j+r+1}(x)\,\delta(x-y) \\
&+ (2r-2i+1)v'_{-r}(x)\,\hat{v}_{-i+j+r+1}(x)\,\delta(x-y)\Big).
\end{aligned} \tag{6.7}
$$

7 Concluding remarks

We start from the Lax pair presentation (3.10), (3.11) of the two-component BKP hierarchy, and arrive at a bihamiltonian representation of this hierarchy. Our method in the construction of the Poisson brackets is to employ the standard R-matrix formalism, which is analogous with that for the two-dimensional Toda hierarchy [1]. In comparison with the two-dimensional Toda hierarchy, we expect to find an infinite-dimensional Frobenius manifold underlying the two-component BKP hierarchy.

As mentioned in the beginning of this article, the two-component BKP hierarchy (3.10), (3.11) is reduced to the Drinfeld-Sokolov hierarchy of type $(D_n^{(1)}, c_0)$ under the constraint $P^{2n-2} = \hat{P}^2$. Whether such a constraint induces a reduction of the bihamiltonian structure is unclear yet. We hope that considering this problem would help to understand the relations between Frobenius manifolds of infinite and finite dimensions.

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