

Sine–Gordon theory in a semi–strip

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Abstract

Initial-boundary value problems for complex sine-Gordon and sine-Gordon equations in a semi-strip are treated. The evolution of the Weyl function and a uniqueness result are obtained for complex sine-Gordon equation. The evolution of the Weyl function as well as an existence result and a procedure to recover solution are given for sine-Gordon equation. It is shown that for a wide class of examples the solutions of the sine-Gordon equation are unbounded in the quarter-plane.

MSC(2010) 35G31, 37K15, 34A55, 34B20, 34L40.

Keywords:

sine-Gordon equation, complex sine-Gordon equation, skew-self-adjoint Dirac system, Weyl function, inverse problem, initial-boundary value problem.

1 Introduction

The well-known sine-Gordon and complex sine-Gordon equations are actively used in study of various physical models and processes: self-induced transparency and coherent optical pulse propagation, relativistic vortices in a superfluid, nonlinear sigma models, the motion of rigid pendula, dislocations in crystals and so on (see, for instance, references in [4, 6, 7, 22, 33, 34, 46]).

Sine-Gordon equation (SGE) in the light cone coordinates has the form

$$\psi_{\xi\eta} = 2 \sin 2\psi, \quad \psi_\xi := \frac{\partial}{\partial \xi} \psi. \quad (1.1)$$

It is the first equation to which the so called auto-Bäcklund transformation was applied. It is also a one of the first equations, for which a Lax pair was found and which was consequently solved by the Inverse Scattering Transform method [2]. The more general *complex sine-Gordon equation* (CSGE) was introduced (and its integrability was treated) only several years later [29, 34]. For further developments of the theory of CSGE and various applications see, for instance, [6, 7, 13, 16, 32, 33] and references therein. CSGE has the form

$$\psi_{\xi\eta} + \frac{4 \cos \psi}{(\sin \psi)^3} \chi_\xi \chi_\eta = 2 \sin 2\psi, \quad \chi_{\xi\eta} - \frac{2}{\sin 2\psi} (\psi_\xi \chi_\eta + \psi_\eta \chi_\xi) = 0, \quad (1.2)$$

where $\psi = \bar{\psi}$, $\chi = \bar{\chi}$, and $\bar{\psi}$ denotes the complex conjugate of ψ .

There are also two constraint equations

$$2(\cos \psi)^2 \chi_\xi - (\sin \psi)^2 \theta_\xi = 2(\sin \psi)^2 \gamma, \quad 2(\cos \psi)^2 \chi_\eta + (\sin \psi)^2 \theta_\eta = 0, \quad (1.3)$$

where γ is a constant ($\gamma = \bar{\gamma} \equiv \text{const}$) and $\theta = \bar{\theta}$.

For the particular case $\chi \equiv 0$ and $\theta = -2\xi\gamma$, CSGE turns into sine-Gordon equation (1.1) and the constraint equations hold automatically. There are also many interesting modifications and generalizations of the sine-Gordon equation: elliptic SGE, matrix SGE, non-abelian SGE, sh-Gordon equation, et cetera, which often enough could be studied in a way similar to SGE.

The introduction of the Inverse Scattering Transform brought a breakthrough in the initial value problems for integrable nonlinear equations. The initial-boundary value problems are more complicated, though the Inverse Scattering Transform and several other methods help to obtain various interesting results. In spite of many interesting developments the rigorous results in this domain are comparatively rare. Nevertheless one could mention, for instance, important uniqueness and existence results in [1, 11, 12, 14, 17, 21, 47] (see also references therein). Here, we apply the Inverse Spectral Transform method [9, 10, 24, 35, 36, 38, 40, 42–45]. In this way we shall obtain a uniqueness result for CSGE and a global existence result for SGE in the semi-strip

$$\mathcal{D} = \{(\xi, \eta) : 0 \leq \xi < \infty, 0 \leq \eta < a\}. \quad (1.4)$$

Notice that the initial-boundary problem for SGE, where the values of ψ are given on the characteristics $\xi = -\infty$ and $\eta = 0$, is treated in [25] (see [50] for the related Cauchy problem for SGE in laboratory coordinates). A local solution of the Goursat problem for SGE, where ψ is given on the characteristics $\xi = 0$ and $\eta = 0$, is described in [26] (see also [28]). Our result for SGE is based on a global existence theorem from [36].

Preliminaries on zero curvature equations for CSGE and SGE and on inverse spectral problem are contained in the next Section 2. Evolution of the Weyl function and uniqueness result for CSGE are given in Section 3, and existence and recovery of solution of the initial-boundary value problem for SGE are treated in Section 4. In the second subsection of Section 4 we construct a class of unbounded in the quarter-plane solutions of the initial-boundary value problem.

As usual, we denote the real part of a scalar or matrix z by $\Re z$ and the imaginary part by $\Im z$. The real axis is denoted by \mathbb{R} , the positive semi-axis by \mathbb{R}_+ , the complex plane is denoted by \mathbb{C} , and the lower semi-plane - by \mathbb{C}_- .

2 Preliminaries

2.1 Zero curvature equations

Zero curvature representation of the integrable nonlinear equations is a well known approach (see [3,18,31,49] and references therein), which was developed soon after the seminal Lax pairs appeared in [27]. We shall need zero curvature equations for CSGE and SGE.

If $\sin 2\psi \neq 0$ the compatibility condition for constraint equations (1.3) is equivalent to the second equation in (1.2). If $\sin 2\psi \neq 0$ and (1.3) is true, then CSGE (i.e., equations (1.2)) is equivalent to the compatibility condition of the systems

$$W_\xi = GW, \quad W_\eta = FW; \quad (2.1)$$

$$G(\xi, \eta, \lambda) := i(\lambda - \gamma)j + V(\xi, \eta), \quad F(\xi, \eta, \lambda) := -\frac{i}{\lambda}g(\xi, \eta)^* jg(\xi, \eta). \quad (2.2)$$

Here

$$j = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad g(\xi, \eta) = D_1(\xi, \eta) \begin{bmatrix} \cos \psi & i \sin \psi \\ i \sin \psi & \cos \psi \end{bmatrix} D_2(\xi, \eta), \quad (2.3)$$

$$D_1 = \exp \left\{ i \left(\chi + \frac{\theta}{2} \right) j \right\}, \quad D_2 = \exp \left\{ i \left(\chi - \frac{\theta}{2} \right) j \right\}, \quad (2.4)$$

$$V = -g^* g_\xi - i\gamma(g^* jg - j). \quad (2.5)$$

In other words, CSGE is equivalent to the zero curvature equation

$$G_\eta - F_\xi + [G, F] = 0, \quad [G, F] := GF - FG. \quad (2.6)$$

We shall consider CSGE in the semi-strip (1.4). According to [33,34] the following statement is true.

Proposition 2.1 *Let $\{\psi(\xi, \eta), \chi(\xi, \eta), \theta(\xi, \eta)\}$ be a triple of real-valued and twice continuously differentiable functions on \mathcal{D} . Assume that $\sin 2\psi \neq 0$ and that equations (1.2) and (1.3) hold. Then the zero curvature equation (2.6) holds too.*

Moreover, g given by (2.3) belongs $SU(2)$ and satisfies relations

$$g^* g_\xi + i\gamma g^* j g = i(\chi_\xi + \frac{1}{2}\theta_\xi + \gamma)g^* j g + i(\chi_\xi - \frac{1}{2}\theta_\xi)j + i\psi_\xi D_2^* \sigma_1 D_2, \quad (2.7)$$

$$g^* j g = D_2^* \begin{bmatrix} \cos 2\psi & i \sin 2\psi \\ -i \sin 2\psi & -\cos 2\psi \end{bmatrix} D_2, \quad \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (2.8)$$

In view of the first constraint in (1.3) and equalities (2.7) and (2.8), the matrix function V introduced by (2.5) has the form

$$V = \begin{bmatrix} 0 & v \\ -\bar{v} & 0 \end{bmatrix}, \quad v = (-i\psi_\xi + 2\chi_\xi \cot \psi) e^{i(\theta - 2x)}. \quad (2.9)$$

According to (2.2) and (2.9) the auxiliary system $W_x = GW$ is a Dirac-type system, which will be used for the study of CSGE.

In the zero curvature representation (2.6) of the sine-Gordon equation (1.1) we put

$$G(\xi, \eta, z) = izj + V(\xi, \eta), \quad v(\xi, \eta) = -\psi_\xi(\xi, \eta), \quad \psi = \bar{\psi}, \quad (2.10)$$

$$V = \begin{bmatrix} 0 & v \\ -v & 0 \end{bmatrix}, \quad F(\xi, \eta, z) = \frac{1}{iz} \begin{bmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{bmatrix}, \quad (2.11)$$

though the pair $\widehat{G} = \widehat{D}G\widehat{D}^{-1}$, $\widehat{F} = \widehat{D}F\widehat{D}^{-1}$, where G and F are defined in (2.10) and (2.11), $\widehat{D} := \text{diag}\{i, 1\}$, and diag means diagonal matrix, would be closer to G and F from (2.2).

2.2 Weyl function and inverse problem

Put $z = \lambda - \gamma$ and write down the auxiliary system given by the first relations in (2.1) and (2.2) in the form

$$\frac{d}{d\xi} w(\xi, z) = G(\xi, z)w(\xi, z), \quad G(\xi, z) = izj + V(\xi), \quad w(0, z) = I_2, \quad (2.12)$$

where V has the form given by the first relations in (2.9) and (2.11), I_2 is the 2×2 identity matrix, and w is the normalized fundamental

solution. In this subsection we adduce some results on the Weyl theory of the skew-self-adjoint Dirac-type system (2.12) from [35, 36] (see also [5, 15, 19, 23, 30, 37, 39, 40]).

Definition 2.2 *Let system (2.12) be given on the semi-axis $[0, \infty)$. Then a function $\varphi(z)$ holomorphic in some semi-plane $\Im z < -M < 0$ is called a Weyl function of this system, if*

$$\sup_{\xi \leq r, \Im z < -M} \left\| e^{i\xi z} w(\xi, z) \begin{bmatrix} \varphi(z) \\ 1 \end{bmatrix} \right\| < \infty \quad \text{for all } 0 < r < \infty. \quad (2.13)$$

We shall consider systems (2.12), where potentials v are bounded on all finite intervals:

$$\sup_{0 < \xi < r} |v(\xi)| < \infty \quad \text{for all } 0 < r < \infty. \quad (2.14)$$

The next statement follows from [37] (see Remark 8.4, p.113).

Proposition 2.3 *There is at most one Weyl function of system (2.12), where v satisfies (2.14).*

If v is bounded on $[0, \infty)$ the Weyl function always exists.

Proposition 2.4 [35–37] *If the inequality*

$$\sup_{0 < \xi < \infty} |v(\xi)| \leq M \quad (2.15)$$

holds, then there is a unique Weyl function φ of system (2.12). Moreover, this Weyl function is the unique function such that

$$\int_0^\infty \begin{bmatrix} \overline{\varphi(z)} & 1 \end{bmatrix} w(\xi, z)^* w(\xi, z) \begin{bmatrix} \varphi(z) \\ 1 \end{bmatrix} dx < \infty, \quad \Im z < -M < 0. \quad (2.16)$$

The Weyl function is given by the formulas

$$\varphi(z) = \lim_{r \rightarrow \infty} \frac{\mathcal{A}_{11}(r, z)P_1(r, z) + \mathcal{A}_{12}(r, z)P_2(r, z)}{\mathcal{A}_{21}(r, z)P_1(r, z) + \mathcal{A}_{22}(r, z)P_2(r, z)} \quad (\Im z < -M), \quad (2.17)$$

$$\mathcal{A}(r, z) = \{\mathcal{A}_{kp}(r, z)\}_{k,p=1}^2 := w(r, \bar{z})^*, \quad (2.18)$$

where $P_1(r, z), P_2(r, z)$ is an arbitrary non-singular pair of meromorphic functions with property-j, namely

$$|P_2|^2 > 0, \quad |P_2|^2 \geq |P_1|^2. \quad (2.19)$$

Finally, we have $|\varphi(z)| \leq 1$ for $\Im z < -M$.

We shall need a scalar version (i.e., the case where $v(\xi)$ is scalar) of Theorem 1 from [35]:

Theorem 2.5 *System (2.12) satisfying condition (2.15) is uniquely defined by its Weyl function.*

The next theorem deals with the inverse problem for φ such that

$$\sup_{\Im z < -M} |z^2(\varphi(z) - \alpha/z)| < \infty, \quad \alpha \in \mathbb{C}. \quad (2.20)$$

Theorem 2.6 [36, 37] *Let holomorphic function $\varphi(z)$ ($\Im z < -M$) admit representation (2.20). Then there is a unique system (2.12) satisfying (2.14) such that φ is its Weyl function.*

To recover $v(\xi)$ notice that according to (2.12) we have

$$w(\xi, z)w(\xi, \bar{z})^* = w(\xi, \bar{z})^*w(\xi, z) = I_2, \quad (2.21)$$

and

$$v(\xi) = w_1'(\xi)w_2(\xi)^*, \quad w_1(\xi) := [1 \quad 0]w(\xi, 0), \quad w_2(\xi) := [0 \quad 1]w(\xi, 0). \quad (2.22)$$

To recover w_1 and w_2 on some interval $[0, c]$ ($c < \infty$) we construct function $s(\xi) \in L^2(0, c)$ via the Fourier transform

$$s(\xi) = \frac{i}{2\pi} e^{-y\xi} \text{l.i.m.}_{b \rightarrow \infty} \int_{-b}^b e^{i\xi x} (x + iy)^{-1} \varphi\left(\frac{x + iy}{2}\right) dx, \quad y < -2M, \quad (2.23)$$

where l.i.m. is the limit in the norm of $L^2(0, c)$. As $|\varphi| \leq 1$ it is easy to see that s does not depend on the choice of $y < -2M$. The function s is absolutely continuous and we have

$$s(0) = 0, \quad \sup_{0 < \xi < c} |s'(\xi)| < \infty, \quad s' := \frac{d}{d\xi} s \quad (2.24)$$

(see (18) in [36], where $\Phi_2 = s$). By [36] the operator

$$S_r f = f(\xi) + \frac{1}{2} \int_0^r \int_{|\xi-u|}^{\xi+u} s' \left(\frac{l+\xi-u}{2} \right) \overline{s' \left(\frac{l+u-\xi}{2} \right)} dl f(u) du, \quad (2.25)$$

where $0 < r \leq c$, is bounded in $L^2(0, r)$. The inequality $S_r \geq I$, where I is the identity operator, holds. Now, we can recover $w_1, w_2 \in \mathbb{C}_2$ by the formulas

$$\omega_1(r) = - \int_0^r \overline{(S_r^{-1} s')(\xi)} d\xi, \quad \omega_2(r) = 1 - \int_0^r \overline{(S_r^{-1} s')(\xi)} s(\xi) d\xi, \quad (2.26)$$

$$w_2(r) = [\omega_1(r) \quad \omega_2(r)], \quad w_1(r) = [\overline{\omega_2(r)} \quad -\overline{\omega_1(r)}]. \quad (2.27)$$

Theorem 2.7 [35–37] *Let function φ be the Weyl function of system (2.12) such that (2.14) holds. Assume that either (2.15) or (2.20) is fulfilled. Then the solution of the inverse problem to recover v is given by the formulas (2.22), (2.26), and (2.27), where s and S_r are constructed using formulas (2.23) and (2.25).*

3 CSGE: evolution of the Weyl function and uniqueness of the solution

In this section we consider the initial-boundary value problem for CSGE:

$$v(\xi, 0) = h_1(\xi), \quad \psi(0, \eta) = h_2(\eta), \quad \chi(0, \eta) = h_3(\eta), \quad \theta(0, 0) = h_4, \quad (3.1)$$

where v is defined by the second equality in (2.9). We assume that the conditions of Proposition 2.1 are fulfilled.

Theorem 3.1 *Let $\{\psi(\xi, \eta), \chi(\xi, \eta), \theta(\xi, \eta)\}$ be a triple of real-valued and twice continuously differentiable functions on \mathcal{D} . Assume that $\sin 2\psi \neq 0$, that v is bounded, that is,*

$$\sup_{(\xi, \eta) \in \mathcal{D}} |\psi_\xi + 2i\chi_\xi \cot \psi| \leq M, \quad (3.2)$$

and that relations (1.2), (1.3), and (3.1) hold. Then the Weyl functions $\varphi(\eta, z)$ of the auxiliary Dirac-type systems $W_\xi = GW$, where $G(\xi, \eta, z)$ ($z = \lambda - \gamma$) is defined via (2.2) and (2.9), exist and have the form

$$\varphi(\eta, z) = \frac{R_{11}(\eta, z)\varphi(0, z) + R_{12}(\eta, z)}{R_{21}(\eta, z)\varphi(0, z) + R_{22}(\eta, z)}. \quad (3.3)$$

Here $R := \{R_{kp}\}_{k,p=1}^2$ is defined by the equalities

$$\begin{aligned} \frac{d}{d\eta}R(\eta, z) &= \frac{1}{i(z + \gamma)}e^{-id(\eta)j} \begin{bmatrix} \cos 2h_2(\eta) & i \sin 2h_2(\eta) \\ -i \sin 2h_2(\eta) & -\cos 2h_2(\eta) \end{bmatrix} \\ &\quad \times e^{id(\eta)j}R(\eta, z), \end{aligned} \quad (3.4)$$

$$R(0, z) = I_2, \quad d(\eta) = h_3(0) - \frac{1}{2}h_4 + \int_0^\eta h_3'(u)(\sin h_2(u))^{-2} du, \quad (3.5)$$

and $\varphi(0, z)$ is the Weyl function of the system

$$\frac{d}{d\xi}W(\xi, z) = (izj + V(\xi))W(\xi, z), \quad V(\xi) = \begin{bmatrix} 0 & h_1(\xi) \\ -h_1(\xi) & 0 \end{bmatrix}. \quad (3.6)$$

Proof. Introduce normalized fundamental solutions of the auxiliary systems by the the equalities:

$$\frac{d}{d\xi}w(\xi, \eta, z) = G(\xi, \eta, z)w(\xi, \eta, z), \quad w(0, \eta, z) = I_2; \quad (3.7)$$

$$\frac{d}{d\eta}R(\xi, \eta, z) = F(\xi, \eta, z)R(\xi, \eta, z), \quad R(\xi, 0, z) = I_2; \quad (3.8)$$

$$R(\eta, z) = R(0, \eta, z); \quad (3.9)$$

where G and F are given by formulas (2.2), (2.4), (2.8), and (2.9). Notice that the definition of $R(\eta, z)$ in (3.9) complies with (3.4) and (3.5). The definition of V in (3.6) complies with (2.9) and (3.1). As G and F are continuously differentiable, we can use factorization formula (1.6) from Chapter 12 in [37]:

$$w(\xi, \eta, z) = R(\xi, \eta, z)w(\xi, 0, z)R(\eta, z)^{-1}. \quad (3.10)$$

From (2.2) and (3.8) we derive

$$(R(\xi, \eta, \bar{z})^* R(\xi, \eta, z))_\eta = 0, \quad \text{and so} \quad R(\xi, \eta, \bar{z})^* = R(\xi, \eta, z)^{-1},$$

$\det R(\xi, \eta, z) \neq 0$. In particular, we have

$$(R(\eta, \bar{z})^{-1})^* = R(\eta, z). \quad (3.11)$$

Taking into account (3.10) and (3.11) we get

$$\mathcal{A}(\xi, \eta, z) := w(\xi, \eta, \bar{z})^* = R(\eta, z)\mathcal{A}(\xi, 0, z)R(\xi, \eta, \bar{z})^*. \quad (3.12)$$

Now, notice that according to (3.2) the conditions of Proposition 2.4 are fulfilled for any $a > \eta \geq 0$. It is immediate also that the pair $P_1 = 0, P_2 = 1$ is non-singular with property- j . Hence (2.17) holds for these P_1 and P_2 . From (2.17) and (3.12) we get

$$\varphi(\eta, z) = \lim_{r \rightarrow \infty} \frac{R_{11}(\eta, z)\tilde{P}_1(r, \eta, z) + R_{12}(\eta, z)\tilde{P}_2(r, \eta, z)}{R_{21}(\eta, z)\tilde{P}_1(r, \eta, z) + R_{22}(\eta, z)\tilde{P}_2(r, \eta, z)}, \quad (3.13)$$

where

$$\begin{bmatrix} \tilde{P}_1(r, \eta, z) \\ \tilde{P}_2(r, \eta, z) \end{bmatrix} := \mathcal{A}(r, 0, z)R(r, \eta, \bar{z})^* \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (3.14)$$

If $a < \infty$ and $|z + \gamma| > \delta > 0$ we have

$$\sup |\eta/(z + \gamma)| = C < \infty, \quad (3.15)$$

and so the inequality

$$\|R(\xi, \eta, z) - I_2 - \int_0^\eta F(\xi, t, z)dt\| \leq \eta^2(z + \gamma)^{-2}e^C \quad (3.16)$$

holds. Indeed, using (2.2), (3.8), and the corresponding *multiplicative integral* representation of R we get

$$R(\xi, \eta, z) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \left(I_2 + \frac{\eta F_k}{n(z + \gamma)} + \frac{\eta^2 C_k(n)}{n^2(z + \gamma)^2} \right), \quad (3.17)$$

$$F_k := -ig\left(\xi, \frac{k}{n}\eta\right)^* jg\left(\xi, \frac{k}{n}\eta\right), \quad \|C_k(n)\| \leq 1 \quad \text{for all } n > n_0(C). \quad (3.18)$$

It follows from (3.17) and (3.18) that for any $\varepsilon > 0$ and sufficiently large values of n we have

$$\begin{aligned} & \left\| R(\xi, \eta, z) - I_2 - \sum_{k=1}^n \frac{\eta F_k}{n(z + \gamma)} \right\| \leq \left(\frac{(1 + \varepsilon)\eta}{z + \gamma} \right)^2 \\ & \times \sum_{p=2}^n \left(\frac{(1 + \varepsilon)\eta}{z + \gamma} \right)^{p-2} \frac{1}{n^p} \binom{n}{p} + \varepsilon. \end{aligned} \quad (3.19)$$

Inequalities (3.15) and (3.19) imply (3.16).

According to (3.16) there is a value $M_1 > M > 0$ such that the pairs

$$\begin{bmatrix} \widehat{P}_1(r, \eta, z) \\ \widehat{P}_2(r, \eta, z) \end{bmatrix} := R(r, \eta, \bar{z})^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (3.20)$$

are non-singular with property- j for all $\eta < a$ and z such that $\Im z < -M_1$. Next, notice that in view of (3.14) and (3.20) we get

$$\begin{bmatrix} \widetilde{P}_1(r, \eta, z) \\ \widetilde{P}_2(r, \eta, z) \end{bmatrix} := \mathcal{A}(r, 0, z) \begin{bmatrix} \widehat{P}_1(r, \eta, z) \\ \widehat{P}_2(r, \eta, z) \end{bmatrix}. \quad (3.21)$$

It is also easy to see that

$$j > \mathcal{A}(r, \eta, z)^* j \mathcal{A}(r, \eta, z) \quad \text{for } \Im z < -M_1, \quad r > 0. \quad (3.22)$$

Indeed, by (2.12) and (3.2) we have

$$\frac{d}{d\xi} (w(\xi, \eta, z)^* j w(\xi, \eta, z)) > 0, \quad \Im z < -M,$$

and so the inequality

$$w(\xi, \eta, z)^* j w(\xi, \eta, z) > j \quad (\Im z < -M, \xi > 0) \quad (3.23)$$

is true. By (2.21) we get $w(\xi, \eta, \bar{z})^* = w(\xi, \eta, z)^{-1}$. Therefore, inequality (3.22) is immediate from (2.18) and (3.23).

As the pair $\widehat{P}_1, \widehat{P}_2$ has property- j and (3.22) holds, it follows from (3.21) that

$$\widetilde{P}_2(r, \eta, z) \neq 0, \quad r > 0. \quad (3.24)$$

Moreover, according to (2.17) and (3.21) we have

$$\lim_{r \rightarrow \infty} (\tilde{P}_1(r, \eta, z) / \tilde{P}_2(r, \eta, z)) = \varphi(0, z). \quad (3.25)$$

The evolution formula (3.3) is immediate from (3.13) and (3.25).

We assumed first that $a < \infty$ but if $a = \infty$ formula (3.3) is still proved for η on all finite intervals on \mathbb{R}_+ , and so (3.3) holds on \mathbb{R}_+ .

■

The idea of the proof above as well as the idea to present the evolution of the Weyl functions in the form of linear-fractional (Möbius) transformations comes from the seminal works [41–43].

Corollary 3.2 *There is at most one triple $\{\psi(\xi, \eta), \chi(\xi, \eta), \theta(\xi, \eta)\}$ of real-valued and twice continuously differentiable functions on \mathcal{D} such that $\sin 2\psi \neq 0$, that v is bounded, and that CSGE (1.2), constraints (1.3), and initial-boundary conditions (3.1) are satisfied.*

Proof. Suppose the triple $\{\psi(\xi, \eta), \chi(\xi, \eta), \theta(\xi, \eta)\}$ satisfies conditions of the corollary. Then the conditions of Theorem 3.1 are fulfilled and the evolution of the Weyl function is given by formula (3.3). Hence, according to Theorem 2.5 the function $v(\xi, \eta)$ is uniquely recovered.

Next, we recover ψ from v . Using (1.2), (1.3), and (2.9) we derive

$$\begin{aligned} v_\eta &= \left(-i\psi_{\xi\eta} + 2\chi_{\xi\eta} \cot \psi - 2\chi_\xi \psi_\eta (\sin \psi)^{-2} \right. \\ &\quad \left. + (\psi_\xi + 2i\chi_\xi \cot \psi)(\theta_\eta - 2\chi_\eta) \right) e^{i(\theta-2\chi)} = \left(-i\psi_{\xi\eta} + 2\chi_{\xi\eta} \cot \psi \right. \\ &\quad \left. - 2\chi_\xi \psi_\eta (\sin \psi)^{-2} - 2(\psi_\xi + 2i\chi_\xi \cot \psi)\chi_\eta (\sin \psi)^{-2} \right) e^{i(\theta-2\chi)} \quad (3.26) \\ &= \left(-2i \sin 2\psi + 2\left(\chi_{\xi\eta} - \frac{1}{\sin \psi \cos \psi} (\chi_\xi \psi_\eta + \chi_\eta \psi_\xi)\right) \cot \psi \right) e^{i(\theta-2\chi)} \\ &= -2ie^{i(\theta-2\chi)} \sin 2\psi. \end{aligned}$$

It follows from (3.26) that $|\sin 2\psi| = |v_\eta|/2$. As ψ is continuous and $\sin 2\psi \neq 0$, the function $\sin 2\psi$ is uniquely recovered from the values of $|\sin 2\psi(\xi, \eta)|$ and the sign of $\sin 2\psi(0, \eta) = \sin 2h_2(\eta)$. In view of (2.9) and (3.26) we get $\psi_\xi = 2(\sin 2\psi)\Re(v/v_\eta)$, and thus we recover

also ψ_ξ . Therefore, the function $\psi(\xi, \eta) = h_2(\eta) + \int_0^\xi \psi_x(x, \eta) dx$ is uniquely recovered too. Moreover, we have

$$\chi(\xi, \eta) = h_3(\eta) + \int_0^\xi \chi_x(x, \eta) dx, \quad \chi_\xi = 2(\sin \psi)^2 \Im(v/v_\eta).$$

Finally, θ is uniquely recovered from the value $\theta(0, 0)$ and constraint equations (2.15). ■

Remark 3.3 *In a way similar to the cases of other nonlinear equations (see [36, 41, 45]) one can show that the evolution of the Weyl function prescribed by CSGE satisfies Riccati equation. Indeed, rewrite (3.3) as*

$$\varphi(\eta, z) = \varkappa_1(\eta, z)/\varkappa_2(\eta, z), \quad \varkappa_1(\eta, z) := [1 \quad 0]R(\eta, z) \begin{bmatrix} \varphi(0, z) \\ 1 \end{bmatrix}, \quad (3.27)$$

$$\varkappa_2(\eta, z) := [0 \quad 1]R(\eta, z) \begin{bmatrix} \varphi(0, z) \\ 1 \end{bmatrix}.$$

It is immediate from (3.8), (3.9), and (3.27) that

$$\begin{aligned} \frac{d}{d\eta} \varphi(\eta, z) &= [1 \quad 0]F(0, \eta, z)R(\eta, z) \begin{bmatrix} \varphi(0, z) \\ 1 \end{bmatrix} \frac{1}{\varkappa_2(\eta, z)} \\ &\quad - \frac{\varkappa_1(\eta, z)}{\varkappa_2(\eta, z)^2} [0 \quad 1]F(0, \eta, z)R(\eta, z) \begin{bmatrix} \varphi(0, z) \\ 1 \end{bmatrix}. \end{aligned} \quad (3.28)$$

Using the expression for $F(0, \eta, z)$ (compare for that purpose (3.4) and (3.8)), we rewrite (3.28) in the final form

$$\begin{aligned} \frac{d}{d\eta} \varphi(\eta, z) &= \frac{1}{z + \gamma} \left(i(\sin 2h_2(\eta)) (\exp 2id(\eta)) \varphi(\eta, z)^2 \right. \\ &\quad \left. + 2(\cos 2h_2(\eta)) \varphi(\eta, z) + i(\sin 2h_2(\eta)) (\exp -2id(\eta)) \right). \end{aligned} \quad (3.29)$$

Another case of Riccati equations for Weyl functions one can find, for instance, in [20].

4 Sine-Gordon equation in a semi-strip

4.1 Existence theorems and construction of solution

Theorem 3 in [36] gives sufficient conditions, under which a solution of sine-Gordon equation in the semi-strip \mathcal{D} exists, and a procedure to recover this solution. The procedure to solve the sine-Gordon equation in the semi-strip is based on the procedure to solve the inverse problem, which is given in Theorem 2.7.

Definition 4.1 *Let $\varphi(z)$ be holomorphic in the semi-plane $\Im z < -M$ and admit representation (2.20). Then, according to Theorems 2.6 and 2.7 there is a unique solution of the inverse problem, that is, there is a unique potential v such that (2.14) holds and φ is the Weyl function of the corresponding system (2.12). Denote this solution of the inverse problem by $\Omega(\varphi)$ (i.e., $v(\xi) = (\Omega(\varphi))(\xi)$).*

Theorem 4.2 [36] *Let the initial–boundary conditions*

$$\psi(\xi, 0) = h_1(\xi), \quad \psi(0, \eta) = h_2(\eta), \quad (h_1(0) = h_2(0), \quad h_k = \overline{h_k}) \quad (4.1)$$

be given. Assume that h_2 is continuous on $[0, a)$ and that h_1 is boundedly differentiable on all the finite intervals on $[0, \infty)$. Moreover, assume that the Weyl function $\varphi_0(z)$ of the system

$$W_\xi = GW, \quad G(\xi, z) = izj + V(\xi), \quad V(\xi) = \begin{bmatrix} 0 & -h'_1(\xi) \\ h'_1(\xi) & 0 \end{bmatrix} \quad (4.2)$$

exists and admits representation (2.20). Then a solution of the initial–boundary value problem (1.1), (4.1) exists and is given by the equality

$$\psi(\xi, \eta) = h_2(\eta) - \int_0^\xi \left(\Omega(\varphi(\eta, z)) \right) (x) dx, \quad (4.3)$$

where

$$\varphi(\eta, z) = \frac{R_{11}(\eta, z)\varphi_0(z) + R_{12}(\eta, z)}{R_{21}(\eta, z)\varphi_0(z) + R_{22}(\eta, z)}, \quad (4.4)$$

and $R = \{R_{kp}\}_{k,p=1}^2$ is defined by the relations

$$\frac{d}{d\eta}R(\eta, z) = \frac{1}{iz} \begin{bmatrix} \cos 2h_2(\eta) & \sin 2h_2(\eta) \\ \sin 2h_2(\eta) & -\cos 2h_2(\eta) \end{bmatrix} R(\eta, z), \quad R(0, z) = I_2. \quad (4.5)$$

Here $\varphi(\eta, z)$ admits representation (2.20), where

$$\alpha(\eta) = \alpha(0) - i \int_0^\eta \sin(2h_2(x)) dx,$$

and we can put $M(\eta) \equiv \widetilde{M}(\tilde{a})$ for all η on the intervals $0 \leq \eta \leq \tilde{a}$ for each $0 < \tilde{a} < a$ and some $\widetilde{M}(\tilde{a}) > 0$. So Ω is well-defined.

Remark 4.3 *The equality*

$$\varphi(\eta, z) = \overline{\varphi(\eta, -\bar{z})}, \quad (4.6)$$

where φ is given by (4.4), is used in the proof of Theorem 4.2 to show that $\psi = \bar{\psi}$. It is also of independent interest. To derive (4.6) notice that the fundamental solution w of (2.12), where V is given in (4.2), and the fundamental solution $R(\eta, z)$ of (4.5) have the properties

$$w(\xi, z) = \overline{w(\xi, -\bar{z})}, \quad R(\eta, z) = \overline{R(\eta, -\bar{z})}. \quad (4.7)$$

It follows from Definition 2.2 and (4.7) that $\overline{\varphi_0(-\bar{z})}$ is the Weyl function of system (4.2) simultaneously with $\varphi_0(z)$. Hence, by Proposition 2.3 we have $\overline{\varphi_0(-\bar{z})} = \varphi_0(z)$. Therefore, equality (4.6) is immediate from (4.4) and the second equality in (4.7).

The next proposition is a particular case of Theorem 6.1 in [8]

Proposition 4.4 *Suppose $v(\xi) \in L^1(\mathbb{R}_+)$. Then there is a fundamental solution W of (2.12) such that we have*

$$\lim_{z \rightarrow \infty} W(\xi, z) e^{-i\xi z j} = I_2, \quad z \in \mathbb{C}_- \quad (4.8)$$

uniformly with respect to ξ . If, in addition, v is two times differentiable and $v'(\xi), v''(\xi) \in L^1(\mathbb{R}_+)$, then there is a matrix E such that

$$W(0, z) = I_2 + \frac{1}{z} E + O\left(\frac{1}{z^2}\right), \quad z \rightarrow \infty, \quad z \in \mathbb{C}_-. \quad (4.9)$$

Corollary 4.5 (i) If $v \in L^1(\mathbb{R}_+)$ then there is a Weyl function of system (2.12) and this Weyl function is given by the formula

$$\varphi(z) = W_{12}(0, z)/W_{22}(0, z), \quad W(0, z) =: \{W_{kp}(0, z)\}_{k,p=1}^2 \quad (4.10)$$

for all z in the semi-plane $\Im z < -M$ for some $M > 0$.

(ii) If v is two times differentiable and $v, v', v'' \in L^1(\mathbb{R}_+)$, then this Weyl function φ admits representation (2.20).

Proof. By (4.8) (see also Theorem A in [8]) there is a value $M > 0$ such that $W(\xi, z)$ is holomorphic for $\Im z < -M$, and we have

$$\det W(0, z) \neq 0, \quad W_{22}(0, z) \neq 0, \quad (4.11)$$

$$\sup_{\xi \geq 0, \Im z < -M} \|W(\xi, z)e^{-i\xi z j}\| < \infty, \quad \sup_{\Im z < -M} |1/W_{22}(0, z)| < \infty. \quad (4.12)$$

Thus, according to (2.12) it is immediate that

$$w(\xi, z) = W(\xi, z)W(0, z)^{-1}.$$

Hence, we derive

$$\begin{aligned} e^{i\xi z} w(\xi, z) \begin{bmatrix} \frac{W_{12}(0, z)}{W_{22}(0, z)} \\ 1 \end{bmatrix} &= \frac{e^{i\xi z}}{W_{22}(0, z)} W(\xi, z)W(0, z)^{-1} \begin{bmatrix} W_{12}(0, z) \\ W_{22}(0, z) \end{bmatrix} \\ &= \frac{1}{W_{22}(0, z)} W(\xi, z)e^{-i\xi z j} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned} \quad (4.13)$$

In view of (4.12) and (4.13) the function φ given by (4.10) satisfies conditions of Definition 2.2, that is, the statement (i) is proved.

If $v, v', v'' \in L^1(\mathbb{R}_+)$, then it follows from (4.9) that

$$W_{12}(0, z) = \frac{1}{z}E_{12} + O\left(\frac{1}{z^2}\right), \quad W_{22}(0, z) = 1 + \frac{1}{z}E_{22} + O\left(\frac{1}{z^2}\right).$$

Therefore, we get for $z \in \mathbb{C}_-, z \rightarrow \infty$ that

$$1/W_{22}(0, z) = 1 - \frac{1}{z}E_{22} + O\left(\frac{1}{z^2}\right), \quad W_{12}(0, z)/W_{22}(0, z) = \frac{1}{z}E_{12} + O\left(\frac{1}{z^2}\right). \quad (4.14)$$

The statement (ii) is immediate from (4.10) and (4.14). ■

The next theorem easily follows from Theorem 4.2 and Corollary 4.5.

Theorem 4.6 *Assume that $h_1(\xi) = \overline{h_1(\xi)}$ is three times differentiable for $\xi \geq 0$, that*

$$h_1', h_1'', h_1''' \in L^1(\mathbb{R}_+),$$

and that $h_2 = \overline{h_2}$ is continuous on $[0, a)$ ($h_1(0) = h_2(0)$). Then the Weyl function $\varphi_0(z)$ of the system (4.2) exists and admits representation (2.20). A solution of the initial-boundary value problem (4.1) for sine-Gordon equation (1.1) exists and is given by the equalities (4.3) and (4.4), where $R = \{R_{kp}\}_{k,p=1}^2$ is defined by the relations (4.5).

Note that a rapid decay of h_1' was required in [48].

Remark 4.7 *If the conditions of Theorem 4.6 hold, then by Lemma 2 from [36] the functions ψ , ψ_ξ , and $\psi_{\xi\eta}$ are continuous.*

4.2 Unbounded solutions in the quarter-plane

The behavior of the solutions of initial-boundary value problems is of interest. Notice also that it is difficult to treat unbounded solutions using the Inverse Scattering Transform method. Here we describe a family of unbounded solutions.

First we formulate Theorem 2 from [36], which is proved in [36] in a way quite similar to the proof of Theorem 3.1. (The equivalence of the definitions of Weyl functions here and in [36] follows from Proposition 2.4 under condition (2.15).)

Theorem 4.8 *Let $\psi = \overline{\psi}$ and ψ_ξ be continuous functions in the semi-strip \mathcal{D} , let $\psi_{\xi\eta}$ exist, and let (1.1) hold. Assume that ψ_ξ is bounded, that is,*

$$\sup |\psi_\xi(\xi, \eta)| \leq M \quad ((\xi, \eta) \in \mathcal{D}). \quad (4.15)$$

Then the evolution $\varphi(\eta, z)$ of the Weyl function of the auxiliary system $W_\xi = GW$, where G is given by (2.10) and (2.11), is expressed by the formula (4.4), where $R(\eta, z)$ is given by (4.5), $h_2(\eta) = \psi(0, \eta)$, and $\varphi_0(z) = \varphi(0, z)$.

Put $a = \infty$ in (1.4), that is, let \mathcal{D} be a quarter-plane. Then the next corollary follows from Theorem 4.8.

Corollary 4.9 *Assume that \mathcal{D} is a quarter-plane and the conditions of Theorem 4.8 hold. Then, for values of z , such that the inequalities*

$$(\cos 2h_2(\eta) - \varepsilon(z))\Im\bar{z} \geq |\sin 2h_2(\eta)||\Re z|, \quad \Im z < -M \quad (4.16)$$

hold for some $\varepsilon(z) > 0$ and for all $\eta \geq 0$, we have

$$\varphi_0(z) = - \lim_{\eta \rightarrow \infty} R_{12}(\eta, z)/R_{11}(\eta, z). \quad (4.17)$$

Proof. Recall that according to Proposition 2.4 the inequality

$$|\varphi(\eta, z)| \leq 1$$

is true. Hence, in view of (4.4) we get

$$[\varphi_0(z)^* \quad 1]R(\eta, z)^*jR(\eta, z) \begin{bmatrix} \varphi_0(z) \\ 1 \end{bmatrix} \leq 0. \quad (4.18)$$

By (4.5) and (4.16) we derive

$$\frac{d}{d\eta} \left(R(\eta, z)^*jR(\eta, z) \right) \geq \frac{2\varepsilon(z)}{|z|^2} (\Im\bar{z}) R(\eta, z)^*R(\eta, z). \quad (4.19)$$

It is immediate from (4.18) and (4.19) that

$$\int_0^\infty [\varphi_0(z)^* \quad 1]R(\eta, z)^*R(\eta, z) \begin{bmatrix} \varphi_0(z) \\ 1 \end{bmatrix} d\eta < \infty. \quad (4.20)$$

As according to (4.19) we have $\frac{d}{d\eta} (R(\eta, z)^*jR(\eta, z)) \geq 0$, it follows that $R(\eta, z)^*jR(\eta, z) \geq j$. In particular, we get

$$|R_{11}(\eta, z)| \geq 1. \quad (4.21)$$

According to (4.5) we have also $\frac{d}{d\eta} \left((\exp 2\eta/|z|)R(\eta, z)^*R(\eta, z) \right) \geq 0$, that is,

$$R(\eta, z)^*R(\eta, z) \geq (\exp 2(\eta_0 - \eta)/|z|)R(\eta_0, z)^*R(\eta_0, z), \quad \eta \geq \eta_0 \geq 0. \quad (4.22)$$

Inequalities (4.20) and (4.22) imply that

$$\lim_{\eta \rightarrow \infty} \left\| R(\eta, z) \begin{bmatrix} \varphi_0(z) \\ 1 \end{bmatrix} \right\| = 0. \quad (4.23)$$

Finally, (4.17) follows from (4.21) and (4.23). ■

Example 4.10 Let $h_2(\eta) \equiv 0$ ($\infty > \eta \geq 0$). Putting $\varepsilon = 1/2$ we see that (4.16) holds for all z in the semi-plane $\Im z < -M < 0$. By (4.5) the equality

$$R(\eta, z) = e^{(\eta/iz)j} \quad (4.24)$$

is true. Thus, if the conditions of Theorem 4.8 hold, by Corollary 4.9 we derive $\varphi_0(z) \equiv 0$. It is immediate from Definition 2.2, that if $\varphi_0(z) \equiv 0$, then $v = \Omega(\varphi_0) \equiv 0$ (see also Definition 4.1 of $\Omega(\varphi)$). Therefore, taking into account that $\varphi_0(z) = \varphi(0, z)$, we have

$$\psi_\xi(\xi, 0) = -\Omega(\varphi_0) \equiv 0. \quad (4.25)$$

From Example 4.10 and Theorem 4.6 we derive the next proposition.

Proposition 4.11 Assume that $h_1(\xi) = \overline{h_1(\xi)} \neq 0$ is three times differentiable for $\xi \geq 0$, that

$$h_1', h_1'', h_1''' \in L^1(\mathbb{R}_+), \quad h_1(0) = 0,$$

and that $h_2 \equiv 0$. Then one can use the procedure given in Theorem 4.6 to construct a solution ψ of the initial-boundary value problem (4.1) for sine-Gordon equation (1.1), and ψ_ξ is always unbounded in the quarter-plane.

Proof. As the conditions of Theorem 4.6 hold, we can construct a solution ψ of (1.1), (4.1). Moreover, by Remark 4.7 the functions ψ , ψ_ξ , and $\psi_{\xi\eta}$ are continuous.

Thus, if (4.15) holds, then the conditions of Theorem 4.8 and Example 4.10 are satisfied. In particular, we get (4.25), which contradicts the initial condition $\psi(\xi, 0) = h_1(\xi)$ and assumptions

$$h_1(0) = 0, \quad h_1(\xi) \neq 0$$

of the proposition. So (4.15) is not true (i.e., the proposition is proved by contradiction). ■

Acknowledgement. The author is grateful to F. Gesztesy for fruitful discussions and for his interest in the existence theorems, which stimulated this paper. The work was supported by the Austrian Science Fund (FWF) under Grant no. Y330.

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