New variables of separation for particular case of the Kowalevski top.

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Abstract

We discuss the polynomial bi-Hamiltonian structures for the Kowalevski top in special case of zero square integral. An explicit procedure to find variables of separation and separation relations is considered in detail.

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To V.V. Kozlov on the occasion of his 60th birthday

1 Introduction

During a century the only cases of integrability of the Euler-Poisson equations were the isotropic case and the cases of Euler (1758) and Lagrange (1788). In 1888 S. Kowalevski found a new highly non-trivial case of integrability [8]. In modern terms, this is an integrable system on the e(3) algebra with quadratic and quartic (in angular momenta) integrals of motion.

Furthermore, by using a mysterious change of variables, she showed that equations of motion for the new case of integrability are linearized on the abelian variety by means of the Jacobi-Abel theorem about the inversion of a system of abelian integrals [8]. At the moment no separation which is alternative to her original separation of variables is known for this system , even though there is a large body of literature dedicated to the problem, including the detailed geometric description of the invariant surfaces on which the motion evolves, see books [1, 2] and references within.

In this paper we discuss the direct method of finding variables of separation without any additional information (ingenious and at times obscure change of variables, Lax matrices, r-matrices, links with soliton equations etc). For example, we apply the machinery of bi-Hamiltonian geometry to the Kowalevski top at zero level of the cyclic integral of motion, which is a particular case of the generic Kowalevski top. The rational Poisson bivector associated with famous Kowalevski variables may be found in [24]. Here we specially do not consider Kowalevski variables in order to get only the new variables of separation and the new underlying polynomial Poisson structures.

The other aim is the construction of different variables of separation lying on the distinct algebraic curves [26]. Relations between such distinct curves give us a lot of new examples of reductions of Abelian integrals and, therefore, they may be the source of new ideas in the number theory, algebraic geometry and modern cryptography [1, 3, 10].

In Section 2 we construct new compatible Poisson bivectors for the Kowalevski top. In Section 3 we find the new corresponding variables of separation and the separated relations. Finally, some concluding remarks can be found in the last Section.

2 The bi-hamiltonian structure

2.1 Description of the model

According to [8], the Kowalevski top is a dynamical system with the following integrals of motion

$$\begin{aligned} H_1 &= J_1^2+J_2^2+2J_3^2+c_1x_1, \qquad c_1\in\mathbb{R}, \\ H_2 &= (J_1^2+J_2^2)^2-2\Big(x_1(J_1^2-J_2^2)+2x_2J_1J_2\Big)c_1+(x_1^2+x_2^2)c_1^2. \end{aligned} \tag{2.1}$$

Here J_i are the components of the angular momentum in the moving frame of coordinates attached to the principal axes of inertia. The position of a rigid body is fixed by the components x_i of the Poisson vector, which are the cosines between the axes of the body frame and the field up to a constant.

Using the Hamilton function H_1 and the Lie-Poisson bracket $\{.,.\}$ on the Euclidean algebra $e^*(3)$ the customary Euler-Poisson equations may be rewritten in the hamiltonian form

$$\dot J_i=\{J_i,H_1\}\,,\qquad \dot x_i=\{x_i,H_1\}\,,\qquad \text{where}\qquad \{f,g\}=\langle Pdf,dg\rangle\,. \tag{2.2}$$

In coordinates $z=(x_1,x_2,x_3,J_1,J_2,J_3)$ on $e^*(3)$ the Lie-Poisson bivector P is the following antisymmetric matrix

$$P=\begin{pmatrix} 0&0&0&0&x_3&-x_2\\ *&0&0&-x_3&0&x_1\\ *&*&0&x_2&-x_1&0\\ *&*&*&0&J_3&-J_2\\ *&*&*&*&0&J_1\\ *&*&*&*&*&0 \end{pmatrix}.$$

It has two Casimir elements

$$PdC_{1,2}=0,\qquad C_1=|x|^2\equiv\sum_{k=1}^3 x_k^2,\qquad C_2=\langle x,J\rangle\equiv\sum_{k=1}^3 x_kJ_k. \tag{2.3}$$

After fixing values of the Casimir elements

$$C_1=a^2,\qquad C_2=b$$

one gets a generic four-dimensional symplectic leaf $\mathcal{O}_{ab}$, which is topologically equivalent to the cotangent bundle $T^*\mathcal{S}^2$ of the sphere $\mathcal{S}^2$ with radius a. However, the symplectic structure of $\mathcal{O}_{ab}$ is different from the standard symplectic structure on $T^*\mathcal{S}^2$ by the magnetic term proportional to b [13].

The Kowalevski top is an integrable system on the phase space $\mathcal{O}_{ab}$ because the two independent integrals of motion $H_{1,2}$ (2.1) are in the involution

$$\{H_1,H_2\}=\langle PdH_1,dH_2\rangle=0. \tag{2.4}$$

In mechanics the Casimir function C_1 is a norm of the unit Poisson vector such as $a=1$, whereas second Casimir function C_2 is called a square or cyclic integral of motion [2, 27].

Remark 1 In original Kowalevski work the first step in the separation of variables method is the complexification: she considers

$$\mathrm{z}_1=J_1+iJ_2,\qquad \mathrm{z}_2=J_1-iJ_2$$

as independent complex variables. Next she makes her famous change of variables

$$s_{1,2}=\frac{R(\mathrm{z}_1,\mathrm{z}_2)\pm\sqrt{R(\mathrm{z}_1,\mathrm{z}_1)R(\mathrm{z}_2,\mathrm{z}_2)}}{2(\mathrm{z}_1-\mathrm{z}_2)^2}.$$

The fourth degree polynomials $R(\mathrm{z}_i,\mathrm{z}_k)$ we will not specify here. It brings the system (2.2) to the form

$$(-1)^k\,(s_1-s_2)\dot s_k=\sqrt{P(s_k)}\,,\qquad k=1,2,$$

where

$$P(s) = 4\left((s-H)^2 - \frac{K}{4}\right)\left[s\left((s-H)^2 + c_1^2a^2 - \frac{K}{4}\right) + c_1^2 b\right] \tag{2.5}$$

Here the pairs $\left(s_k, \eta_k = \sqrt{P(s_k)}\right)$, can be regarded as coordinates of points on the Kowalevski curve of genus two

$$\mathcal{C}_{kow}: \qquad \eta^2 - P(s) = 0\,. \tag{2.6}$$

At zero level of the cyclic integral of motion $C_2 = 0$ the Kowalevski curve has the same genus 2.

We address the problem of separation of variables for the Hamilton-Jacobi equation as well. At $C_2 = 0$ the symplectic leaves $\mathcal{O}_{a0}$ are completely symplectomorphic to $T^*\mathcal{S}^2$ [13]. We will only consider such symplectic leaves and, therefore, all the formulae below hold true up to $C_2 = 0$.

For this Kowalevski top on the two-dimensional sphere we want to calculate different variables of separation and, according to the general usage of the bi-hamiltonian geometry, firstly we have to find the second dynamical Poisson bivector P' equipped with some necessary properties [5, 9].

2.2 Dynamical Poisson bivectors

According to [21, 22, 23, 25] let us suppose that the desired second Poisson bivector P' is the Lie derivative of P along some unknown Liouville vector field X

$$P' = \mathcal{L}_X(P)\,. \tag{2.7}$$

In addition it has to satisfy the following equations

$$[P', P'] \equiv [\mathcal{L}_X(P), \mathcal{L}_X(P)] = 0, \tag{2.8}$$

and

$$\{H_1, H_2\}' = \langle P' dH_1, dH_2\rangle = 0, \tag{2.9}$$

where $[.,.]$ is the Schouten bracket.

The first assumption (2.7) guarantees that this dynamical bivector P' is compatible with the given kinematic Poisson bivector P, i.e. $[P, P'] = 0$. In geometry such bivector P' is said to be the 2-coboundary associated with the Liouville vector field X in the Poisson-Lichnerowicz cohomology defined by P.

The second condition (2.8) means that P' is the Poisson bivector, i.e. that the Jacobi identity is true. The third equation (2.9) relates P' with the given integrable system. In the wake of this agreement the foliation defined by the $H_{1,2}$ is the bi-Lagrangian foliation [5, 9].

The system of equations (2.8-2.9) has infinitely many solutions with respect to X [20, 23]. So, in order to get some particular solution we have to narrow the search space. In this paper we suppose that

$$P' dC_{1,2} = 0, \tag{2.10}$$

and that the components X_j of the Liouville vector field $X = \sum X_j\,\partial_j$ are non-homogeneous polynomials in momenta J_k

$$X_j = \sum_{m=0}^{N}\sum_{k=0}^{m} \mathrm{g}_{jkm}^N(x_1, x_2, x_3)\, J_1^k\, J_2^{m-k}$$

with unknown coefficients $\mathrm{g}(x_1, x_2, x_3)$ [21, 22, 25]. Here we explicitly use the restriction $C_2 = 0$, i.e. that $J_3 = -(x_1J_1 + x_2J_2)/x_3$.

Upon substituting this polynomial *ansätze* into the equations (2.8,2.9-2.10) and demanding that all the coefficients at powers of J_k vanish one gets the over determined system of algebro-differential equations. Such systems are solved on personal computer by using modern software in a few seconds. So, the only real problem is the classification and the analysis of the received computer results.

The first three nontrivial solutions arise only in the cubic case $N=3$. Components of the first real vector field $X^{(1)}$ are equal to

$$
\begin{aligned}
X_1^{(1)} &= -\frac{\sqrt{x_1^2+x_2^2}\Big(x_1J_1-x_2J_2\Big)J_3}{2x_1x_3}, \qquad X_2^{(1)}=\frac{\sqrt{x_1^2+x_2^2}\Big(x_1J_1-x_2J_2\Big)J_3}{2x_2x_3}, \qquad X_3^{(1)}=0,\\
X_4^{(1)} &= -\sqrt{x_1^2+x_2^2}\left(\frac{(x_1^2+x_2^2)J_1^3}{6x_2^2x_3^2}-\frac{J_2^2J_3}{2x_1x_3}\right)+\frac{c_1x_3J_3}{4\sqrt{x_1^2+x_2^2}}\,,\\
X_5^{(1)} &= -\sqrt{x_1^2+x_2^2}\left(\frac{(x_1^2+x_2^2)J_2^3}{6x_1^2x_3^2}-\frac{J_1^2J_3}{2x_2x_3}\right)+\frac{c_1(x_1J_2-x_2J_1)}{4\sqrt{x_1^2+x_2^2}}\,,\\
X_6^{(1)} &= -\sqrt{x_1^2+x_2^2}\frac{(x_1^2+x_2^2)J_3^3}{6x_1^2x_2^2}+\frac{c_1\sqrt{x_1^2+x_2^2}J_1}{4x_3}\,.
\end{aligned}
\tag{2.11}
$$

The components of the second real vector field $X^{(2)}$ read as

$$
\begin{aligned}
X_1^{(2)} &= \frac{2(x_1^2+x_2^2)}{x_3}\,J_1J_3\,, \qquad X_2^{(2)}=-\frac{2x_1(x_1^2+x_2^2)}{x_2x_3}\,J_1J_3\,, \qquad X_3^{(2)}=0\,,\\
X_4^{(2)} &= \frac{(x_1^2+x_2^2)^2}{3x_2^2x_3^2}\,J_1^3-\left(J_1+\frac{x_1x_3}{3x_2^2}J_3\right)J_3^2+\frac{x_1^2+x_2^2}{x_2x_3}\,J_1J_2J_3+\frac{c_1x_2J_2}{2}\,,\\
X_5^{(2)} &= \frac{(x_1^2+x_2^2)^2}{3x_1^2x_3^2}\,J_2^3-\left(J_2-\frac{(2x_1^2-x_2^2)x_3}{3x_1^2x_2}\,J_3-\frac{2(x_1^2+x_2^2)}{x_1x_2}J_1\right)J_3^2\\
&+ \frac{x_1^2+x_2^2}{x_1x_3}\,J_1J_2J_3-\frac{c_1(2x_1J_2-x_2J_1)}{2}\\
X_6^{(2)} &= \frac{2x_1^2+x_2^2}{3x_2^2}\,J_3^3+\frac{c_1x_2(x_1J_2-x_2J_1)}{2x_3}
\end{aligned}
\tag{2.12}
$$

The components of the third vector field $X^{(3)}$ are the complex functions on initial variables

$$
\begin{aligned}
X_1^{(3)} &= -\frac{\mathrm{i}x_2(x_1+\mathrm{i}x_2)^2}{x_1^2}\,J_2^2+\frac{2x_2(x_1+\mathrm{i}x_2)}{x_1}\,J_1J_2, \qquad \mathrm{i}=\sqrt{-1}\,,\\
X_2^{(3)} &= \frac{\mathrm{i}(x_1+\mathrm{i}x_2)^2}{x_1}J_2^2-2(x_1+\mathrm{i}x_2)J_1J_2, \qquad X_3^{(3)}=0\,,\\
X_4^{(3)} &= (J_1-\mathrm{i}J_2)J_2^2-\frac{1}{3}J_1^3+\frac{2(2x_1+\mathrm{i}x_2)x_3}{3x_1^2}\,J_3^3+\frac{4x_2}{3x_1}J_2^3+\frac{\mathrm{i}x_2(x_1J_1-x_3J_3)}{x_1^2}J_2^2\\
&+ c_1x_3J_3\,,\\
X_5^{(3)} &= \frac{2\mathrm{i}}{3}J_1^3-(J_1-\mathrm{i}J_2)J_1J_2-\frac{1}{3}J_2^3-\frac{2\mathrm{i}x_3}{3x_1}J_3^3+\frac{x_2^2(2x_1-\mathrm{i}x_2)}{3x_1^3}J_2^3-\frac{\mathrm{i}x_2^2x_3}{x1^3}J_2^2J_3\\
&+ \mathrm{i}c_1x_3J_3\,,\\
X_6^{(3)} &= \frac{2}{3}\frac{(x_1+\mathrm{i}x_2)x_3^2-2x_1^3}{x_1^3}\,J_3^3-(J_1^2+J_2^2)\,J_3-c_1(x_1+\mathrm{i}x_2)\,J_3\,.
\end{aligned}
\tag{2.13}
$$

The quartic ansätze yields a lot of solutions, which will be classified and studied in future.

Let us show the simplest part of these real and complex Poisson brackets explicitly

$$
\begin{aligned}
&\{x_i,x_j\}=\varepsilon_{ijk}\,x_k\,, && \{x_i,x_j\}^{(1)}=\varepsilon_{ijk}\frac{\sqrt{x_1^2+x_2^2}\,J_3}{x_3}\,x_k\,,\\
&\{x_i,x_j\}^{(2)}=-\varepsilon_{ijk}\frac{2(x_1^2+x_2^2)\,J_3}{x_3}\,x_k\,, && \{x_i,x_j\}^{(3)}=2\mathrm{i}\varepsilon_{ijk}\,(\mathrm{i}x_3J_3-x_1J_2+x_2J_1)x_k\,.
\end{aligned}
$$

Here ε_{ijk} is the totally skew-symmetric tensor. Other brackets are appreciably more tedious expressions. The complex Poisson structure may be rewritten in the lucid form by using the 2×2 Lax matrices [7, 18] and the bi-hamiltonian structure associated with the reflection equation algebra [24].

It is easy to prove that the corresponding Poisson bivectors $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ have the following properties

$$
[P^{(1)},P^{(2)}]=0,\qquad [P^{(1)},P^{(3)}]\neq 0,\qquad [P^{(2)},P^{(3)}]\neq 0 \tag{2.14}
$$

with respect to the Schouten brackets. It means that $P^{(1)}$ and $P^{(2)}$ are compatible bivectors, whereas the complex bivector $P^{(3)}$ is incompatible with the real bivectors.

Remark 2 For any bivectors P and P' there are a lot of vector fields X, such as $P' = \mathcal{L}_X(P)$. Above we put $X_3 = 0$ in order to restrict this freedom. It may be the origin of some non-symmetry and irregularity in expressions (2.11,2.12) and (2.13).

Remark 3 There are two *rational* Poisson bivectors P' for the Kowalevski top. The first bivector is associated with the Kowalevski variables of separation and the underlying reflection equation algebra [24]. The second bivector is related with the Lax matrix of Reyman-Semenov-Tian-Shansky and the linear r-matrix algebra [19]. The components of the corresponding vector fields X are logarithmic functions in momenta.

To sum up, using the applicable polynomial ansätze for the Liouville vector field X we got two compatible real cubic bivectors $P^{(1,2)} = \mathcal{L}_{X^{(1,2)}}(P)$ and one complex cubic bivector $P^{(3)} = \mathcal{L}_{X^{(3)}}(P)$ for the Kowalevski top. Although these bivectors are defined by arbitrary value of C_2, they are compatible with the initial Poisson bivector P only for $C_2 = 0$. The application of this Poisson bivectors will be given in the next section.

3 Calculation of the variables of separation and the separation relations

A system of canonical variables $(q_1, \dots, q_n, p_1, \dots, p_n)$

$$\{q_i, q_k\} = \{p_i, p_k\} = 0, \qquad \{q_i, p_k\} = \delta_{ik} \tag{3.1}$$

will be called *separated* if there are n relations of the form

$$\Phi_i(q_i, p_i, H_1, \dots, H_n) = 0 \ , \quad i = 1, \dots, n \ , \qquad \text{with } \det \left[\frac{\partial \Phi_i}{\partial H_j} \right] \neq 0 \, , \tag{3.2}$$

binding together each pair (q_i, p_i) and $H_1, \dots, H_n$.

The reason for this definition is that the stationary Hamilton-Jacobi equations for the Hamiltonians $H_i = \alpha_i$ can be collectively solved by the additively separated complete integral

$$W(q_1, \dots, q_n; \alpha_1, \dots, \alpha_n) = \sum_{i=1}^{n} W_i(q_i; \alpha_1, \dots, \alpha_n) \, , \tag{3.3}$$

where W_i are found by the quadratures as the solutions of the ordinary differential equations.

The integrals of motion $(H_1, \dots, H_n)$ are the Stäckel separable integrals if the separation relations (3.2) are given by affine equations in H_j, that is,

$$\sum_{j=1}^{n} S_{ij}(q_i, p_i) H_j - U_i(q_i, p_i) = 0 \ , \qquad i = 1, \dots, n \ , \tag{3.4}$$

with an invertible matrix S. The functions S_{ij} and U_i depend only on one pair (q_i, p_i) of the canonical variables of separation, it means that

$$\{S_{ik}, q_j\} = \{S_{ik}, p_j\} = \{S_{ik}, S_{jm}\} = 0, \qquad i \neq j, \tag{3.5}$$

and similar to U

$$\{U_i, q_j\} = \{U_i, p_j\} = \{U_i, U_j\} = 0, \qquad i \neq j. \tag{3.6}$$

In this case S is called the *Stäckel matrix*, and U the *Stäckel potential*.

Remark 4 We have to point out that the definition of the Stäckel separability depends on the choice of H_i. Indeed, if $(H_1, \dots, H_n)$ are Stäckel-separable, then $\widehat{H}_i = \widehat{H}_i(H_1, \dots, H_n)$ will not, in general, fulfill the affine relations of the form (3.4).

Remark 5 The method of the separation of variables for a long time served an important, but technical role in solving Liouville integrable systems of classical mechanics. A new, and much more exciting application of the method came with the development of quantum integrable systems. Because of the fact that the quantization of the action variables seems to be a rather formidable task, quantum separation of variables became an inevitable refuge. In fact, it can be successfully performed for many families of integrable systems with affine separated relations (3.4) only [16, 17].

So, our second step is the calculation of the canonical variables of separation (q_i, p_i) and of the separation relations Φ_i (3.2). According to [5, 9], the coordinates of separation q_i are eigenvalues of the recursion operator, which are the so called Darboux-Nijenhuis variables. In order to get the recursion operator $N = \widehat{P}'\widehat{P}^{-1}$ we have to find restrictions $\widehat{P}, \widehat{P}'$ of the Poisson bivectors P and P' onto the symplectic leaves.

We can avoid the procedure of restriction using the $n \times n$ control matrix F defined by

$$P'\mathbf{dH} = P\big(F\mathbf{dH}\big), \qquad \text{or} \qquad P'dH_i = P\sum_{j=1}^{n} F_{ij}\,dH_j, \qquad i = 1,\ldots,n. \tag{3.7}$$

The bi-involutivity of the integrals of motion (2.4,2.9) is equivalent to the existence of F, whereas the imposed condition (2.10) ensures that F is a non-degenerate matrix. In this case eigenvalues of this matrix coincide with the Darboux-Nijenhuis variables and we can easily *calculate* the desired coordinates of separation q_i.

Moreover, for the Stäckel separable systems the suitable normalized left eigenvectors of the control matrix F form the Stäckel matrix S

$$F = S^{-1}\,\mathrm{diag}\,(q_1,\ldots,q_n)\,S$$

which would allow us to get separated relations (3.4).

So, the main problems are the finding of the conjugated momenta p_i and the construction of the separation relations ϕ_j (3.2) for the generic non-Stäckel separable systems. Below we show how we can solve these problems using the same control matrix F and some additional useful observations.

3.1 The real compatible Poisson bivectors

For the first Poisson bivector $P^{(1)}$ (2.11) the entries of the control matrix $F^{(1)}$ read as

$$\begin{aligned}
F^{(1)}_{11} &= \frac{(2x_1^2+2x_2^2+x_3^2)(J_1^2+J_2^2)}{4x_3^2\sqrt{x_1^2+x_2^2}} \qquad F^{(1)}_{12} = -\frac{1}{8\sqrt{x_1^2+x_2^2}}\\
F^{(1)}_{21} &= \frac{(2x_1^2+2x_2^2+x_3^2)(J_1^2+J_2^2)}{2x_3^2\sqrt{x_1^2+x_2^2}} - \frac{c_1\sqrt{x_1^2+x_2^2}\big(x_1(J_1^2-J_2^2)+2x_2J_1J_2\big)}{x_3^2} - \frac{c_1^2\sqrt{x_1^2+x_2^2}}{2}\\
F^{(1)}_{22} &= -\frac{J_1^2+J_2^2}{2\sqrt{x_1^2+x_2^2}}
\end{aligned}$$

The eigenvalues $q_{1,2}$ of this matrix are the required variables of separation $q_{1,2}$

$$\begin{aligned}
\det(F^{(1)}-\lambda I) &= (\lambda-q_1)(\lambda-q_2)\\
&= \lambda^2 - \frac{\sqrt{x_1^2+x_2^2}(J_1^2+J_2^2)}{2x_3^2}\lambda - \frac{c_1\Big(2x_1(J_1^2-J_2^2)+4x_2J_1J_2+c_1x_3^2\Big)}{16x_3^2}.
\end{aligned}$$

The matrix of normalized eigenvectors of $F^{(1)}$ does not form the Stäckel matrix, because property (3.5) is missed, and the underlying separation relations differ from the Stäckel affine equations (3.4) in $H_{1,2}$.

For the second Poisson bivector $P^{(2)}$ (2.12) the entries of the control matrix $F^{(2)}$ are equal to

$$
\begin{aligned}
F^{(2)}_{11} &= -\frac{J_1^2+J_2^2-c_1x_1}{2}+\frac{(x_1J_2-x_2J_1)^2}{x_3^2}\,, \qquad F^{(2)}_{12}=\frac{1}{4}\,,\\
F^{(2)}_{21} &= -(J_1^2+J_2^2)^2\left(1+\frac{2(x_1^2+x_2^2)}{x_3^2}\right)+c_1(x_1^2+x_2^2)\left(\frac{2\Big(x_1(J_1^2-J_2^2)+2x_2J_1J_2\Big)}{x_3^2}+c_1\right)\,,\\
F^{(2)}_{22} &= \frac{J_1^2+J_2^2+2J_3^2+c_1x_1}{2}\,.
\end{aligned}
$$

The eigenvalues $f_{1,2}$ of the matrix $F^{(2)}$ are the roots of the equation

$$
\begin{aligned}
\det(F^{(2)}-\lambda I) &= (\lambda-f_1)(\lambda-f_2)\\
&= \lambda^2-\left(c_1x_1-\frac{(x_2J_1-x_1J_2-x_3J_3)(x_2J_1-x_1J_2+x_3J_3)}{x_3^2}\right)\lambda\\
&- \frac{(2(x_2J_1-x_1J_2)J_3-c_1x_2x_3)^2}{4x_3^2}\,.
\end{aligned}
$$

Remark 6 According to [5, 9] the compatibility of $P^{(1,2)}$ (2.14) ensures that in the Darboux-Nijenhuis variables q,p the corresponding restrictions of $\widehat{P}^{(1,2)}$ look like

$$
\widehat{P}^{(1)}=\begin{pmatrix}0&0&q_1&0\\0&0&0&q_2\\-q_1&0&0&0\\0&-q_2&0&0\end{pmatrix},\qquad
\widehat{P}^{(2)}=\begin{pmatrix}0&0&f_1&0\\0&0&0&f_2\\-f_1&0&0&0\\0&-f_2&0&0\end{pmatrix}.
$$

where $f_{1,2}$ are the functions on q,p such as

$$
\{q_i,f_j\}=\{p_i,f_j\}=0,\qquad i\neq j\,.
$$

So, f_1 is the function only on q_1 and p_1 and similar f_2 is the function on q_2 and p_2.

We can find these functions $f_{1,2}$ using the Poisson bracket. Namely, it is easy to see that the recurrence chain

$$
\phi_1=\{f_1(q_1,p_1),q_1\},\quad \phi_2=\{\phi_1,q_1\},\quad \ldots,\quad \phi_i=\{\phi_{i-1},q_1\} \tag{3.8}
$$

breaks down on the third step $\phi_3=0$. It means that f_1 is the second order polynomial in momenta p_1 and, therefore, we can define this unknown momenta in the following way

$$
p_1=\frac{\phi_1}{\phi_2}=\frac{2x_3\Big(4(x_2J_1-x_1J_2)\,q_1+c_1\sqrt{x_1^2+x_2^2}J_2\Big)}{(4\sqrt{x_1^2+x_2^2}(J_1^2+J_2^2)\,q_1+c_1\Big(x_1(J_1^2-J_2^2)+2x_2J_1J_2\Big)} \tag{3.9}
$$

up to canonical transformations $p_1\to p_1+\mathrm{g}(q_1)$.

The similar calculation for the function $f_2(q_2,p_2)$ yields the definition of the second momenta

$$
p_2=\frac{2x_3\Big(4(x_2J_1-x_1J_2)\,q_2+c_1\sqrt{x_1^2+x_2^2}J_2\Big)}{(4\sqrt{x_1^2+x_2^2}(J_1^2+J_2^2)\,q_2+c_1\Big(x_1(J_1^2-J_2^2)+2x_2J_1J_2\Big)}\,. \tag{3.10}
$$

In fact we have to substitute q_2 instead of q_1 only.

So, one gets the canonical transformation from the initial physical variables (x,J) to the variables of separation (q,p) using a pair of compatible bivectors $P^{(1,2)}$ and the corresponding control matrices $F^{(1,2)}$.

In these separated variables entries of the matrix S of normalized eigenvectors of $F^{(1)}$ depend on the pair of variables (q_i,p_i) and the Hamilton function

$$
S_{i1}=-2H_1-\left(4q_i^2-\frac{c_1^2}{4}\right)p_i^2,\qquad S_{1,2}=1,\qquad i=1,2.
$$

In this case S may be called the *generalized* Stäckel matrix. The separated relations (3.4) look like

$$S_{i1}H_1 + H_2 - \left(H_1^2 - \frac{(c_1^2-16q_i)^2p_i^4}{64} - a^2(c_1^2-16q_i^2)\right) = 0, \qquad i=1,2.$$

and, therefore, the *generalized* Stäckel potential U_i depends on the Hamilton function too.

So, we can say that the variables of separation (q_i,p_i) lie on the algebraic hyperelliptic curve $\mathcal{C}$ of genus three defined by

$$\begin{aligned}\mathcal{C}: \qquad \Phi(q,p) &= \left(\frac{(c_1^2-16q^2)p^2}{8} - H_1 - \sqrt{H_2}\right)\left(\frac{(c_1^2-16q^2)p^2}{8} - H_1 + \sqrt{H_2}\right)\\ &- a^2(c_1^2-16q^2) = 0\,. \end{aligned}\tag{3.11}$$

This equation is invariant with respect to involution $(q,p)\to(-q,p)$. Factorization with respect to this involution give rise to elliptic curve

$$\begin{aligned}\mathcal{E}: \qquad \Phi(z,p) &= \left(\frac{(c_1^2-16z)p^2}{8} - H_1 - \sqrt{H_2}\right)\left(\frac{(c_1^2-16z)p^2}{8} - H_1 + \sqrt{H_2}\right)\\ &- a^2(c_1^2-16z) = 0\,, \qquad z=q^2 \end{aligned}\tag{3.12}$$

Due to the standard formalism we have to calculate differential on this curve

$$\Omega = \frac{dz}{Z(z,p)}, \qquad Z(z,p) = p\,(c_1^2-16z)\Big(8H_1 - p^2(c_1^2-16z)\Big).$$

Then it's easy to prove that

$$\frac{\dot q_1}{Z(q_1^2,p_1)} + \frac{\dot q_2}{Z(q_2^2,p_2)} = 0\,, \qquad \frac{(c_1^2-16q_1^2)p_1^2\dot q_1}{Z(q_1^2,p_1)} + \frac{(c_1^2-16q_2^2)p_2^2\dot q_2}{Z(q_2^2,p_2)} = -\frac{1}{4} \tag{3.13}$$

and

$$\begin{aligned}&\int^{q_1}\frac{dq}{Z(q^2,p)} + \int^{q_2}\frac{dq}{Z(q^2,p)} = \beta_1\,,\\ &\int^{q_1}\frac{(c_1^2-16q^2)p^2dq}{Z(q^2,p)} + \int^{q_2}\frac{(c_1^2-16q^2)p^2dq}{Z(q^2,p)} = -\frac{t}{4} + \beta_2\end{aligned}$$

where p has to be obtained from (3.11).

Remark 7 The equations of motion are linearized on an abelian variety, which is roughly spiking the *complexified* of the corresponding Liouville real torus. So, even though $q_{1,2}$ are the real variables of separation we have to solve the Jacobi inversion problem over the complex field, see more detailed discussion in [1, 4].

Remark 8 We have to point out the Kowalevski separation of variables leading to hyperelliptic quadratures, whereas in the new variables of separation $q_{1,2}$ equations of motion are integrable by quadratures in terms of elliptic functions.

The third part of the Jacobi method consists of the construction of new integrable systems starting with known variables of separation and some other separated relations [6]. If we substitute our variables of separation (q,p) into the following deformation of (3.11)

$$\Phi^{(d)}(p,q) = \Phi(p,q) - 8d_1q - 16d_2q^2 = 0, \qquad d_1,d_2\in\mathbb{R}, \tag{3.14}$$

we get the following generalization of the initial Hamilton function

$$H_1^{(d)} = J_1^2 + J_2^2 + 2J_3^2 + c_1x_1^2 + \frac{d_1}{\sqrt{x_1^2+x_2^2}} + \frac{d_2}{x_3^2}\,(J_1^2+J_2^2)\,. \tag{3.15}$$

Here the main problem is how to get the Hamiltonian to be interesting to physics. For example, in our case we obtained the natural Hamiltonian at $d_2=0$ only. For this system only the integrals of motion have been known [27].

The apparent problem is that generalized equations (3.14) have not involution $(q,p)\to(-q,p)$ at $d_1\neq 0$. Thereby equations of motion are related with the hyperelliptic curve of genus three [11, 12] instead of elliptic curve. Nevertheless, at $d_1\neq 0$, $d_2=0$ we have the same equations (3.13) as above

$$\begin{aligned}\frac{\dot{q}_1}{Z(q_1^2,p_1)}+\frac{\dot{q}_2}{Z(q_2^2,p_2)} &= 0\,,\\ \frac{(c_1^2-16q_1^2)p_1^2\dot{q}_1}{Z(q_1^2,p_1)}+\frac{(c_1^2-16q_2^2)p_2^2\dot{q}_2}{Z(q_2^2,p_2)} &= -\frac{1}{4}\end{aligned}\qquad(3.16)$$

where $p_{1,2}$ satisfy to the deformed equations (3.14).

3.2 The complex Poisson bivector

For the cubic in momenta Poisson bivector $P^{(3)}$ (2.13) the control matrix is equal to

$$F^{(3)}=\begin{pmatrix}2(J_1^2+J_2^2+2J_3^2)+c_1(x_1+\mathrm{i}x_2) & -\dfrac{1}{2}\\ 2(J_1^2+J_2^2)^2-2c_1(x_1-\mathrm{i}x_2)(J_1+\mathrm{i}J_2)^2 & 0\end{pmatrix}$$

and the Darboux-Nijenhuis coordinates $\lambda_{1,2}$ are the roots of the characteristic polynomial

$$\det(F-\lambda I)=(\lambda-\lambda_1)(\lambda-\lambda_2)=\lambda^2-F_{11}^{(3)}\lambda+\frac{F_{21}^{(3)}}{2}\,.\qquad(3.17)$$

As above we can get the conjugated momenta $\mu_{1,2}$ by using compatible with $P^{(3)}$ bivector of fourth order in momenta J_k. However we can do it without such calculations as well.

It is easy to see that in this case matrix S of normalized eigenvectors of $F^{(3)}$ is the standard Stäckel matrix

$$S_{i1}=-2\lambda_i,\qquad S_{i,2}=1,\qquad i=1,2,$$

and, therefore, the Stäckel potentials

$$U_{1,2}=-(S_{i1}H_1+H_2)$$

are some functions on (λ_1,μ_1) and (λ_2,μ_2), respectively.

In fact notion of the Stäckel potentials allows us to find the unknown conjugated momenta $\mu_{1,2}$ using the Poisson brackets only. Namely, the following recurrence chain of the Poisson brackets

$$\phi_1=\{\lambda_1,U_1\},\qquad \phi_2=\{\lambda_1,\phi_1\},\ldots,\qquad \phi_i=\{\lambda_1,\phi_{i-1}\}\qquad(3.18)$$

is a quasi-periodic chain

$$\phi_3=16\lambda_1\phi_1.$$

It means that the Stäckel potential U_1 is a trigonometric function on momenta μ_1 and, therefore, we can determine this desired momenta

$$\mu_1=\varphi(\lambda_1)\ln\Bigl(\sqrt{16\lambda_1}\,\phi_1+\phi_2\Bigr)$$

up to canonical transformations $\mu_1\to\mu_1+\mathrm{g}(\lambda_1)$. Here the function $\varphi(\lambda_1)$ is easily calculated from $\{\lambda_1,\mu_1\}=1$.

These variables of separation (λ_i,μ_i) lie on the hyperelliptic curve of genus three

$$\widetilde{\mathcal{C}}:\qquad \widetilde{\Phi}(\lambda,\mu)=\mathrm{e}^{4\mathrm{i}\sqrt{\lambda}\mu}+\frac{a^4c_1^4}{16}\,\mathrm{e}^{-4\mathrm{i}\sqrt{\lambda}\mu}+\lambda^2-2H_1\lambda+H_2=0\,.\qquad(3.19)$$

According to [10], this curve $\widetilde{\mathcal{C}}$ are related with an elliptic curve $\widetilde{\mathcal{E}}$ and equations of motion for the Kowalevski top are linearized on the corresponding abelian variety.

One main difference is that the variables of separation $\lambda_{1,2}$ are complex functions on the initial variables (x,J), whereas $q_{1,2}$ are real functions on them. It will be important when we express initial real variables via real or complex variables of separation after solving of the Jacobi inversion problem over the complex field.

The other difference is that the affine relations of separations (3.19) allows us to study quantum counterpart of the Kowalevski top [7, 17]. For real variables of separation the procedure of quantization is unknown.

Remark 9 In framework of the Sklyanin formalism [16] variables of separation are the poles of the Baker-Akhiezer function with suitable normalization. In [7, 18] we find such variables of separation $u_{1,2}$ for the Kowalevski-Goryachev-Chaplygin gyrostat

$$\widehat{H}_1 = J_1^2 + J_2^2 + 2J_3^2 + \rho J_3 + c_1 x_1 + c_2(x_1^2 - x_2^2) + c_3 x_1 x_2 + \frac{c_4}{x_3^2} \tag{3.20}$$

using 2×2 Lax matrix, its Baker-Akhiezer vector-function and the reflection equation algebra.

It is easy to prove that the Darboux-Nijenhuis variables $\lambda_{1,2}$ (3.17) are related with the poles $u_{1,2}$ of the Baker-Akhiezer function by the following point transformation

$$\lambda_{1,2} = u_{1,2}^2\,, \tag{3.21}$$

which gives rise to a ramified two-sheeted covering of $\widetilde{\mathcal{C}}$, see [10].

4 Conclusion

Starting with the integrals of motion for the Kowalevski top we found three polynomial in momenta Poisson bivectors, which are compatible with the canonical Poisson bivector on the cotangent bundle $T^*\mathcal{S}^2$ of two-dimensional sphere.

Then in framework of the bi-hamiltonian geometry we get new real variables of separation (q,p) for the Kowalevski top on the sphere and reproduce known complex variables (λ,μ). These variables are related by the canonical transformation

$$\lambda_{1,2} = \lambda_{1,2}(q_1,q_2,p_1,p_2), \qquad \mu_{1,2} = \mu_{1,2}(q_1,q_2,p_1,p_2),$$

which may be rewritten as a quasi-point canonical transformation [14]

$$\lambda_{1,2} = \lambda_{1,2}(q_1,q_2,H_1,H_2),$$

which relates two hyperelliptic curves of genus three. We can assume that it is no rational cover and that these Jacobians are non-isogeneous in Richelot sense [15]. Similar transformations relate these curves with the Kowalevski curve of genus two. Further inquiry of such relations between hyperelliptic curves goes beyond the scope of this paper, see discussion in [1, 3, 10].

The proposed approach may be useful for the investigation of other integrable systems with integrals of motion higher order in momenta, for instance, the search of another real variables of separation for the Kowalevski-Goryachev-Chaplygin gyrostat and its various generalizations [27].

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