Deformations of Poisson structures by closed 3-forms[1]

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Abstract

We prove that an arbitrary Poisson structure $\omega^{ij}(u)$ and an arbitrary closed 3-form $T_{ijk}(u)$ generate the local Poisson structure $A^{ij}(u, u_x) = M^i_s(u, u_x)\omega^{sj}(u)$, where $M^i_s(u, u_x)(\delta^s_j + \omega^{sp}(u)T_{pjk}(u)u^k_x) = \delta^i_j$, on the corresponding loop space. We obtain also a special graded ε-deformation of an arbitrary Poisson structure $\omega^{ij}(u)$ by means of an arbitrary closed 3-form $T_{ijk}(u)$.

In this paper we prove that an arbitrary Poisson structure $\omega^{ij}(u)$ and an arbitrary closed 3-form $T_{ijk}(u)$ generate the local Poisson structure

$$A^{ij}(u, u_x) = B^i_s(u, u_x)\omega^{sj}(u), \tag{1}$$

where

$$B^i_s(u, u_x)M^s_j(u, u_x) = \delta^i_j, \qquad M^s_j(u, u_x) = \delta^s_j + \omega^{sp}(u)T_{pjk}(u)u^k_x, \tag{2}$$

i.e., the matrix operator $A^{ij}(u, u_x)$ gives the Poisson bracket

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} A^{ij}(u, u_x) \frac{\delta J}{\delta u^j(x)} dx \tag{3}$$

on the space of functionals on the corresponding loop space.

Let M^N be an arbitrary smooth N-dimensional manifold with the local coordinates $u = (u^1, \ldots, u^N)$. By the *loop space* ΩM of the manifold M^N we mean, in this paper, the space of all smooth parametrized mappings of the circle S^1 into M^N, $\gamma : S^1 \to M^N$, $\gamma(x) = \{u^i(x)\}$, $x \in S^1$. The tangent space $T_\gamma \Omega M$ of the loop space ΩM at the point γ consist of all smooth vector fields $\xi = \{\xi^i,\ 1 \le i \le N\}$, defined along the loop γ with $\xi(\gamma(x)) \in T_{\gamma(x)}M$, $\forall x \in S^1$, where $T_{\gamma(x)}M$ is a tangent space of the manifold M at the

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point $\gamma(x)$. All closed 2-forms (presymplectic structures) on the loop space ΩM that are given by matrix operators of the form $\omega_{ij}(u, u_x, \ldots, u_{(k)})$, i.e., all closed 2-forms of the form

$$\omega(\xi, \eta) = \int_{S^1} \xi^i \omega_{ij}(u, u_x, \ldots, u_{(k)}) \eta^j dx, \tag{4}$$

where $\xi, \eta \in T_\gamma \Omega M$, were completely described in [1] (see also descriptions of various differential-geometric classes of symplectic (presymplectic) and Poisson structures in [2]–[9]).

Theorem 1 [1]. *A bilinear form* (4) *is a closed skew-symmetric 2-form* (*a presymplectic structure*) *on the loop space* ΩM *if and only if*

$$\omega_{ij}(u, u_x, \ldots, u_{(k)}) = T_{ijk}(u) u^k_x + \Omega_{ij}(u), \tag{5}$$

where $T_{ijk}(u)$ *is an arbitrary closed 3-form on the manifold* M^N *and* $\Omega_{ij}(u)$ *is an arbitrary closed 2-form on* M^N*.*

If the matrix $\omega_{ij}(u, u_x, \ldots, u_{(k)})$ is nondegenerate, $\det(\omega_{ij}(u, u_x, \ldots, u_{(k)})) \neq 0$, then the corresponding presymplectic form (4), (5) is symplectic and the inverse matrix $\omega^{ij}(u, u_x, \ldots, u_{(k)})$, $\omega^{is}(u, u_x, \ldots, u_{(k)}) \omega_{sj}(u, u_x, \ldots, u_{(k)}) = \delta^i_j$, gives the Poisson structure

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \omega^{ij}(u, u_x, \ldots, u_{(k)}) \frac{\delta J}{\delta u^j(x)} dx \tag{6}$$

on the loop space ΩM, i.e., the bracket (6) is skew-symmetric and satisfy the Jacobi identity. Therefore Theorem 1 gives the complete description of all nondegenerate Poisson structures on the loop space ΩM that are given by matrix operators of the form $\omega^{ij}(u, u_x, \ldots, u_{(k)})$, i.e., all the nondegenerate Poisson brackets of the form (6), $\det(\omega^{ij}(u, u_x, \ldots, u_{(k)})) \neq 0$ (such nondegenerate Poisson structures were studied by Astashov and Vinogradov in [9], see also [7]–[8] and [1]–[6]). We note that if the closed 2-form $\Omega_{ij}(u)$ is nondegenerate, $\det(\Omega_{ij}(u)) \neq 0$, i.e, the form $\Omega_{ij}(u)$ is symplectic on M^N, then the 2-form (5) is a nondegenerate form on ΩM for any closed 3-form $T_{ijk}(u)$ on the manifold M^N since it is obvious that in this case $\det(T_{ijk}(u) u^k_x + \Omega_{ij}(u)) \neq 0$. Thus we can define, on the loop space of an arbitrary symplectic manifold M^N, the Poisson bracket

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \omega^{ij}(u, u_x) \frac{\delta J}{\delta u^j(x)} dx, \tag{7}$$

where

$$\omega^{li}(u, u_x)(T_{ijk}(u) u^k_x + \Omega_{ij}(u)) = \delta^l_j, \tag{8}$$

I and J being arbitrary functionals on ΩM. The Poisson bracket (7), (8) is a partial case of the bracket (1)–(3), namely, the case when the Poisson structure $\omega^{ij}(u)$ is nondegenerate, $\det(\omega^{ij}(u)) \neq 0$, $\omega^{ij}(u) = \Omega^{ij}(u)$, $\Omega^{is}(u) \Omega_{sj}(u) = \delta^i_j$, since

$$\omega^{ij}(u, u_x) = C^i_s(u, u_x) \Omega^{sj}(u), \tag{9}$$

where

$$C^l_s(u,u_x)(\delta^s_j + \Omega^{si}(u)T_{ijk}(u)u^k_x) = \delta^l_j. \tag{10}$$

The case of degenerate Poisson structures $\omega^{ij}(u)$, $\det(\omega^{ij}(u)) = 0$, is much more complicated. We note that in contrast to the case of all closed 2-forms (presymplectic structures) of the form (4) (Theorem 1) the problem of description of all degenerate Poisson structures of the form (6) is a very complicated and unsolved problem.

Theorem 2. *An arbitrary Poisson structure $\omega^{ij}(u)$ and an arbitrary closed 3-form $T_{ijk}(u)$ give the local Poisson bracket* (1)–(3).

First of all, we note that obviously the matrix operator $A^{ij}(u,u_x)$ (1), (2) is skew-symmetric.

Lemma. *A skew-symmetric matrix operator $A^{ij}(u,u_x)$* (1) *gives a Poisson bracket* (3) *if and only if the following relations hold:*

$$\omega^{ij}(u)\omega^{rp}(u)\frac{\partial M^s_r}{\partial u^i_x} = \omega^{is}(u)\omega^{rj}(u)\frac{\partial M^p_r}{\partial u^i_x}, \tag{11}$$

$$\omega^{ij}(u)\omega^{rp}(u)\frac{\partial M^s_r}{\partial u^i} - \omega^{ij}(u)\frac{d}{dx}\left(\frac{\partial M^s_r}{\partial u^i_x}\omega^{rp}(u)\right) + \frac{\partial \omega^{ij}}{\partial u^r}\omega^{rp}(u)M^s_i(u) +$$
$$+\omega^{is}(u)\omega^{rj}(u)\frac{\partial M^p_r}{\partial u^i} + \frac{d}{dx}\left(\omega^{is}(u)\right)\frac{\partial M^p_r}{\partial u^i_x}\omega^{rj}(u) + \frac{\partial \omega^{is}}{\partial u^r}\omega^{rj}(u)M^p_i(u) +$$
$$+\omega^{ip}(u)\omega^{rs}(u)\frac{\partial M^j_r}{\partial u^i} + \frac{d}{dx}\left(\omega^{ip}(u)\right)\frac{\partial M^j_r}{\partial u^i_x}\omega^{rs}(u) + \frac{\partial \omega^{ip}}{\partial u^r}\omega^{rs}(u)M^j_i(u) = 0. \tag{12}$$

If $M^i_s(u,u_x) = \delta^s_j + \omega^{sp}(u)T_{pjk}(u)u^k_x$, then relations (11), (12) hold for an arbitrary Poisson structure $\omega^{ij}(u)$ and an arbitrary closed 3-form $T_{ijk}(u)$.

Let us add an arbitrary parameter ε in the formula for our Poisson structure:

$$A^{ij}(\varepsilon,u,u_x) = B^i_s(\varepsilon,u,u_x)\omega^{sj}(u), \tag{13}$$

where

$$B^i_s(\varepsilon,u,u_x)(\delta^s_j + \varepsilon\omega^{sp}(u)T_{pjk}(u)u^k_x) = \delta^i_j. \tag{14}$$

We can now expand the Poisson structure $A^{ij}(\varepsilon,u,u_x)$ in series in ε:

$$A^{ij}(\varepsilon,u,u_x) = \omega^{ij}(u) - \varepsilon\omega^{is}(u)T_{srk}(u)\omega^{rj}(u)u^k_x + \cdots. \tag{15}$$

This expansion give an ε-deformation of an arbitrary Poisson structure $\omega^{ij}(u)$ by means of an arbitrary closed 3-form $T_{ijk}(u)$.

We note that this ε-deformation of an arbitrary Poisson structure $\omega^{ij}(u)$ belongs to a special class of graded ε-deformations of Poisson structures (see, for example, [10], [11]).

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