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Chaotic Maps, Hamiltonian Flows, and Holographic Methods

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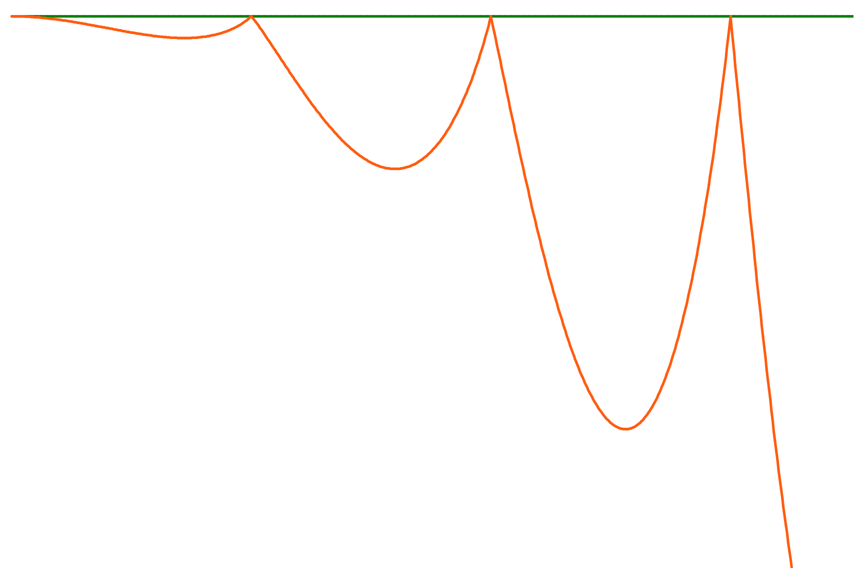
Abstract

Holographic functional methods are introduced as probes of discrete time-stepped maps that lead to chaotic behavior. The methods provide continuous time interpolation between the time steps, thereby revealing the maps to be splintered Hamiltonian systems underlain by novel potentials. A sequence of successively deepening switchback potentials for a particle driven by Hamiltonian dynamics explains, in very physical terms, the frequency doubling and trajectory folding that occur on the particular route to chaos revealed by the logistic map, $x \mapsto 4x(1-x)$.



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1. INTRODUCTION

In a previous paper [1] we have discussed how functions of position, defined on a discrete lattice of time points, may be smoothly interpolated in time by functions of both x and t, defined on a continuum of time points, through the use of solutions to Schröder's nonlinear functional equation [2]. If the effect of the first discrete time step is given as the map $x \mapsto f_1(x,s)$, for some parameter s, Schröder's functional equation is

$$s\Psi(x,s) = \Psi(f_1(x,s), s) \ , \tag{1}$$

with Ψ to be determined. So, $f_1(x,s) = \Psi^{-1}(s\Psi(x,s), s)$, where the inverse function Ψ^{-1} obeys Poincaré's equation,

$$\Psi^{-1}(sx,s) = f_1\left(\Psi^{-1}(x,s), s\right) \ . \tag{2}$$

The nth iterate of (1) gives $s^n\Psi(x,s) = \Psi(f_1(\cdots f_1(f_1(x,s),s)\cdots,s),s)$, with f_1 acting n times, and thus the nth order functional composition, $f_n(x,s) \equiv f_1(\cdots f_1(f_1(x,s),s)\cdots,s) = \Psi^{-1}(s^n\Psi(x,s),s)$ — the so-called "splinter" of the functional equation. A continuous interpolation between the integer lattice of time points is then, for *any* t,

$$f_t(x,s) = \Psi^{-1}\left(s^t\Psi(x,s), s\right) \ . \tag{3}$$

This can be a well-behaved and single-valued function of x and t provided that $\Psi^{-1}(x,s)$ is a well-behaved, single-valued function of x, even though $\Psi(x,s)$ might be, and typically is, multi-valued.

As discussed in [1], the interpolation can be envisioned as the trajectory of a particle,

$$x(t) = f_t(x,s) \ , \tag{4}$$

where the particle is moving under the influence of a potential according to Hamiltonian dynamics. The velocity of the particle is then found by differentiating (3) with respect to t,

$$\frac{dx(t)}{dt} = (\ln s)\, s^t\Psi(x)\left(\Psi^{-1}\left(s^t\Psi(x)\right)\right)' \ , \tag{5}$$

where any dependence of Ψ on s is implicitly understood. Therefore, the velocity will inherit and exhibit any multi-valuedness possessed by $\Psi(x)$.

Indeed, suppose that Hamiltonian dynamics have been specified, trajectories have been computed, and $f_1 = x(t=1)$ has emerged in terms of initial velocity and initial position $x(t=0) \equiv x$. A solution of the functional equation (1) can then be constructed, and expressed in very physical terms, as just an exponential of the time elapsed along any such particle trajectory that passes through position x. That solution is

$$\Psi(x) = s^{T(x)}\Psi_0 \ , \quad T(x) = \int^x \frac{dy}{v(y)} \ . \tag{6}$$

Here $v(x)$ is the velocity as a function of the position along the trajectory. For a particle moving in a potential $V(x)$ at fixed energy, E, it can be expressed in the usual way as $v(x) = \pm\sqrt{E - V(x)}$, with suitably chosen mass units.

When written in a form more closely related to Schröder's functional equation, the solution (6) is simply

$$\Psi\left(f_t\left(x\right)\right) = s^{\int_x^{f_t(x)} \frac{dy}{v(y)}}\ \Psi\left(x\right)\ . \tag{7}$$

At $t = 1$, $\int_x^{f_1(x)} \frac{dy}{v(y)} = 1$, and Schröder's equation (1) re-emerges. But, as this construction clearly shows, one must be careful at turning points where $v = 0$, especially if these are encountered at finite times along the trajectory. Typically these turning points produce branch points in Ψ so that it is multi-valued.

The interpolation (3) can also be viewed as a holographic specification on the x, t plane [3], determining $f_t\left(x\right)$ in the surrounded "bulk" from the total data given at the bounding times, $\{x\} \cup \{f_1\left(x\right)\}$. In this point of view, *fixed points* in x complete the boundary data and facilitate the solution of Schröder's equation through power series in x, hence leading to $V\left(x\right)$ *which was not known a priori* [1].

In this holographic approach, the potential first appears as a quadratic in Ψ/Ψ',

$$V\left(x\right) = -\left(\ln s\right)^2 \left(\frac{\Psi\left(x\right)}{\Psi'\left(x\right)}\right)^2\ , \tag{8}$$

up to an additive constant, where any dependence of Ψ on s is again implicit. The x dependence of the potential, $V\left(x\right) \equiv -v^2\left(x\right)$, follows from that of the velocity profile of the interpolation, defined and given by [4]

$$v\left(x\right) \equiv \left.\frac{df_t\left(x\right)}{dt}\right|_{t=0} = \left(\ln s\right)\Psi\left(x\right)\left(\Psi^{-1}\left(\Psi\left(x\right)\right)\right)' = \frac{\ln s}{\frac{d}{dx}\ln\Psi\left(x\right)}\ . \tag{9}$$

Monotonic motions between two fixed points, such as occur for the Ricker model [5], provide the most elementary examples [1]. However, situations where one or both of the fixed points are absent were not fully addressed in our previous work. This is precisely the situation that occurs when turning points are encountered at finite times in the particle dynamics interpretation, and leads to an intriguing modification in the physical picture involving the potential and its effect on the particle trajectories. We consider here a specific example of such a situation involving the well-studied logistic map of chaotic dynamics, namely, $x \mapsto sx\left(1-x\right)$ *[6–10]*.

Applying our functional methods to that example, the resulting interpolations from a discrete lattice of time points to a time continuum then allow us to appreciate analytic features of the logistic map, and to derive the governing differential evolution laws — indeed, subtly time-translation-invariant Hamiltonian dynamical laws — of the underlying physical system, hence to obtain potentials that were not previously known.

For the all-familiar logistic map illustrating transition to chaos, the functional interpolation reveals that there are well-defined expressions for the continuous time evolution of this map for *all* parametric values of the map, whether chaotic or not. We obtain agreement with the explicit closed-form solutions of (1) for the special values of the parameters: $s = -2,\ 2,$ and 4. (These explicit solutions have been known for almost a century and a half [2].) Moreover, as indicated, from (8) we can now find the potentials needed to produce these explicit, continuously evolving trajectories in Hamiltonian dynamics.

A new feature for the potentials so obtained for the $s = 4$ case is that they must change at discrete intervals of the envisioned Hamiltonian particle's motion, to be consistent with the evolution trajectories: *Every time the particle hits a turning point, the potential changes!*

We therefore call the corresponding Vs "switchback potentials." These potentials deepen successively and thus lead to more rapid, higher frequency motion, in agreement with the familiar chaotic behavior of the discrete $s=4$ logistic map that they interpolate. The familiar frequency-doubling-and-folding behavior of the chaotic discrete map is thus understood from — indeed, *explained* by — the properties of the switchback potentials.

2. THE LOGISTIC MAP

Consider in detail the logistic map $x \mapsto sx\,(1-x)$ on the unit interval, $x \in [0,1]$. Schröder's equation for this map is

$$s\Psi\,(x,s) = \Psi\,(sx\,(1-x)\,,s)\ . \tag{10}$$

The inverse function satisfies the corresponding Poincaré equation,

$$\Psi^{-1}\,(sx,s) = s\Psi^{-1}\,(x,s)\,\big(1-\Psi^{-1}\,(x,s)\big)\ . \tag{11}$$

As originally obtained by Schröder, there are three closed-form solutions known, for $s=-2$, 2, and 4:

$$\begin{aligned}
&\Psi\,(x,-2) = \frac{\sqrt{3}}{6}\left(2\pi - 3\arccos\left(x-\frac{1}{2}\right)\right)\ ,\quad \Psi^{-1}\,(x,-2) = \frac{1}{2} - \cos\left(\frac{2x}{\sqrt{3}}+\frac{\pi}{3}\right)\ ,\\
&\Psi\,(x,2) = -\frac{1}{2}\ln\,(1-2x)\ ,\quad \Psi^{-1}\,(x,2) = \frac{1}{2}\left(1-e^{-2x}\right)\ ,\\
&\Psi\,(x,4) = \left(\arcsin\sqrt{x}\right)^2\ ,\quad \Psi^{-1}\,(x,4) = \left(\sin\sqrt{x}\right)^2\ .
\end{aligned} \tag{12}$$

Note that, while the Ψ are multi-valued, the inverse functions are all single-valued. More generally, consider a power series for any s.

$$\Psi^{-1}\,(x,s) = x + x\sum_{n=1}^{\infty} x^n c_n\,(s)\ . \tag{13}$$

The Poincaré equation then leads to a recursion relation for the s-dependent coefficients.

$$c_{n+1} = \frac{1}{1-s^{n+1}}\sum_{j=0}^{n} c_j c_{n-j}\ , \tag{14}$$

with $c_0 = 1$, $c_1 = \frac{1}{1-s}$, $c_2 = \frac{2}{(1-s)(1-s^2)}$, etc. The explicit coefficients are easily recognized for $s=-2$, 2, and 4, and immediately yield the three closed-form cases. Similarly let

$$\Psi\,(x,s) = x + x\sum_{n=1}^{\infty}(-x)^n\,d_n\,(s)\ . \tag{15}$$

Then, as a consequence of Schröder's equation, $d_1 = 1/\,(1-s)$, and for $n \geq 2$,

$$d_n = \frac{1}{1-s^n}\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1-k}{k} s^{n-k} d_{n-k}\ . \tag{16}$$

where $\lfloor \cdots \rfloor$ is the (integer-valued) floor function. In principle, these series solve (11) and (10) for any s, within their radii of convergence. The first few terms for Ψ and Ψ^{-1} for generic s are given explicitly by

$$\begin{aligned}\Psi(x,s) = x + \frac{x^2}{s-1} + \frac{2s}{(s+1)}\frac{x^3}{(s-1)^2} + \frac{s\left(1+5s^2\right)}{\left(s^2+s+1\right)(s+1)}\frac{x^4}{(s-1)^3} \\ +\frac{2s^3\left(3+2s+7s^3\right)}{\left(s^2+1\right)\left(s^2+s+1\right)(s+1)^2}\frac{x^5}{(s-1)^4} \\ +\frac{2s^3\left(1+3s+14s^3+14s^4+7s^5+21s^7\right)}{\left(s^4+s^3+s^2+s+1\right)\left(s^2+1\right)\left(s^2+s+1\right)(s+1)^2}\frac{x^6}{(s-1)^5} + O\left(x^7\right)\ ,\end{aligned} \tag{17}$$

$$\begin{aligned}\Psi^{-1}(x,s) = x + \frac{x^2}{1-s} + \frac{2}{(s+1)}\frac{x^3}{(s-1)^2} + \frac{5+s}{(s+1)\left(s^2+s+1\right)}\frac{x^4}{(1-s)^3} \\ +\frac{2\left(7+3s+2s^2\right)}{\left(s^2+1\right)\left(s^2+s+1\right)(s+1)^2}\frac{x^5}{(s-1)^4} \\ +\frac{2\left(21+14s+14s^2+8s^3+3s^4\right)}{\left(s^4+s^3+s^2+s+1\right)\left(s^2+1\right)\left(s^2+s+1\right)(s+1)^2}\frac{x^6}{(1-s)^5} + O\left(x^7\right)\ ,\end{aligned} \tag{18}$$

from which we infer that $\Psi(x,s)/x$ and $\Psi^{-1}(x,s)/x$ are actually series in $x/(1-s)$ with s-dependent coefficients that are analytic near $s=1$.

The trajectories interpolating the splinter (integer t) of the logistic map are then,

$$\begin{aligned}x(t) = \Psi^{-1}\left(s^t\Psi(x,s),s\right) = s^t\ x + \frac{s^t\left(1-s^t\right)}{s-1}\ x^2 + \frac{2s^t\left(1-s^t\right)\left(s-s^t\right)}{(s+1)(s-1)^2}\ x^3 \\ +\frac{s^t\left(1-s^t\right)\left(s-s^t\right)\left(1+5s^2-(s+5)\,s^t\right)}{(s+1)\left(s^2+s+1\right)(s-1)^3}\ x^4 \\ +\frac{2s^t\left(1-s^t\right)\left(s-s^t\right)\left(s^2-s^t\right)\left(7s^3+2s+3-s^t\left(2s^2+3s+7\right)\right)}{(s+1)^2\left(s^2+1\right)\left(s^2+s+1\right)(s-1)^4}\ x^5 + O\left(x^6\right)\ .\end{aligned} \tag{19}$$

The trajectories are single-valued functions of the time so long as Ψ^{-1} is single-valued, and in fact, they exist even for $s\to 1$ as series solutions. Explicitly,

$$\begin{aligned}\lim_{s\to 1}\Psi^{-1}\left(s^t\Psi(x,s),s\right) = x - t\ x^2 + t(t-1)\ x^3 - \frac{1}{2}t(t-1)(2t-3)\ x^4 \\ +\frac{1}{3}t(t-1)(t-2)(3t-4)\ x^5 - \frac{1}{12}t(t-1)(t-2)\left(12t^2-41t+31\right)\ x^6 \\ +\frac{1}{30}t(t-1)(t-2)\left(30t^3-171t^2+302t-157\right)\ x^7 + O\left(x^8\right)\ .\end{aligned} \tag{20}$$

For the three special cases, $s=-2,\ 2,$ and 4, there are closed-form results for various quantities of interest. For example, for $s=4$, the trajectory and velocity are given by

$$x(t)\big|_{s=4} = \Psi^{-1}\left(4^t\ \Psi(x,4),4\right) = \left(\sin\left(2^t\arcsin\sqrt{x}\right)\right)^2\ , \tag{21}$$

(as originally presented in [2], p 306) and by

$$\frac{d}{dt}x(t) = \left(2^{1+t}\ln 2\right)\sin\left(2^t\arcsin\sqrt{x}\right)\cos\left(2^t\arcsin\sqrt{x}\right)\arcsin\sqrt{x} \tag{22a}$$

$$= (\ln 4)\sqrt{x(t)\left(1-x(t)\right)}\arcsin\sqrt{x(t)}\ . \tag{22b}$$

The last expression evinces a continuous time-translational invariance. However, this velocity function has branch points (i.e. turning points) at $x(t)=0$ and $x(t)=1$, so some care is needed to determine which branch of the function is involved, particularly when the turning points are encountered at finite times, as explained below.

With this caveat in mind, we may thus deduce the velocity profile $v(x)=\left.\frac{dx(t)}{dt}\right|_{t=0}$, effective potential, and force for an underlying Hamiltonian system:

$$v(x)=\left.\frac{dx(t)}{dt}\right|_{s=4,t=0}=(\ln 4)\sqrt{x(1-x)}\arcsin\sqrt{x}\ , \tag{23}$$

$$V(x)=-v^2(x)=(\ln 4)^2\,x(x-1)\arcsin^2\sqrt{x}\ , \tag{24}$$

$$F(x)=-\frac{d}{dx}V(x)=(\ln 4)^2\left(\arcsin\sqrt{x}\right)\left(\sqrt{x(1-x)}-(2x-1)\arcsin\sqrt{x}\right)\ . \tag{25}$$

Note $x=0$ is an unstable fixed point for the system. Similar closed-form results hold for the other two solvable cases, $s=-2$ and $s=2$.

In principle, there are also potentials underlying the trajectories for other s by dint of the series constructions. From the general expression for $x(t)$ in terms of Ψ and Ψ^{-1}, and with use of the chain rule, the potential may be expressed entirely in terms of $(\ln\Psi)'$, as given in the Introduction by (8). The Schröder auxiliary function is then recognized as just an exponential of the time function

$$T(x)=\int^x\frac{dy}{\sqrt{-V(y)}}\ , \tag{26}$$

computed along a zero-energy trajectory, as given by (6).

3. NOVEL POTENTIALS AND SWITCHBACK EFFECTS

The effective potentials for all three closed-form cases are somewhat unusual:

$$V(x,s=4)=(\ln 4)^2\,x(x-1)\arcsin^2\sqrt{x}\ , \tag{27}$$

$$V(x,s=2)=-\left(\ln\sqrt{2}\right)^2(1-2x)^2\ln^2(1-2x)\ \ , \tag{28}$$

$$V(x,s=-2)=\frac{1}{36}\left(\ln(-2)\right)^2(2x+1)(2x-3)\left(2\pi-3\arccos\left(x-\frac{1}{2}\right)\right)^2\ . \tag{29}$$

Another way to express the potential for $s=4$ is similar in form to that for $s=-2$, namely,

$$V(x,s=4)=(\ln 2)^2\,x(x-1)\left(\pi-\arccos(2x-1)\right)^2\ . \tag{30}$$

Indeed, it is well-known that the logistic maps for $s=4$ and $s=-2$ are intimately related through the functional conjugacy of the underlying Schröder equations. But note the potential for $s=-2$ is in fact complex, since $\ln(-2)=\ln 2+i\pi$, as are the trajectories under real time evolution for this case. Of course, since the complexity of $V(x,s=-2)$ is solely a multiplicative factor, if we switch to complex time, $\tau=(\ln 2+i\pi)\times t$, then $\frac{d^2x}{d\tau^2}=-\frac{\partial}{\partial x}\frac{V(x,s=-2)}{2(\ln 2+i\pi)^2}$ again yields real trajectories $x(\tau)$. Nevertheless, this does raise several questions about complex x, and about the behavior of V in the complex plane. We discuss only one aspect of this behavior here, and consider non-principal values for the multi-valued functions $v\left(x, s=4\right)$ and $V\left(x, s=4\right)$. In particular, *all* branches of the arcsin function are important to understand the behavior of the explicit trajectories (21).

Switchbacks on the road to chaos. An interesting *new* feature, which we shall call the *switchback effect*, appears for a particle moving in the $s=4$ effective potential. This is a distinguishing feature that we encounter for the $s=4$ chaotic map, but not, say, for the non-chaotic $s=2$ map. The effect is traceable to the fact that, while the trajectory is single-valued as a function of the time, the velocity and hence the effective potential are *not* single-valued as functions of *position*. Moreover, the branch points of the multi-valued functions are encountered by zero-energy trajectories at *finite* times for the $s=4$ case, unlike, say, for the $s=2$ case. For the latter case, it takes an infinite time for a particle with zero energy to reach a turning point.

Switchbacks are essentially transitions from one branch of the position-dependent velocity function to another, and occur at the turning points encountered at finite times by the interpolating particle trajectories, (21). The effect is easily seen upon viewing animations of these trajectories [11].

Consider a particle with zero total energy that starts in the $V\left(x, s=4\right)$ potential given above, in the region $0<x<1$. If the particle is initially moving to the left, it will continue towards $x=0$, taking an infinite amount of time to reach that turning point. But if the particle is initially moving to the right, it will reach the $x=1$ turning point in a *finite* amount of time that depends on its initial x, as given by

$$\Delta t_0\left(x\right)=\frac{1}{\ln 4}\int_x^1\frac{dy}{\sqrt{y\left(1-y\right)}\left(\arcsin\sqrt{y}\right)}=\frac{1}{\ln 2}\;\ln\left(\frac{\pi/2}{\arcsin\sqrt{x}}\right)\;. \tag{31}$$

Upon reaching $x=1$, the explicit form of the time-dependent solution (21) exhibits the classical counterpart of *a sudden transition* that keeps $E=0$, but *changes the potential* for the return trip towards $x=0$, exactly as follows from the particle moving on a different branch of the arcsin function. Explicitly, the potential deepens to

$$V\left(x, s=4\right)\Longrightarrow V_1\left(x\right)=\left(\ln 4\right)^2 x\left(x-1\right)\left(-\pi+\arcsin\sqrt{x}\right)^2\;, \tag{32}$$

with the particle's speed changing accordingly as a function of x. The return velocity profile is now negative, and given by

$$v_1\left(x\right)=\left(\ln 4\right)\sqrt{x\left(1-x\right)}\left(-\pi+\arcsin\sqrt{x}\right)\;. \tag{33}$$

The arcsin in this last expression, as well as in V_1, is understood to be the principal value.

Moving in this modified negative potential, it now takes the zero-energy particle a *finite* amount of time to travel from $x=1$ down to $x=0$, as given by $\Delta t_1=1$. Upon reaching $x=0$, the exact solution (21) exhibits another sudden transition that keeps $E=0$, but again alters the potential for the return trip towards $x=1$. The potential becomes

$$V_1\left(x\right)\Longrightarrow V_2\left(x\right)=\left(\ln 4\right)^2 x\left(x-1\right)\left(\pi+\arcsin\sqrt{x}\right)^2\;, \tag{34}$$

with corresponding changes in the velocity profile. Again, a finite amount of time is needed for the particle to go from $x=0$ to $x=1$ in the potential V_2, as given by $\Delta t_2=\ln\left(\frac{3}{2}\right)/\ln 2$. Upon reaching $x=1$ the switchback process continues. The total energy remains at zero, but the potential for the second return trip deepens again,

$$V_2\left(x\right)\Longrightarrow V_3\left(x\right)=\left(\ln 4\right)^2 x\left(x-1\right)\left(-2\pi+\arcsin\sqrt{x}\right)^2\;, \tag{35}$$

etc. In order for the particle to follow the interpolating trajectory (21) specified by the Schröder equation for the $s = 4$ case, it has to move under the influence of successively deepening potentials.

At later times, the effective potential seen by the particle on its zigzag path between the turning points will depend on the total number of previous encounters with those turning points. In this sense, the particle remembers its past. Let P be the total number of turning points previously encountered on the particle's trajectory. The motion of the particle before encountering the next, $(P+1)$st, turning point is then completely determined, for $s = 4$, by the effective potential (again, arcsin here is understood to be the principal value)

$$V_P\left(x\right) = \left(\ln 4\right)^2 x\left(x-1\right)\left(\left(-\right)^P \left\lfloor \frac{1+P}{2} \right\rfloor \pi + \arcsin\sqrt{x}\right)^2 , \tag{36}$$

where $\lfloor \cdots \rfloor$ is again the floor function [12]. That is to say, the potential deepens as P increases. The corresponding velocity profile speeds up:

$$v_P\left(x\right) = \left(\ln 4\right)\sqrt{x\left(1-x\right)}\left(\left(-\right)^P \left\lfloor \frac{1+P}{2} \right\rfloor \pi + \arcsin\sqrt{x}\right) . \tag{37}$$

The particle will either be traveling to the left, with $v_P\left(x\right) < 0$ for odd P, or traveling to the right, with $v_P\left(x\right) > 0$ for even P, with its speed increasing with P. As mentioned previously, this effect is clearly evident upon viewing numerical animations of the $s = 4$ trajectories [11].

Alternatively, the motion may be visualized as a trajectory on the sheets of a Riemann surface. Consider the particle moving on the complex x plane, and not just the real line segment $[0,1]$, the endpoints of which are now branchpoints. There are cuts from $+1$ to $+\infty$ and from 0 to $-\infty$. The particle first moves along the real axis, approaches the cut at $+1$, and then goes around the branchpoint such that $\sqrt{x} \to \sqrt{x}$, $\sqrt{1-x} \to -\sqrt{1-x}$, and $\arcsin \to \pi - \arcsin$. The particle returns along the real axis to the origin and encircles the branchpoint at 0, such that $\sqrt{x} \to -\sqrt{x}$, $\sqrt{1-x} \to \sqrt{1-x}$, and $\arcsin\sqrt{x} \to -\arcsin\sqrt{x}$. The particle then goes back to $+1$ and goes around it once more such that again $\sqrt{x} \to \sqrt{x}$, $\sqrt{1-x} \to -\sqrt{1-x}$, and $\arcsin \to \pi - \arcsin$. The trajectory continues in this way, flipping signs and adding πs according to the formulae (36) and (37).

The time for the zero-energy particle to traverse the complete unit interval in x while moving through the V_P potential is always finite, for $P \neq 0$:

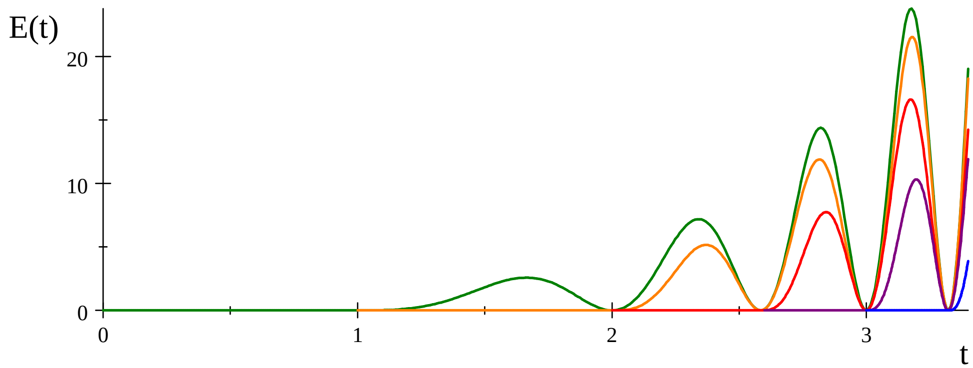
$$\Delta t_P = \left|\int_0^1 \frac{dx}{v_P\left(x\right)}\right| = \frac{1}{\ln 2}\left|\ln\left(1 + \frac{1}{2\left(-\right)^P \left\lfloor \frac{1+P}{2} \right\rfloor}\right)\right| = \frac{1}{\ln 2}\ln\left(\frac{P+1}{P}\right) . \tag{38}$$

This *transit time* decreases monotonically as P increases, with $\Delta t_P \underset{P\to\infty}{\sim} \frac{1}{P\ln 2}$. Starting from an initial x, with initial $v > 0$, the times at which changes in the potential occur, i.e. the times at which the particle encounters turning points, are obtained by summing these transit times and then adding $\Delta t_0\left(x\right)$. The transit time sums are simple enough [13]:

$$\sum_{N=1}^{P-1} \Delta t_N = \frac{\ln P}{\ln 2} . \tag{39}$$

Thus, a particle beginning at x, with initial $v > 0$, will encounter its Pth turning point, and the potential V_P will switch on, at time $t_{P\ \mathrm{on}}\left(x\right) = \Delta t_0\left(x\right) + \frac{\ln P}{\ln 2}$. The potential V_P

will remain in effect for a time span equal to the particle's transit time, $\Delta t_P = t_{P+1\text{ on}}(x) - t_{P\text{ on}}(x)$, i.e. until the next potential V_{P+1} switches on. It is instructive to plot $E(t) = v^2(x(t)) + V_P(x(t))$ versus t for various P, with $t \geq t_{P\text{ on}}$, to check $E = 0$, as well as to see how the energy would *not* be conserved if the potentials were *not* switched.



$E(t)$ for initial $x = 1/2$, using potentials V_P, with $P = 0$, 1, 2, 3, & 4.

For particles with initial $v > 0$, a third way to picture the dynamics is in terms of the total distance traveled by the particle. In this point of view, the successive V_P patch together to form a continuous potential $V(X)$ on the real half-line, $X \geq 0$, as shown in the *cover graphic* ($V(X)$ is the orange curve). Indeed, the half-line may be thought of as a *covering manifold* of the unit interval in x, with the previous multi-valued functions of x now single-valued functions of X. To avoid any hang-ups at the cusps of $V(X)$, either physical *or* conceptual, in this picture the $E = 0$ right-moving trajectories $X(t)$ may be considered as limits of positive energy right-movers, $X(t) = \lim_{E\downarrow 0} X_E(t)$. This limiting process corresponds to the encirclement of the branch points in the Riemann surface picture.

Finally, in terms of the coordinate transformation introduced in [2], namely, the angle

$$\theta = \arcsin\sqrt{x}\ , \tag{40}$$

the motion is completely unraveled, at least for $s = 4$. In that case $E = 0$ implies $\left(\frac{d\theta(t)}{dt}\right)^2 - (\ln 2)^2\,\theta^2(t) = 0$, and the trajectory for all t is just that of a particle moving in a repulsive quadratic potential (inverted SHO). This is the easiest example solvable in closed-form by the Schröder method [1]. The exponentially growing solution, $\theta(t) = 2^t\theta(0)$, whose values in the real-line cover of the circle are all physically distinct, immediately leads to $x(t)$ for the particle moving through the various $V_P(x)$, as is evident from (21).

4. CONCLUSION

Any first order differential equation $df(t)/dt = F(f(t))$ can be approximated by a finite difference equation $f_{n+1} = f_n + \Delta t\ F(f_n)$ where the time-step index n supplants the continuous variable t. In the usual sense, this difference equation is equivalent to the original differential equation only as the time step vanishes, $\Delta t \to 0$.

However, in many situations as described and illustrated in [1] and in this paper, for any fixed time step there may also exist a second, *interpolating* differential equation whose continuous time solutions fill in the "bulk" region between the "boundary data" f_{n+1} and

f_n, for any n. This second differential equation is found by Schröder's functional method, thereby interpreted as a holographic technique; but the interpolating equation is then seen to follow from a Hamiltonian system *sub rosa* whose solution set, for selected energy, provides all solutions to the finite difference equation without approximation.

For nontrivial systems such as the chaotic ($s = 4$) logistic map illustrated here, in order to produce the global features of the trajectories the potential of the underlying Hamiltonian system *must* change at each switchback of the motion, as occurs when turning points are reached in finite time. The trajectories dictate and completely determine the succession of switchback potentials, in a richly extended analogy of inverse scattering techniques.

For values of s between 2 and 4, a *preliminary* analysis using the techniques of this paper leads to Hamiltonian systems involving switchbacks among sequences of potentials, producing trajectories that eventually move between finite sets of turning points, in cycles. However, much more work is needed to completely understand these intermediate cases.

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[12] For odd P, $\lfloor \frac{1+P}{2} \rfloor$ corresponds to a Morse index for the $E = 0$ phase space cycle: $(x = 1, p = 0) \underset{V_1}{\rightarrow} (x = 0, p = 0) \underset{V_2}{\rightarrow} (x = 1, p = 0) \underset{V_3}{\rightarrow} \cdots \underset{V_{P+1}}{\rightarrow} (x = 1, p = 0)$. Cf. Ch. 14 in M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer, New York (1990).

[13] As Feynman used to say, "It's just like a zipper!"