On Darboux Integrable Semi-Discrete Chains

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Abstract

Differential-difference equation $\frac{d}{dx}t(n+1,x) = f(x,t(n,x),t(n+1,x),\frac{d}{dx}t(n,x))$ with unknown $t(n,x)$ depending on continuous and discrete variables x and n is studied. We call an equation of such kind Darboux integrable, if there exist two functions F and I of a finite number of arguments n, x, $\{t(n+k,x)\}_{k=-\infty}^{\infty}$, $\left\{\frac{d^k}{dx^k}t(n,x)\right\}_{k=1}^{\infty}$, such that $D_xF=0$ and $DI=I$, where D_x is the operator of total differentiation with respect to x, and D is the shift operator: $Dp(n)=p(n+1)$. It is proved that the chain is Darboux integrable if and only if its characteristic Lie algebras in both directions are of finite dimension. Structure of the integrals is described. Numerous examples of Darboux integrable chains are given together with their integrals and characteristic Lie algebras.

Keywords: semi-discrete chain, classification, x-integral, n-integral, characteristic Lie algebra, integrability conditions.

1 Introduction

In this paper we study Darboux integrable semi-discrete chains of the form

$$\frac{d}{dx}t(n+1,x) = f(x,t(n,x),t(n+1,x),\frac{d}{dx}t(n,x))\,. \tag{1}$$

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Here uknown functions $t = t(n, x)$ and $t_1 = t(n + 1, x)$ depend on discrete and continuous variables n and x respectively; function $f = f(x, t, t_1, t_x)$ is assumed to be locally analytic, and $\frac{\partial f}{\partial t_x}$ is not identically zero. The last two decades the discrete pfenomena have become very popular due to various important applications (for more details see [1]-[3] and references therein). The article deals with a class of the chains (1) admitting in a sense a closed formula for the general solution.

Below we use a subindex to indicate the shift of the discrete argument: $t_k = t(n+k, x)$, $k \in \mathbb{Z}$, and derivatives with respect to x: $t_{[1]} = t_x = \dfrac{d}{dx}t(n, x)$, $t_{[2]} = t_{xx} = \dfrac{d^2}{dx^2}t(n, x)$, $t_{[m]} = \frac{d^m}{dx^m}t(n, x)$, $m \in \mathbb{N}$. Introduce *the set of dynamical variables* containing $\{t_k\}_{k=-\infty}^{\infty}$; $\{t_{[m]}\}_{m=1}^{\infty}$. We denote through D and D_x the shift operator and the operator of the total derivative with respect to x correspondingly. For instance, $Dh(n, x) = h(n + 1, x)$ and $D_x h(n, x) = \frac{d}{dx}h(n, x)$.

Functions I and F, both depending on x, n, and a finite number of dynamical variables, are called respectively *n- and x-integrals* of (1), if $DI = I$ and $D_x F = 0$ (see also [4]). Clearly, any function depending on n only, is an x-integral, and any function, depending on x only, is an n-integral. Such integrals are called *trivial* integrals. One can see that any n-integral I does not depend on variables t_m, $m \in \mathbb{Z}\backslash\{0\}$, and any x-integral F does not depend on variables $t_{[m]}$, $m \in \mathbb{N}$.

Chain (1) is called *Darboux integrable* if it admits a nontrivial n-integral and a nontrivial x-integral.

One Darboux integrable chain is $t_{1x} = t_1 t_x / t$ with $F = \ln(t_1/t)$ and $I = t_x/t$ as some of many nontrivail x- and n-integrals.

The basic ideas on integration of partial differential equations of the hyperbolic type go back to classical works by Laplace, Darboux, Goursat, Vessio, Monge, Ampere, Legendre, Egorov, etc. Notice that understanding of integration as finding an explicit formula for a general solution was later replaced by other, in a sense less obligatory, definitions. For instance, the Darboux method for integration of hyperbolic type equations consists of searching for integrals in both directions followed by the reduction of the equation to two ordinary differential equations. In order to find integrals, provided that they exist, Darboux used the Laplace cascade method. An alternative, more algebraic approach based on the characteristic vector fields was used by Goursat and Vessio. Namely this method allowed Goursat to get a list of integrable equations [5]. An important contribution to the development of the algebraic method investigating Darboux integrable equations was made by A.B.Shabat who introduced the notion of the characteristic Lie algebra of the hyperbolic equation

$$u_{x,y} = f(x, y, u, u_x, u_y) \,. \tag{2}$$

It turned out that the operator of total differentiation, with respect to the variable y, defines a derivative in the characteristic Lie algebra in the direction of x. Moreover, the operator ad_{D_y} defined according to the rule $ad_{D_y}X = [D_y, X]$ acts on the generators of the algebra in a very simple way. This makes it possible to obtain effective integrability conditions for the equation (2).

A.V. Zhiber and F.Kh. Mukminov investigated the structures of the characteristic Lie algebras for the so-called quadratic systems containing the Liouville equation and the sine-Gordon equation (see [6]). In [6] and [7] the very nontrivial connection between characteristic Lie algebras and Lax pairs of the hyperbolic S-integrable equations and systems of equations is studied, and perspectives on the application of the characteristic algebras to classify such kinds of equations are discussed.

Recently the concept of the characteristic Lie algebras has been defined for discrete models. In our articles [8]-[9] an effective algorithm was worked out to classify Darboux integrable models. By using this algorithm some new classification results were obtained. It is remarkable that in the discrete case an automorphism generated by the shift operator plays an important role.

Due to the requirement of $\frac{\partial}{\partial t_x} f(x, t, t_1, t_x) \neq 0$, we can rewrite (at least locally) chain (1) in the inverse form $t_x(n-1, x) = g(x, t(n, x), t(n-1, x), t_x(n, x))$. Since x-integral F does not depend on variables $t_{[k]}$, $k \in \mathbb{N}$, then the equation $D_x F = 0$ becomes $KF = 0$, where

$$K = \frac{\partial}{\partial x} + t_x \frac{\partial}{\partial t} + f \frac{\partial}{\partial t_1} + g \frac{\partial}{\partial t_{-1}} + f_1 \frac{\partial}{\partial t_2} + g_{-1} \frac{\partial}{\partial t_{-2}} + \dots . \tag{3}$$

Also, $XF = 0$, with $X = \frac{\partial}{\partial t_x}$. Therefore, any vector field from Lie algebra generated by K and X annulates F. This algebra is called *the characteristic Lie algebra L_x of chain (1) in the x-direction.* Stress that L_x is the Lie algebra over the field of the locally analitic functions, depending on x and a finite number of dynamical variables, but not over the field of numbers. The relation between Darboux integrability of chain (1) and its Lie algebra L_x is given by the following important criterion.

Theorem 1.1 *Chain (1) admits a nontrivial x-integral if and only if its Lie algebra L_x is of finite dimension.*

The equation $DI = I$, defining an n-integral I, in an enlarged form becomes

$$I(x, n+1, t_1, f, f_x, ...) = I(x, n, t, t_x, t_{xx}, ...). \tag{4}$$

The left hand side contains the variable t_1 while the right hand side does not. Hence we have $D^{-1} \frac{d}{dt_1} DI = 0$, i.e. the n-integral is in the kernel of the operator

$$Y_1 = D^{-1} Y_0 D,$$

where

$$Y_1 = \frac{\partial}{\partial t} + D^{-1}(Y_0 f)\frac{\partial}{\partial t_x} + D^{-1}Y_0(f_x)\frac{\partial}{\partial t_{xx}} + D^{-1}Y_0(f_{xx})\frac{\partial}{\partial t_{xxx}} + ..., \tag{5}$$

and

$$Y_0 = \frac{d}{dt_1}\,. \tag{6}$$

One can show that $D^{-j}Y_0D^jI = 0$ for any natural j. Direct calculations show that

$$D^{-j}Y_0D^j = X_{j-1} + Y_j, \qquad j \geq 2,$$

where

$$Y_{j+1} = D^{-1}(Y_j f)\frac{\partial}{\partial t_x} + D^{-1}Y_j(f_x)\frac{\partial}{\partial t_{xx}} + D^{-1}Y_j(f_{xx})\frac{\partial}{\partial t_{xxx}} + ..., \quad j \geq 1\,, \tag{7}$$

$$X_j = \frac{\partial}{\partial_{t_{-j}}}, \qquad j \geq 1. \tag{8}$$

Define by N^* the dimension of the linear space spanned by the opreators $\{Y_j\}_1^\infty$. The Lie algebra over the field of the locally analitic functions generated by the opreators $\{Y_j\}_1^{N^*} \cup \{X_j\}_1^{N^*}$ is called *the characteristic algebra L_n of chain (1) in n-direction.*

Theorem 1.2 *Equation (1) admits a nontrivial n-integral if and only if its Lie algebra L_n is of finite dimension.*

The article is organized as follows. In section 2 we give the complete description or all n-integrals and x-integrals of the Darboux integrable chains. Then we show that one can choose the minimal order n-integral and the minimal order x-integral of a special canonical form, convenient for the purpose of classification. In section 3 the Darboux integrebility property of the chain is reformulated in an algebraic form in terms of the characteristic Lie algebras. Particularly, we prove that the chain is Darboux integrable if and only if its both characteristic Lie algebras L_x and L_n are of finite dimension. Theorems in sections 2 and 3 are considered as a basis for further investigations of the classification problem for the chain (1) by using characterictic Lie algebras. Section 4 studies examples of Darboux integrable chains. For each such a chain of the form $t_{1x} = t_x + d(t, t_1)$ the coresponding algebras L_n and L_x are given. Remind that for the exponential type Darboux integrable systems of partial diffirential equations the characteristic Lie algebras are semi-simple [10]. Our examples show, that for the general situation it is not the case.

2 On the structure of nontrivial x- and n- integrals

We define the *order of a nontrivial n-integral* $I = I(x, n, t, t_x, \dots, t_{[k]})$ with $\frac{\partial I}{\partial t_{[k]}} \neq 0$, as the number k.

Lemma 2.1 *Assume equation (1) admits a nontrivial n-integral. Then for any nontrivial n-integral $I^*(x, n, t, t_x, \dots, t_{[k]})$ of the smallest order and any n-integral I we have*

$$I = \phi(x, I^*, D_x I^*, D_x^2 I^*, \dots), \tag{9}$$

where ϕ is some function.

Proof: Denote by $I^* = I^*(x, n, t, \dots, t_{[k]})$ an n-integral of the smallest order. Let I be any other n-integral, $I = I(x, n, t, \dots, t_{[r]})$. Clearly $r \geq k$. Let us introduce new variables x, n, t, t_x, $\dots$, $t_{[k-1]}$, I^*, $D_x I^*$, $\dots$, $D_x^{r-k} I^*$ instead of the variables x, n, t, t_x, $\dots$, $t_{[k-1]}$, $t_{[k]}$, $t_{[k+1]}$, $\dots$, $t_{[r]}$. Now, $I = I(x, n, t, t_x, \dots, t_{[k-1]}, I^*, D_x I^*, \dots, D_x^{r-k} I^*)$. We write the power series for function I in the neighbourhood of the point $((I^*)_0, (D_x I^*)_0, \dots, (D_x^{r-k} I^*)_0)$:

$$I = \sum_{i_0, i_1, \dots, i_{r-k}} E_{i_0, i_1, \dots, i_{r-k}} (I^* - (I^*)_0)^{i_0} (D_x I^* - (D_x I^*)_0)^{i_1} \dots (D_x^{r-k} I^* - (D_x^{r-k} I^*)_0)^{i_{r-k}}. \tag{10}$$

Then

$$DI = \sum_{i_0, i_1, \dots, i_{r-k}} DE_{i_0, i_1, \dots, i_{r-k}} (DI^* - (I^*)_0)^{i_0} (DD_x I^* - (D_x I^*)_0)^{i_1} \dots (DD_x^{r-k} I^* - (D_x^{r-k} I^*)_0)^{i_{r-k}}.$$

Since $DI = I$, $DD_x^j I^* = D_x^j DI^* = D_x^j I^*$ and the power series representation for function I is unique, then $DE_{i_0, i_1, \dots, i_{r-k}} = E_{i_0, i_1, \dots, i_{r-k}}$, i.e. $E_{i_0, i_1, \dots, i_{r-k}}(x, n, t, \dots, t_{[k-1]})$ are all n-integrals. Due to the fact that minimal n-integral depends on x, n, t, $\dots$, $t_{[k]}$, we conclude that all $E_{i_0, i_1, \dots, i_{r-k}}(x, n, t, \dots, t_{[k-1]})$ are trivial n-integrals, i.e. functions depending only on x. Now equation (9) follows immeadiately from (10).

We define the *order of a nontrivial x-integral* $F = F(x, n, t_k, t_{k+1}, \dots t_m)$ with $\frac{\partial F}{\partial t_{[m]}} \neq 0$, as the number $m - k$.

Lemma 2.2 *Assume equation (1) admits a nontrivial x-integral. Then for any nontrivial x-integral $F^*(x, n, t, t_1, \dots, t_m)$ of the smallest order and any x-integral F we have*

$$F = \xi(n, F^*, DF^*, D^2 F^*, \dots), \tag{11}$$

where ξ is some function.

Proof: Denote by $F^* = F^*(x, n, t, t_1, \ldots, t_m)$ an x-integral of the smallest order. Let F be any other x-integral, $F = F(x, n, t, t_1, \ldots, t_l)$. Clearly, $l \geq m$. Let us introduce new variables x, n, t, t_1, $\ldots$, t_{m-1}, F^*, DF^*, $\ldots$, $D^{l-m}F^*$ instead of variables x, n, t, t_1, $\ldots$,t_{m-1}, t_m, $\ldots$, t_l. Now, $F = F(x, n, t, t_1, \ldots, t_{m-1}, F^*, DF^*, \ldots, D^{l-m}F^*)$. We write the power series representation of function F in the neighbourhood of point $((F^*)_0, (DF^*)_0, \ldots, (D^{l-m}F^*)_0)$:

$$F = \sum_{i_0, i_1, \ldots, i_{l-m}} K_{i_0, i_1, \ldots, i_{l-m}} (F^* - (F^*)_0)^{i_0} (DF^* - (DF^*)_0)^{i_1} \ldots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}} \,. \quad (12)$$

Then

$$D_x F = \sum_{i_0, i_1, \ldots, i_{l-m}} D_x\{K_{i_0, i_1, \ldots, i_{l-m}}\} (F^* - (F^*)_0)^{i_0} (DF^* - (DF^*)_0)^{i_1} \ldots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}}$$

$$+ \sum_{i_0, i_1, \ldots, i_{l-m}} K_{i_0, i_1, \ldots, i_{l-m}} D_x\{(F^* - (F^*)_0)^{i_0} (DF^* - (DF^*)_0)^{i_1} \ldots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}}\}$$

Since $D_x D^j F^* = D^j D_x F^* = 0$, then $D_x\{(F^* - (F^*)_0)^{i_0} (DF^* - (DF^*)_0)^{i_1} \ldots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}}\} = 0$. Therefore,

$$0 = D_x F = \sum_{i_0, i_1, \ldots, i_{l-m}} D_x\{K_{i_0, i_1, \ldots, i_{l-m}}\} (F^* - (F^*)_0)^{i_0} (DF^* - (DF^*)_0)^{i_1} \ldots (D^{l-m}F^* - (D^{l-m}F^*)_0)^{i_{l-m}} \,.$$

Due to the unique representation of the zero power series we have that $D_x\{K_{i_0, i_1, \ldots, i_{l-m}}\} = 0$, i.e. all $K_{i_0, i_1, \ldots, i_{l-m}}(x, n, t, \ldots, t_{m-1})$ are x-integrals. Since the minimal nontrivial x-integral is of order m, then all $K_{i_0, i_1, \ldots, i_{l-m}}$ are trivial x-integrals, i.e. functions depending on n only. Now the equation (11) follows from (12).

The next two lemmas are just discrete versions of Lemma 1.2 from [11].

Lemma 2.3 *Among all nontrivial n-integrals $I^*(x, n, t, t_x, \ldots, t_{[k]})$ of the smallest order, with $k \geq 2$, there is an n-integral $I^0(x, n, t, t_x, \ldots, t_{[k]})$ such that*

$$I^0(x, n, t, t_x, \ldots, t_{[k]}) = a(x, n, t, t_x, \ldots, t_{[k-1]}) t_{[k]} + b(x, n, t, t_x, \ldots, t_{[k-1]}) \,. \quad (13)$$

Proof: Consider nontrivial minimal n-integral $I^*(x, n, t, t_x, \ldots, t_{[k]})$ with $k \geq 2$. Equality $DI^* = I^*$ can be rewritten as

$$I^*(x, n+1, t_1, f, f_x, \ldots, f_{[k-1]}) = I^*(x, n, t, t_x, \ldots, t_{[k]}).$$

We differentiate both sides of the last equality wit respect to $t_{[k]}$:

$$\frac{\partial I^*(x, n+1, t_1, f, \ldots, f_{[k-1]})}{\partial f_{[k-1]}} \cdot \frac{\partial f_{[k-1]}}{\partial t_{[k]}} = \frac{\partial I^*(x, n, t, \ldots, t_{[k]})}{\partial t_{[k]}} \,. \quad (14)$$

In virtue of $\frac{\partial f_{[j]}}{\partial t_{[j+1]}}=f_{t_x}$, the equation (14) can be rewritten as

$$\frac{\partial I^*(x,n+1,t_1,f,\dots,f_{[k-1]})}{\partial f_{[k-1]}}f_{t_x}=\frac{\partial I^*(x,n,t,\dots,t_{[k]})}{\partial t_{[k]}}\,. \tag{15}$$

Let us differentiate once more with respect to $t_{[k]}$ both sides of the last equation, we have:

$$\frac{\partial^2 I^*(x,n+1,t_1,f,\dots,f_{[k-1]})}{\partial^2 f_{[k-1]}}f_{t_x}^2=\frac{\partial^2 I^*(x,n,t,\dots,t_{[k]})}{\partial t_{[k]}^2}\,,$$

or the same,

$$D\left\{\frac{\partial^2 I^*}{\partial t_{[k]}^2}\right\}f_{t_x}^2=\frac{\partial^2 I^*}{\partial t_{[k]}^2}\,,$$

where $I^*=I^*(x,n,t,\dots,t_{[k]})$. It follows from (15) that

$$D\left\{\frac{\partial^2 I^*}{\partial t_{[k]}^2}\right\}\left\{\frac{\partial I^*}{\partial t_{[k]}}\right\}^2=\frac{\partial^2 I^*}{\partial t_{[k]}^2}D\left\{\left(\frac{\partial I^*}{\partial t_{[k]}}\right)^2\right\}\,,$$

or the same, function

$$J:=\frac{\frac{\partial^2 I^*}{\partial t_{[k]}^2}}{\left(\frac{\partial I^*}{\partial t_{[k]}}\right)^2}$$

is an n-integral, and by Lemma 2.1, we have that $J=\phi(x,I^*)$. Therefore,

$$\frac{\partial^2 I^*}{\partial t_{[k]}^2}=\frac{\partial H(x,I^*)}{\partial I^*}\left(\frac{\partial I^*}{\partial t_{[k]}}\right)^2,\quad \text{where}\quad \frac{\partial H}{\partial I^*}=J,$$

or

$$\frac{\partial}{\partial t_{[k]}}\left\{\ln\frac{\partial I^*}{\partial t_{[k]}}-H(x,I^*)\right\}=0.$$

Hence, $e^{-H(x,I^*)}\frac{\partial I^*}{\partial t_{[k]}}=e^g$ for some function $g(x,n,t,t_x,\dots,t_{[k-1]})$. Introduce W in such a way that $\frac{\partial W}{\partial I^*}=e^{-H(x,I^*)}$. Then $\frac{\partial W}{\partial t_{[k]}}=e^g$ and $W=e^{g(x,n,t,\dots,t_{[k-1]})}t_{[k]}+l(x,n,t,\dots,t_{[k-1]})$ is an n-integral, where $l(x,n,t,\dots,t_{[k-1]})$ is some function.

Lemma 2.4 *Among all nontrivial x-integrals $F^*(x,n,t_{-1},t,t_1,\dots,t_m)$ of the smallest order, with $m\geq 1$, there is x-integral $F^0(x,n,t_{-1},t,t_1,\dots,t_m)$ such that*

$$F^0(x,n,t_{-1},t,t_1,\dots,t_m)=A(x,n,t_{-1},t,\dots,t_{m-1})+B(x,n,t,t_1,\dots,t_m). \tag{16}$$

Proof: Consider nontrivial x-integral $F^*(x,n,t_{-1},t,t_1,\dots,t_m)$ of minimal order. Since $D_xF^*=0$, then

$$\frac{\partial F^*}{\partial x}+g\frac{\partial F^*}{\partial t_{-1}}+t_x\frac{\partial F^*}{\partial t}+f\frac{\partial F^*}{\partial t_1}+Df\frac{\partial F^*}{\partial t_2}+\ldots+D^{m-1}f\frac{\partial F^*}{\partial t_m}=0. \tag{17}$$

We differentiate both sides of (17) with respect to t_m and with respect to t_{-1} separately and have the following two equations:

$$\{D_x + \frac{\partial}{\partial t_m}(D^{m-1}f)\}\frac{\partial F^*}{\partial t_m} = 0, \tag{18}$$

$$\{D_x + \frac{\partial g}{\partial t_{-1}}\}\frac{\partial F^*}{\partial t_{-1}} = 0. \tag{19}$$

Let us differentiate (18) with respect to t_{-1}, we have,

$$D_x\frac{\partial^2 F^*}{\partial t_m \partial t_{-1}} + \frac{\partial g}{\partial t_{-1}}\frac{\partial^2 F^*}{\partial t_m \partial t_{-1}} + \frac{\partial}{\partial t_m}(D^{m-1}f)\frac{\partial^2 F^*}{\partial t_m \partial t_{-1}} = 0\,. \tag{20}$$

It follows from (18) and (19) that $\frac{\partial}{\partial t_m}(D^{m-1}f) = -\frac{D_x F^*_{t_m}}{F^*_{t_m}}$, $\frac{\partial g}{\partial t_{-1}} = -\frac{D_x F^*_{t_{-1}}}{F^*_{t_{-1}}}$. Equation (20) becomes

$$D_x\left\{\ln\frac{F^*_{t_m t_{-1}}}{F^*_{t_m}F^*_{t_{-1}}}\right\} = 0.$$

By Lemma 2.2 we have, $\frac{F^*_{t_m t_{-1}}}{F^*_{t_m}F^*_{t_{-1}}} = \xi(n, F^*)$, or

$$\frac{F^*_{t_m t_{-1}}}{F^*_{t_m}} = F^*_{t_{-1}}\xi(n, F^*) = H'(F^*)F^*_{t_{-1}} = \frac{\partial}{\partial t_{-1}}H(F^*), \quad \text{where} \quad \xi(n, F^*) = H'(n, F^*)\,.$$

Thus, $\frac{\partial}{\partial t_{-1}}\{\ln F^*_{t_m} - H(n, F^*)\} = 0$, or $e^{-H(n,F^*)}F^*_{t_m} = C(x, n, t, t_1, \ldots, t_m)$ for some function $C(x, n, t, t_1, \ldots, t_m)$. Denote by $\tilde{H}^*(n, F)$ such a function that $\tilde{H}'(n, F^*) = e^{-H(n,F^*)}$. Then $\frac{\partial \tilde{H}(n,F^*)}{\partial t_m} = C(x, n, t, t_1, \ldots, t_m)$. Hence, $\tilde{H}(n, F^*) = B(x, n, t, t_1, \ldots, t_m) + A(x, n, t_{-1}, t, \ldots, t_{m-1})$. Since $D_x\tilde{H}(F^*) = \tilde{H}'(n, F^*)D_x(F^*) = 0$, then $\tilde{H}(n, F^*)$ is an x-integral in the desired form (16).

Corollary 2.5 *Among all nontrivial x-integrals $F(x, n, t, \ldots, t_m)$ of the smallest order with $m \geq 2$, there is x-integral $F^0(x, n, t, \ldots, t_m)$ such that*

$$F^0(x, n, t, \ldots, t_m) = A(x, n, t, \ldots, t_{m-1}) + B(x, n, t_1, \ldots, t_m)\,.$$

3 Algebraic criterion of Darboux integrability

In this section we give complete proof of the Theorems 1.1 and 1.2.

Let us prove Theorem 1.1. Assume equation (1) admits a nontrivial x-integral. Take one such integral $F = F(x, n, t, t_1, \ldots, t_m)$ with $\frac{\partial F}{\partial t_m} \neq 0$ identically. Introduce

$$L_x^{(m)} = \{T^{(m)} = P_m(T) : T \in L_x\},$$

where P_m is the projection operator defined as follows

$$P_i\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} + \sum_{k=1}^{\infty} a_k\frac{\partial}{\partial t_k}\right) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} + \sum_{k=1}^{i} a_k\frac{\partial}{\partial t_k}, \qquad i = 1, 2, 3, \ldots. \tag{21}$$

Denote by N_1 the dimension of $L_x^{(m)}$. Clearly, $N_1 \leq m+2$. Let the set $\{T_{01}, T_{02}, \dots, T_{0N_1}\}$ form a basis in $L_x^{(m)}$. For any $j = 1, 2, \dots, N_1$, denote by $T_j = \sum\limits_{k=1}^{\infty} \alpha_k(T_j)\frac{\partial}{\partial t_k}$ a vector field from L_x such that $P_m(T_j) = T_{0j}$. Let us show that the set $\{T_1, T_2, \dots, T_{N_1}\}$ forms a basis in L_x. Take arbitrary vector field $T = a(T)\frac{\partial}{\partial x} + b(T)\frac{\partial}{\partial t} + \sum_{j=1}^{\infty} a_j(T)\frac{\partial}{\partial t_j}$ from L_x. Since $P_m(T) \in L_x^{(m)}$, then $P_m(T) = \sum\limits_{j=1}^{N_1} \beta_j T_{0j}$. Let us show that $T = \sum\limits_{j=1}^{N_1} \beta_j T_j$, or the same, $Z \equiv 0$, where $Z = T - \sum\limits_{j=1}^{N_1} \beta_j T_j$. We have, $P_m(Z) \equiv 0$. Since F is an x-integral depending on x, n, t, t_1, …, t_m , then DF is an x-integral depending on variables x, $n+1$, t_1, t_2, …, t_m, t_{m+1}. Therefore,

$$0 = Z(DF) = P_m(Z)DF + \left(\alpha_{m+1}(T) - \sum_{j=1}^{N_1} \beta_j \alpha_{m+1}(T_j)\right)\frac{\partial}{\partial t_{m+1}}DF =$$

$$\left(\alpha_{m+1}(T) - \sum_{j=1}^{N_1} \beta_j \alpha_{m+1}(T_j)\right)\frac{\partial}{\partial t_{m+1}}DF.$$

Since $\frac{\partial}{\partial t_{m+1}}DF = D\frac{\partial}{\partial t_m}F \neq 0$, then $\alpha_{m+1}(T) = \sum\limits_{j=1}^{N_1} \beta_j \alpha_{m+1}(T_j)$, that is $P_{m+1}(Z) \equiv 0$. Applying successively the operator Z to x-integrals D^2F, D^3F, …, one can see that $\alpha_{m+i}(T) = \sum\limits_{j=1}^{N_1} \beta_j \alpha_{m+i}(T_j)$ for any $i = 1, 2, 3, \dots$, that is $P_{m+i}(Z) \equiv 0$ for any natural number i. Therefore, $Z \equiv 0$. Hence, any vector field T from L_x can be represented as a linear combination of T_1, T_2, …, T_{N_1}. Thus, L_x is of finite dimension.

Assume that the dimension of the characteristic algebra Lie L_x is finite, denote it by N. Let T_1, T_2, …, T_N form a basis in L_x. Denote by $T_{0j} = P_N(T_j)$, $j = 1, 2, \dots, N$. Then we have N equations $T_{0j}F = 0$ for a function F depending on $N+4$ variables: x, n, t_x, t, t_1, …, t_N. By Jacobi Theorem, such nonconstant function $F = F(x, n, t_x, t, t_1, \dots, t_N)$ exists. Moreover, it does not depend on variable t_x and satisfies the equation $TF = 0$ for any $T \in L_x$. This function F is a nontrivial x-integral of equation (24). This completes the proof of Theorem 1.1.

Let us prove Theorem 1.1. Assume equation (1) admits a nontrivial n-integral. Take one such integral $I = I(x, n, t, t_x, t_{[2]}, \dots, t_{[m]})$ with $\frac{\partial I}{\partial t_{[m]}} \neq 0$ identically. Introduce Lie algebra M generated by vector fields $\{Y_j\}_1^{\infty} \cup \{X_j\}_1^{N_2}$, where number N_2 will be specified later. Define

$$M^{(m)} = \{T^{(m)} = P_m^*(T) : \, T \in M\},$$

where P_m^* is the projection operator defined as follows

$$P_i^*\left(\sum_{k=-N_2}^{-1} a_k \frac{\partial}{\partial t_k} + \sum_{k=0}^{\infty} a_k \frac{\partial}{\partial t_{[k]}}\right) = \sum_{k=-N_2}^{-1} a_k \frac{\partial}{\partial t_k} + \sum_{k=0}^{i} a_k \frac{\partial}{\partial t_{[k]}}, \qquad i = 1, 2, 3, \dots. \tag{22}$$

Denote by N_1 the dimension of $M^{(m)}$. Clearly, $N_1 \le m + N_2 + 1$. Let the set $\{T_{01}, T_{02}, \dots, T_{0N_1}\}$ form a basis in $M^{(m)}$. Denote by $T_j = \sum\limits_{k=-N_2}^{-1} \alpha_k(T_j)\frac{\partial}{\partial t_k} + \sum\limits_{k=0}^{\infty} \alpha_k(T_j)\frac{\partial}{\partial t_{[k]}}$ a vector field from M such that $P_m^*(T_j) = T_{0j}$, $j = 1, 2, \dots, N_1$. Let us show that the set $\{T_1, T_2, \dots, T_{N_1}\}$ forms a basis in M. Take arbitrary vector field $T = \sum_{j=-N_2}^{-1} a_j(T)\frac{\partial}{\partial t_j} + \sum_{j=0}^{\infty} a_j(T)\frac{\partial}{\partial t_{[j]}}$ from M. Since $P_m^*(T) \in M^{(m)}$, then $P_m^*(T) = \sum\limits_{j=1}^{N_1} \beta_j T_{0j}$. Let us show that $T = \sum\limits_{j=1}^{N_1} \beta_j T_j$, or the same, $Z \equiv 0$, where $Z = T - \sum\limits_{j=1}^{N_1} \beta_j T_j$. We have, $P_m^*(Z) \equiv 0$. Since I is an n-integral depending on x, n, t, t_x, $t_{[2]}$, …, $t_{[m]}$, then $D_x I$ is an n-integral depending on variables x, n, t, t_x, $t_{[2]}$, …, $t_{[m]}$, $t_{[m+1]}$. Therefore,

$$0 = Z(D_x I) = P_m^*(Z) D_x I + \left(\alpha_{m+1}(T) - \sum_{j=1}^{N_1} \beta_j \alpha_{m+1}(T_j)\right) \frac{\partial}{\partial t_{[m+1]}} D_x I =$$

$$\left(\alpha_{m+1}(T) - \sum_{j=1}^{N_1} \beta_j \alpha_{m+1}(T_j)\right) \frac{\partial}{\partial t_{[m+1]}} D_x I.$$

Since $\frac{\partial}{\partial t_{[m+1]}} D_x I = \frac{\partial}{\partial t_{[m]}} I \ne 0$, then $\alpha_{m+1}(T) = \sum\limits_{j=1}^{N_1} \beta_j \alpha_{m+1}(T_j)$, that is $P_{m+1}^*(Z) \equiv 0$. Applying successively the operator Z to n-integrals $D_x^2 I$, $D_x^3 I$, …, one can see that $\alpha_{m+i}(T) = \sum\limits_{j=1}^{N_1} \beta_j \alpha_{m+i}(T_j)$ for any $i = 1, 2, 3, \dots$, that is $P_{m+i}^*(Z) \equiv 0$ for any natural number i. Therefore, $Z \equiv 0$. Thus, Lie algebra M is of finite dimension. Then linear envelope of the vector fields $\{Y_j\}_1^{\infty}$ is of finite dimension, say N. Let N_2 be any number satisfying $N_2 \ge N$. We have, algebra L_n generated by $\{Y_j\}_1^N \cup \{X_j\}_1^N$ is a subalgebra of M, and therefore L_n of finite dimension.

Assume that the dimension of the characteristic algebra Lie L_n is finite, denote it by N_1. Let N be the dimension of linear envelope of the vector fields $\{Y_j\}_1^{\infty}$. Set $N_2 = N_1 - N$. Introduce

$$L_x^{(N_2)} = \{T^{(m)} = P_{N_2}^{(N)}(T) : T \in L_x\},$$

where

$$P_{N_2}^{(N)}\left(\sum_{k=-N}^{-1} a_k \frac{\partial}{\partial t_k} + \sum_{k=0}^{\infty} a_k \frac{\partial}{\partial t_{[k]}}\right) = \sum_{k=-N}^{-1} a_k \frac{\partial}{\partial t_k} + \sum_{k=0}^{N_2} a_k \frac{\partial}{\partial t_{[k]}}. \tag{23}$$

Let $\{T_{0j}\}_{j=1}^{N_1}$ form a basis in $L_x^{(N_2)}$. Then we have N_1 equations $T_{0j} G = 0$ for a function G depending on $N_1 + 3$ variables: x, n, t, t_x, …, $t_{[N_2]}$, t_{-1}, …, t_{-N}. By Jacobi Theorem, such nonconstant function G exists. Moreover, it does not depend on variables t_{-j}, $j = 1, 2, \dots N$, and satisfies the equation $TG = 0$ for any $T \in L_n$. Such function $G = G(x, n, t, t_x, \dots, t_{[N_2]})$ is not unique, but any other solution of these equations, depending on the same set of the variables, can be represented as $h(x, G)$ for some function h.

Since $D^{-1} Y_j D = Y_{j+1}$, $j = 0, 2, 3, \dots$, $D^{-1} Y_1 D = Y_2 + X_1$, $D^{-1} X_j D = X_{j+1}$, then for any vector field Z from L_n, we have $D^{-1} Z D = Z^* + \lambda X_{N+1}$ for some vector field Z^* from L_n and some function

λ. So,

$$ZDG = D(D^{-1}ZDG) = D(Z^* + \lambda X_{N+1})G = 0$$

for any $Z \in L_n$. Therefore, DG is also a solution of the aforementioned system of partial differential equations. That is why $DG = h(x, G)$.

By solving ordinary difference equation $DG = h(x, G)$ of the first order we have $G = H(x, n, c)$, where c is an arbitrary constant. By solving the equation $G = H(x, n, c)$ with respect to c one gets $c = I(G, x, n)$. This function $I(G, x, n)$ is a nontrivial n-integral of equation (24). Indeed, $DI(G, x, n) = Dc = c = I(G, x, n)$. So DI=I. This completes the proof of Theorem 1.2.

4 Particular case: $t_{1x} = t_x + d(t, t_1)$

Finiteness of Lie algebras L_x and L_n was used in [8] and [9] to classify Darboux integrable semi-discrete chains of special form

$$t_{1x} = t_x + d(t, t_1)\,. \tag{24}$$

The statement of this classification result is given by the next theorem from [9].

Theorem 4.1 *Chain (24) admits nontrivial x- and n-integrals if and only if $d(t, t_1)$ is one of the kind:*

(1) $d(t, t_1) = A(t_1 - t)$, where $A(t_1 - t)$ is given in implicit form $A(t_1 - t) = \frac{d}{d\theta}P(\theta)$, $t_1 - t = P(\theta)$, with $P(\theta)$ being an arbitrary quasipolynomial, i.e. a function satisfying an ordinary differential equation

$$P^{(N+1)} = \mu_N P^{(N)} + \ldots + \mu_1 P' + \mu_0 P \tag{25}$$

with constant coefficients μ_k, $0 \le k \le N$,

(2) $d(t, t_1) = C_1(t_1^2 - t^2) + C_2(t_1 - t)$,

(3) $d(t, t_1) = \sqrt{C_3 e^{2\alpha t_1} + C_4 e^{\alpha(t_1+t)} + C_3 e^{2\alpha t}}$,

(4) $d(t, t_1) = C_5(e^{\alpha t_1} - e^{\alpha t}) + C_6(e^{-\alpha t_1} - e^{-\alpha t})$,

where $\alpha \neq 0$, C_i, $1 \le i \le 6$, are arbitrary constants. Moreover, some nontrivial x-integrals F and n-integrals I in each of the cases are

i) $F = x - \int^{t_1 - t} \frac{ds}{A(s)}$, $I = L(D_x)t_x$, where $L(D_x)$ is a differential operator which annihilates $\frac{d}{d\theta}P(\theta)$ where $D_x\theta = 1$.

ii) $F = \frac{(t_3-t_1)(t_2-t)}{(t_3-t_2)(t_1-t)}$, $I = t_x - C_1t^2 - C_2t$,

iii) $F = \int^{t_1-t} \frac{e^{-\alpha s}ds}{\sqrt{C_3e^{2\alpha s}+C_4e^{\alpha s}+C_3}} - \int^{t_2-t_1} \frac{ds}{\sqrt{C_3e^{2\alpha s}+C_4e^{\alpha s}+C_3}}$, $I = 2t_{xx} - \alpha t_x^2 - \alpha C_3 e^{2\alpha t}$,

iv) $F = \frac{(e^{\alpha t}-e^{\alpha t_2})(e^{\alpha t_1}-e^{\alpha t_3})}{(e^{\alpha t}-e^{\alpha t_3})(e^{\alpha t_1}-e^{\alpha t_2})}$, $I = t_x - C_5e^{\alpha t} - C_6e^{-\alpha t}$.

Equation of the form $\tau_x = A(\tau)$, where $\tau = t_1 - t$, is integrated in quadratures. But to get the final answer one should evaluate the integral and then find the inverse function. The general solution is given in an explicit form

$$t(n,x) = t(0,x) + \sum_{j=0}^{n-1} P(x + c_j), \tag{26}$$

where $t(0,x)$ and c_j are arbitrary functions of x and j respectively, and $A(\tau) = P'(\theta)$, $t_1 - t = P(\theta)$. Actually we have $\tau_x = P_\theta(\theta)\theta_x = P_\theta(\theta)$, which implies $\theta_x = 1$, so that $\tau(n,x) = P(x + c_n)$. By solving the equation $t(n+1,x) - t(n,x) = P(x+c_n)$ one gets the answer above. Requirement for $\tau_x = A(\tau)$ to be Darboux integrable induces condition on function P to satisfy a linear ordinary differential equation with constant coefficients.

The classication Theorem 4.1 contains all Darboux integrable equations of special form (24) together with the corresponding nontrivial x- and n- integrals. However, the characteristic algebras L_x and L_n for Darboux integrable equations (24) were not specified in [9]. In the next two subsections we present characteristic Lie algebras L_x and L_n in each of the four classes given by Theorem 4.1.

4.1 Lie algebras L_x for Darboux integrable equations $t_{1x} = t_x + d(t, t_1)$

It was proved (see [8]) that if equation $t_{1x} = t_x + d(t,t_1)$ admits a nontrivial x-integral, then it admits a nontrivial x-integral not depending on x. Introduce new vector fields

$$\tilde{X} = [X, K] = \sum_{k=-\infty,}^{\infty} \frac{\partial}{\partial t_k} \qquad t_0 := t,$$

$$J := [\tilde{X}, K]\,.$$

4.1.1 Case 1: $t_{1x} = t_x + A(t_1 - t)$

Direct calculations show that the multiplication table for Lie algebra L_x is the following

L_x	X	K	$\tilde{X}$
X	0	$\tilde{X}$	0
K	$-\tilde{X}$	0	0
$\tilde{X}$	0	0	0

4.1.2 Case 2: $t_{1x} = t_x + C_1(t_1^2 - t^2) + C_2(t_1 - t)$

Direct calculations show that

$$J = 2C_1 \sum_{k=-\infty, k\neq 0}^{\infty} (t_k - t)\frac{\partial}{\partial t_k}$$

and

$$[J, K] = 2C_1^2 \sum_{k=-\infty, k\neq 0}^{\infty} (t_k - t)^2 \frac{\partial}{\partial t_k} = 2C_1(K - t_x\tilde{X}) - (2C_1 t + C_2)J$$

and the multiplication table for Lie algebra L_x is the following

L_x	X	K	$\tilde{X}$	J
X	0	$\tilde{X}$	0	0
K	$-\tilde{X}$	0	$-J$	$-2C_1(K - t_x\tilde{X}) + (2C_1 t + C_2)J$
$\tilde{X}$	0	J	0	0
J	0	$2C_1(K - t_x\tilde{X}) - (2C_1 t + C_2)J$	0	0

4.1.3 Case 3: $t_{1x} = t_x + \sqrt{C_3 e^{2\alpha t_1} + C_4 e^{\alpha(t_1+t)} + C_3 e^{2\alpha t}}$

Direct calculations show that $[\tilde{X}, K] = \alpha K - \alpha t_x \tilde{X}$, and the multiplication table for Lie algebra L_x is the following

L_x	X	K	$\tilde{X}$
X	0	$\tilde{X}$	0
K	$-\tilde{X}$	0	$-\alpha K + \alpha t_x \tilde{X}$
$\tilde{X}$	0	$\alpha K - \alpha t_x \tilde{X}$	0

4.1.4 Case 4: $t_{1x} = t_x + C_5(e^{\alpha t_1} - e^{\alpha t}) + C_6(e^{-\alpha t_1} - e^{-\alpha t})$

Direct calculations show that

$$J = \alpha \sum_{k=-\infty, k\neq 0}^{\infty} \{C_5(e^{\alpha t_k} - e^{\alpha t}) - C_6(e^{-\alpha t_k} - e^{-\alpha t})\}\frac{\partial}{\partial t_k}$$

and

$$[J, K] = 2C_5C_6\alpha^2 \sum_{k=-\infty, k\neq 0}^{\infty} \{e^{\alpha(t-t_k)} + e^{\alpha(t_k - t)} - 2\}\frac{\partial}{\partial t_k}$$
$$= \alpha^2(C_5 e^{\alpha t} + C_6 e^{-\alpha t})(K - t_x\tilde{X}) + \alpha(C_6 e^{-\alpha t} - C_5 e^{\alpha t})J.$$

Denote by

$$\beta_1 = \alpha^2(C_5 e^{\alpha t} + C_6 e^{-\alpha t}), \qquad \beta_2 = \alpha(C_6 e^{-\alpha t} - C_5 e^{\alpha t}).$$

The multiplication table for Lie algebra L_x is

L_x	X	K	$\tilde{X}$	J
X	0	$\tilde{X}$	0	0
K	$-\tilde{X}$	0	$-J$	$-\beta_1(K - t_x\tilde{X}) - \beta_2 J$
$\tilde{X}$	0	J	0	$\alpha^2 K - \alpha^2\tilde{X}$
J	0	$\beta_1(K - t_x\tilde{X}) + \beta_2 J$	$\alpha^2\tilde{X} - \alpha^2 K$	0

4.2 Lie algebras L_n for Darboux integrable equation $t_{1x} = t_x + d(t, t_1)$

4.2.1 Case 1: $t_{1x} = t_x + A(t_1 - t)$

Lie algebra L_n is generated only by two vector fields X_1 and Y_1, and can be of any finite dimension. If $A(t_1 - t) = t_1 - t + c$, where c is some constant, then Lie algebra L_n is trivial, consisting of X_1 and Y_1 only, with commutativity relation $[X_1, Y_1] = 0$. If $A(t_1 - t) \neq t_1 - t + c$, one can choose a basis in L_n consisting of $W = \frac{\partial}{\partial\theta}$, $Z = \sum_{k=0}^{k=\infty} D_x^k p(\theta)\partial/\partial t_{[k]}$, with $\theta = x + \alpha_n$, $C_1 = [W, Z]$, $C_{k+1} = [W, C_k]$, $1 \leq k \leq N - 1$. Its multiplication table for L_n is the following

L_n	W	Z	C_1	C_2	$\ldots$	C_k	$\ldots$	C_{N-1}	C_N
W	0	C_1	C_2	C_3	$\ldots$	C_{k+1}	$\ldots$	C_N	K
Z	$-C_1$	0	0	0	$\ldots$	0	$\ldots$	0	0
C_1	$-C_2$	0	0	0	$\ldots$	0	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_N	$-K$	0	0	0	$\ldots$	0	$\ldots$	0	0

where $K = \mu_0 Z + \mu_1 C_1 + \ldots + \mu_N C_N$.

4.2.2 Cases 2 and 4: $t_{1x} = t_x + C_1(t_1^2 - t^2) + C_2(t_1 - t)$ and $t_{1x} = t_x + C_5(e^{\alpha t_1} - e^{\alpha t}) + C_6(e^{-\alpha t_1} - e^{-\alpha t})$

In both cases Lie algebra L_n is trivial, consisting of X_1 and Y_1 only, with commutativity relation $[X_1, Y_1] = 0$.

4.2.3 Case 3: $t_{1x} = t_x + \sqrt{C_3 e^{2\alpha t_1} + C_4 e^{\alpha(t_1+t)} + C_3 e^{2\alpha t}}$

Denote by $\tilde{X}_1 = A(\tau_{-1})e^{-\alpha\tau_{-1}}\frac{\partial}{\partial\tau_{-1}}$ and $\tilde{Y}_1 = A(\tau_{-1})Y_1$, $C_2 = [\tilde{X}_1, \tilde{Y}_1]$. Direct calculations show that the multiplication table for algebra L_n is the following

L_n	$\tilde{X}_1$	$\tilde{Y}_1$	C_2
$\tilde{X}_1$	0	C_2	$\alpha^2 C_3 \tilde{Y}_1 + C_4/(2C_3)\tilde{X}_1$
$\tilde{Y}_1$	$-C_2$	0	K
C_2	$-\alpha^2 C_3 \tilde{Y}_1 - C_4/(2C_3)\tilde{X}_1$	$-K$	0

where $K = -(\alpha^2 C_4/2)\tilde{Y}_1 + (2\alpha^2 C_4 e^{\alpha\tau_{-1}} - \alpha^2 C_3)\tilde{X}_1$.

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