Classification of integrable Weingarten surfaces possessing an $\mathfrak{sl}(2)$-valued zero curvature representation

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Abstract. In this paper we classify Weingarten surfaces integrable in the sense of soliton theory. The criterion is that the associated Gauss equation possesses an $\mathfrak{sl}(2)$-valued zero curvature representation with a nonremovable parameter. Under certain restrictions on the jet order, the answer is given by a third order ordinary differential equation to govern the functional dependence of the principal curvatures. Employing the scaling and translation (offsetting) symmetry, we give a general solution of the governing equation in terms of elliptic integrals. We show that the instances when the elliptic integrals degenerate to elementary functions were known to nineteenth century geometers. Finally, we characterize the associated normal congruences.

Keywords: integrable surfaces, Weingarten surfaces, zero curvature representation, spectral parameter

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1. Introduction

Already the classical works of nineteenth-century geometers established a major connection between differential geometry and the theory of partial differential equations. Powerful solution-generating techniques such as the Bäcklund and Darboux transformations [33] have origins in the prototypical relationship between pseudospherical surfaces and solutions of the sine-Gordon equation.

Methods available for solving nonlinear partial differential equations were substantially extended in the seventies to include the inverse scattering transform and its numerous developments; see, e.g., [8, 14, 26, 39]. An important open problem is to describe the class of partial differential equations solvable by these powerful methods. Indirect detectors such as the symmetry analysis have been involved in obtaining extensive complete classifications of integrable evolution equations and systems; see [28] and references therein. The known theoretical answer given in terms of the existence of the associated one-parametric zero curvature representation

$$A_y - B_x + [A, B] = 0$$

has been considered as a classification tool in conjunction with the gauge cohomology by one of us [25]. These methods are not limited to evolution equations, although the necessary computations are rather complex, resource consuming, and unthinkable without substantial use of computer algebra. However, certain partial differential equations of geometric origin are particularly well suited for this classification method, namely, the Gauss–Mainardi–Codazzi equations of immersed surfaces. These equations always possess an associated linear zero curvature representation, albeit without the spectral parameter.

Since their introduction by Weingarten [36], immersed surfaces in $\mathbb{R}^3$ that satisfy a functional relation between the principal curvatures have been of continuing interest in differential geometry, see, e.g., [18, 20, 22]. It is therefore not surprising that attempts have been made to identify classes of Weingarten surfaces such that the corresponding Gauss equation is integrable in the sense of soliton theory. The work of Wu [38] and Finkel [16] indicated that all integrable cases are classical, characterized by a linear relation between the Gauss and the mean curvatures (linear Weingarten surfaces [12, §812]; see also [17] and references therein). In other words, the integrable Weingarten surfaces were conjectured to be either minimal or parallel to surfaces of constant Gaussian curvature. This conjecture was, however, disproved by the present authors in [1], henceforth referred to as Part I. In Part I we found another integrable class, consisting of surfaces with a constant difference between the principal radii of curvature, which we called surfaces of constant astigmatism. Surprisingly enough, this extra class turned out to be classical as well, apparently first mentioned by Beltrami [3, Ch. 9, §20], covered by Bianchi [4] and Darboux [12], see also [31], yet forgotten today.

In this paper we continue the work begun in Part I and complete the classification of integrable classes in the simplest possible case. The integrability criterion we adopt is the existence of an $\mathfrak{sl}(2)$-valued zero curvature representation depending on a nonremovable parameter. We apply the same method of formal spectral parameter, introduced in [25] and briefly reproduced in Part I. The underlying symbolic computations, done with the help of *Maple* and our own package *Jets* [2], are omitted. To stay within the limits given by available computing resources we had to restrict the jet order (order of derivatives).

The answer is given by a third order nonlinear ordinary differential equation (10) to govern the functional dependence of the principal curvatures. Incorporation of the actual spectral parameter is achieved in Section 3. This can be considered a proof of integrability, opening up the possibility to obtain explicit solutions by the methods of soliton theory [8, 14, 39]. However, we had to resign ourselves to following this road. Neither were we able to establish a Bäcklund or Darboux transformation [26, 33], which would allow us to construct families of exact solutions depending on an arbitrary number of parameters. We only remark that seed solutions could be conveniently found among the rotational surfaces, see [22, eq. (1)].

The governing equation (10) is explored in Section 4. We identify two basic symmetries, scaling and translation (offsetting), and solve equation (10) in terms of elliptic integrals. The generic class of integrable Weingarten surfaces we obtained depends on one essential parameter (apart from the scaling and offsetting parameters) and is believed to be new. In Sect. 5 we establish the integrable Gauss equation (39) in the generic case as well as in a number of special cases when the elliptic integrals degenerate to elementary functions. All of these special cases could be located in the nineteenth century literature.

Geometrically, surfaces are related by an offsetting symmetry if they are parallel to each other, i.e., if they share the same normal line congruence. Therefore, the offsetting symmetry indicates that the concept of integrability naturally extends from surfaces to their normal line congruences. Section 7 grew out of our attempt to characterize the normal congruences of the integrable Weingarten surfaces. We obtain certain relations satisfied by suitably chosen metric invariants of the pair of the focal surfaces. Naturally, we expect the corresponding focal surfaces to be integrable as well, but a detailed investigation had to be postponed to the next paper.

2. Preliminaries

We consider surfaces $\mathbf{r}(x, y)$, parameterized by the lines of curvature. This is a regular parameterization except at umbilic points. The umbilic points are isolated by the Hartman–Wintner theorem [18] except for spheres and planes, which are, therefore, the only surfaces excluded from consideration.

The fundamental forms can be written as

$$\begin{aligned} \mathrm{I} &= u^2\,\mathrm{d}x^2 + v^2\,\mathrm{d}y^2, \\ \mathrm{II} &= \frac{u^2}{\rho}\,\mathrm{d}x^2 + \frac{v^2}{\sigma}\,\mathrm{d}y^2, \end{aligned} \tag{1}$$

where ρ, σ are the principal radii of curvature. The radii transform in a very simple way under the offsetting symmetry (21) of the integrability problem (unlike the principal curvatures $p = 1/\rho$, $q = 1/\sigma$ we used in Part I).

Choosing the orthonormal frame $\Psi = (\mathbf{r}_x/u, \mathbf{r}_y/v, \mathbf{n})$, we consider the Gauss–Weingarten equations

$$\Psi_x = \begin{pmatrix} 0 & -\dfrac{u_y}{v} & \dfrac{u}{\rho} \\ \dfrac{u_y}{v} & 0 & 0 \\ -\dfrac{u}{\rho} & 0 & 0 \end{pmatrix} \Psi, \qquad \Psi_y = \begin{pmatrix} 0 & \dfrac{v_x}{u} & 0 \\ -\dfrac{v_x}{u} & 0 & \dfrac{v}{\sigma} \\ 0 & -\dfrac{v}{\sigma} & 0 \end{pmatrix} \Psi \tag{2}$$

or, more explicitly,

$$\begin{aligned}
\mathbf{r}_{xx} &= \frac{u_x}{u}\mathbf{r}_x - \frac{uu_y}{v^2}\mathbf{r}_y + \frac{u^2}{\rho}\mathbf{n}, \qquad \mathbf{n}_x = -\frac{1}{\rho}\mathbf{r}_x, \\
\mathbf{r}_{xy} &= \frac{u_y}{u}\mathbf{r}_x + \frac{v_x}{v}\mathbf{r}_y, \\
\mathbf{r}_{yy} &= -\frac{vv_x}{u^2}\mathbf{r}_x + \frac{v_y}{v}\mathbf{r}_y + \frac{v^2}{\sigma}\mathbf{n}, \qquad \mathbf{n}_y = -\frac{1}{\sigma}\mathbf{r}_y.
\end{aligned} \tag{3}$$

Consequently, the Gauss–Mainardi–Codazzi equations, which are the compatibility conditions for (3), read

$$uu_{yy} + vv_{xx} - \frac{v}{u}u_x v_x - \frac{u}{v}u_y v_y + \frac{u^2v^2}{\rho\sigma} = 0, \tag{4}$$

and

$$\frac{u_y}{u} + \frac{\sigma\rho_y}{\rho(\rho-\sigma)} = 0, \quad \frac{v_x}{v} + \frac{\rho\sigma_x}{\sigma(\sigma-\rho)} = 0. \tag{5}$$

As with Part I, we concentrate on Weingarten surfaces, which are characterized by the existence of a functional dependence between ρ and σ. We often resort to a parametric representation $\rho(w), \sigma(w)$ of the dependence.

Recall that parameters x, y label the lines of curvature; otherwise they are arbitrary. In line with Finkel's approach [16], we use this reparameterization freedom to solve the Mainardi–Codazzi subsystem (5). The following proposition is a mixture of classical and new results.

Proposition 1. *Away from umbilic points, a Weingarten surface can be parameterized by the lines of curvature in such a way that*

$$u = \exp\int \frac{\rho'\sigma}{(\sigma-\rho)\rho}\,dw, \qquad v = \exp\int \frac{\rho\sigma'}{(\rho-\sigma)\sigma}\,dw. \tag{6}$$

The Mainardi–Codazzi subsystem (5) *is then identically satisfied, while the remaining Gauss equation can be written in the compact form*

$$R_{yy} + S_{xx} + T = 0, \tag{7}$$

where R, S, T are appropriate functions of the unknown w. Moreover, the constraint

$$\Big(\frac{1}{\rho} - \frac{1}{\sigma}\Big)uv = 1 \tag{8}$$

can be imposed as an additional condition, and then $T = 1/(\sigma - \rho)$.

Proof. Writing $\rho(w), \sigma(w)$ for some function $w(x, y)$, the general solution of the Mainardi–Codazzi subsystem (5) is

$$u = u_0(x)\exp\int \frac{\rho'\sigma}{(\sigma-\rho)\rho}\,dw, \qquad v = v_0(y)\exp\int \frac{\rho\sigma'}{(\rho-\sigma)\sigma}\,dw.$$

Obviously from formulas (1), the multipliers $u_0(x)$, $v_0(y)$ can be removed by an appropriate relabelling $\tilde{x} = \tilde{x}(x)$, $\tilde{y} = \tilde{y}(y)$ of the surface's curvature lines. With $u_0 = v_0 = 1$, we have

$$uv = \exp \int \Big(\frac{\rho'\sigma}{(\rho - \sigma)\rho} + \frac{\rho\sigma'}{(\sigma - \rho)\sigma} \Big)\, dw = c \frac{\rho\sigma}{\sigma - \rho},$$

where c is an arbitrary constant multiplier. Setting $c = 1$ by the same relabelling argument proves the last relation.

Having resolved the Mainardi–Codazzi subsystem, we are left with the Gauss equation (4) alone. Multiplied by $1/\rho - 1/\sigma$, equation (4) can be written in the compact form (7), where

$$R = \int \frac{\rho'}{\rho^2} u^2\, dw, \qquad S = -\int \frac{\sigma'}{\sigma^2} v^2\, dw, \qquad T = u^2 v^2 \frac{\sigma - \rho}{\rho^2 \sigma^2}. \tag{9}$$

Substituting $1/(1/\rho - 1/\sigma)$ for uv finishes the proof. □

3. The classification result

Employing the Maple package *Jets* [2], we completed the computer-aided cohomological classification outlined in Part I. We have no computer-independent proof of the following result.

Proposition 2. *The third-order ordinary differential equation*

$$\rho''' = \frac{3}{2\rho'} \rho''^2 - \frac{\rho' - 1}{\rho - \sigma} \rho'' + 2 \frac{(\rho' - 1)\rho'(\rho' + 1)}{(\rho - \sigma)^2}. \tag{10}$$

determines a unique maximal class of Gauss–Mainardi–Codazzi equations of Weingarten surfaces whose initial $\mathfrak{sl}(2, \mathbb{C})$*-valued zero curvature representation*

$$A_0 = \begin{pmatrix} \dfrac{\mathrm{i} u_y}{2v} & -\dfrac{u}{2\rho} \\ \dfrac{u}{2\rho} & -\dfrac{\mathrm{i} u_y}{2v} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} -\dfrac{\mathrm{i} v_x}{2u} & -\dfrac{\mathrm{i} v}{2\sigma} \\ -\dfrac{\mathrm{i} v}{2\sigma} & \dfrac{\mathrm{i} v_x}{2u} \end{pmatrix} \tag{11}$$

admits a second order formal spectral parameter under the condition that the normal form of the zero curvature representation can depend on derivatives of u, v, σ, ρ *of no higher than the first order.*

Here and in what follows we assume that ρ is a function of σ and the prime refers to derivatives with respect to σ. A kth order formal parameter λ means a power series in terms of λ up to order k. Part I should be consulted for the other unexplained notions.

Remark 1. (1) The last proposition provides a complete classification of integrable Weingarten surfaces under the following assumptions: The one-parametric zero curvature representation takes values in the Lie algebra $\mathfrak{sl}(2)$, includes the initial zero curvature representation (11) as a member, depends analytically on the parameter, and its normal form involves derivatives of no higher than the first order. All these limitations can be overcome, in principle [24], at the cost of requiring significantly more computational resources.

(2) We would like to stress that the only part relying on machine computations is the completeness of the classification. All the other proofs in this paper are traditional.

In the rest of this section we establish integrability of the class determined by equation (10). The equation itself will be solved in the next section.

Proposition 3. *The nonremovable spectral parameter exists for all dependences $\rho(\sigma)$ allowed by the governing equation* (10).

Proof. Inspired by the results of the computer-aided classification, we depart from the following ansatz for the parameter-dependent zero curvature representation:

$$A = \begin{pmatrix} a_{111}\dfrac{u_y}{v} + a_{110}\sigma_x & a_{12}u \\ a_{21}u & -a_{111}\dfrac{u_y}{v} - a_{110}\sigma_x \end{pmatrix},$$

$$B = \begin{pmatrix} b_{111}\dfrac{v_x}{u} + b_{110}\sigma_y & b_{12}v \\ b_{12}v & -b_{111}\dfrac{v_x}{u} - b_{110}\sigma_y \end{pmatrix},$$

with $a_{111}, b_{111}, a_{110}, b_{110}, a_{12}, a_{21}, b_{12}$ being the unknown functions of σ. The problem is to solve the zero curvature condition $D_yA - D_xB + [A, B] = 0$ for matrix functions A, B of u, v, σ, ρ and their derivatives. However, the derivatives are not independent quantities, being subject to the Gauss–Mainardi–Codazzi equations. The proper way to deal with this situation is to introduce the manifold determined by the equation and its derivatives (a diffiety [9]). This is fairly easy if the order of derivatives is restricted as it is. Initially the derivatives are considered to be independent (jet space coordinates). Considering ρ as a function of σ and resolving the Mainardi–Codazzi equations (5) with respect to u_y, v_x, we can express u_y, v_x as functions of $u, v, \sigma, \sigma_x, \sigma_y$. Similarly, the derivatives of the Mainardi–Codazzi equations (5) can be resolved with respect to $u_{xy}, u_{yy}, v_{xx}, v_{xy}$, giving $u_{xy}, u_{yy}, v_{xx}, v_{xy}$ as functions of $u, u_x, v, v_y, \sigma, \sigma_x, \sigma_y$. Consequently, the Gauss equation (4) can be written in terms of $u, u_x, v, v_y, \sigma, \sigma_x, \sigma_y, \sigma_{xx}, \sigma_{yy}$, and then resolved with respect to σ_{yy}. The explicit formulas are somewhat cumbersome, hence omitted.

With A, B chosen as above, the left-hand side $S := D_yA - D_xB + [A, B]$ of the zero curvature condition $S = 0$ is a matrix function of $u, u_x, v, v_y, \sigma, \sigma_x, \sigma_y, \sigma_{xx}, \sigma_{xy}$. From $\partial S/\partial\sigma_{xx} = 0$ and $\partial S/\partial\sigma_{xy} = 0$ we obtain

$$b_{111} = -a_{111}, \qquad b_{110} = a_{110}.$$

From either $\partial^2 S/\partial\sigma_x^2 = 0$ or $\partial^2 S/\partial\sigma_y^2 = 0$ we get $a'_{111} = 0$. Hence, a_{111} is a constant, which we rename λ in anticipation of its role as the spectral parameter.

Now, $\partial S/\partial \sigma_x = 0$ if and only if

$$a_{110} = \frac{\lambda\rho}{2\sigma(\sigma-\rho)}\,\frac{a_{12}+a_{21}}{b_{12}}, \qquad b_{12}' = \frac{\rho}{\sigma(\sigma-\rho)}[b_{12}+\lambda(a_{21}-a_{12})], \tag{12}$$

while $\partial S/\partial \sigma_y = 0$ can be rewritten as

$$\begin{aligned} a_{12}' &= 2a_{110}a_{12} + \frac{\sigma\rho'}{\rho(\rho-\sigma)}(a_{12}+2\lambda b_{12}), \\ a_{21}' &= -2a_{110}a_{21} + \frac{\sigma\rho'}{\rho(\rho-\sigma)}(a_{21}-2\lambda b_{12}), \end{aligned} \tag{13}$$

Modulo these relations, vanishing of S is equivalent to

$$b_{12} = \frac{\lambda}{\rho\sigma(a_{12}-a_{21})}. \tag{14}$$

We claim that the governing equation (10) arises as the condition that the system (12), (13), and (14) be compatible for arbitrary $\lambda \neq 0$. To prove this, we denote $P = a_{12} + a_{21}$, $Q = a_{12} - a_{21}$. With a_{110} and b_{12} taken from formulas (12) and (14), respectively, equations (13) turn into

$$P' = P\frac{\sigma\rho' - Q^2\rho^3}{\rho(\rho-\sigma)}, \qquad Q' = Q\frac{\sigma\rho' - P^2\rho^3}{\rho(\rho-\sigma)} + \frac{4\lambda^2\rho'}{\rho^2(\rho-\sigma)}\,\frac{1}{Q}, \tag{15}$$

and the second equation in (12) into

$$\rho^4(Q^2-P^2)Q^2 + \rho^2(\rho'-1)P^2 + 4\lambda^2\rho' = 0. \tag{16}$$

Now the question is whether equations (15) and (16) are compatible. Modulo eq. (15), the derivative of (16) with respect to σ is

$$\begin{aligned} &2\rho^6(P^2-Q^2)P^2Q^2 + 2(1-3\rho')\rho^4P^2Q^2 - 4\rho'\lambda^2P^2 \\ &\quad + (4\lambda^2+\rho^2Q^2)[4\rho'\rho^2Q^2 + (\rho-\sigma)\rho'' + 2\rho'^2 - 2\rho'] = 0. \end{aligned} \tag{17}$$

This is equivalent to

$$[(\rho-\sigma)\rho'' - 2\rho'^2 + 2(1+8\lambda^2)\rho']\rho^2Q^2 + 4\lambda^2[(\rho-\sigma)\rho'' - 2\rho'^2 - 2\rho'] = 0 \tag{18}$$

modulo (16), since (18) is the remainder after division of (17) by (16) as polynomials in P. Similarly, dividing (16) by (18) as polynomials in Q, we get

$$\begin{aligned} &[(\rho-\sigma)\rho'' - 2\rho'^2 - 2\rho'][(\rho-\sigma)\rho'' - 2\rho'^2 + 2(1+8\lambda^2)\rho']\rho^2P^2 \\ &\quad - 4(1+4\lambda^2)[(\sigma-\rho)^2\rho''^2 - 4\rho'^4 + 8(1+8\lambda^2)\rho'^3 - 4\rho'^2] = 0. \end{aligned} \tag{19}$$

Differentiating (17) once more and taking the result modulo (15), (19) and (18), we get the governing equation (10) immediately.

Summing up, we obtain a zero curvature representation

$$A = \begin{pmatrix} -\dfrac{\lambda\sigma\rho'}{\rho(\rho-\sigma)}\dfrac{u}{v}\sigma_y - \dfrac{1}{2}\dfrac{\rho^2}{\rho-\sigma}PQ\sigma_x & \frac{1}{2}(P+Q)u \\ \frac{1}{2}(P-Q)u & \dfrac{\lambda\sigma\rho'}{\rho(\rho-\sigma)}\dfrac{u}{v}\sigma_y + \dfrac{1}{2}\dfrac{\rho^2}{\rho-\sigma}PQ\sigma_x \end{pmatrix},$$

$$B = \begin{pmatrix} -\dfrac{\lambda\rho}{\sigma(\rho-\sigma)}\dfrac{v}{u}\sigma_x - \dfrac{1}{2}\dfrac{\rho^2}{\rho-\sigma}PQ\sigma_y & \dfrac{\lambda}{\sigma\rho Q}v \\ \dfrac{\lambda}{\sigma\rho Q}v & \dfrac{\lambda\rho}{\sigma(\rho-\sigma)}\dfrac{v}{u}\sigma_x + \dfrac{1}{2}\dfrac{\rho^2}{\rho-\sigma}PQ\sigma_y \end{pmatrix},$$

where P and Q are the square roots to be determined from equations (19) and (18), respectively. Away from umbilic points (where $\rho = \sigma$), matrices A, B actually exist unless $(\rho - \sigma)\rho'' - 2\rho'^2 - 2\rho' = 0$ when P is undefined. This excludes exactly spheres and the linear Weingarten surfaces. The latter surfaces are, however, well known to be integrable, being parallel to surfaces of constant curvature (either Gaussian or mean), see [38] or [33, §1.5.2].

If $\lambda = \mathrm{i}/2$, then we have $P = 0$ and $Q = 1/r^2$, which reproduces the parameterless zero curvature representation (11) we started with. □

Non-removability of the parameter is ensured by the method [25] (follows from nontriviality of the first gauge cohomology group).

4. Solution of the governing equation

Apart from the discrete symmetry $\rho \leftrightarrow \sigma$, the governing equation (10) has two obvious continuous symmetries, which should be expected in every integrable class of surfaces: the *scaling symmetry*

$$\rho \mapsto \mathrm{e}^T\rho, \qquad \sigma \mapsto \mathrm{e}^T\sigma \tag{20}$$

and the *translational symmetry*

$$\rho \mapsto \rho + T, \qquad \sigma \mapsto \sigma + T. \tag{21}$$

The geometric meaning of the latter symmetry is *offsetting*, also known as taking the *parallel surface*. In terms of position vectors, $\mathbf{r}$ is transformed to $\mathbf{r} + T\mathbf{n}$, where $\mathbf{n}$ is the unit normal vector and T is the distance.

With the help of these symmetries we can reduce the order of equation (10) by two. This can be done by rewriting the equation in terms of the symmetry invariants. Since rescaling applies also to the offset, the translational reduction should precede the scaling reduction. For the two lowest-order translational invariants we choose

$$\xi = \rho - \sigma, \qquad \eta = \rho' \tag{22}$$

(recall that the prime denotes the derivative with respect to σ).

1. If $\xi' = 0$ (equivalently, $\rho' = 1$), then $\rho - \sigma = \text{const}$, which are the surfaces of constant astigmatism we dealt with in Part I.

2. Otherwise, more translational invariants can be computed as derivatives of η with respect to ξ:

$$\eta_\xi = \frac{\eta'}{\xi'} = \frac{\rho''}{\rho' - 1}, \quad \eta_{\xi\xi} = \frac{\rho'''}{(\rho' - 1)^2} - \frac{\rho''^2}{(\rho' - 1)^3}, \tag{23}$$

etc. In terms of these invariants, the governing equation (10) reduces to the second-order equation

$$2\xi^2(\eta - 1)\eta\eta_{\xi\xi} - \xi^2(\eta - 3)\eta_\xi^2 + 2\xi(\eta - 1)\eta\eta_\xi - 4(\eta + 1)\eta^2 = 0. \tag{24}$$

As expected, this equation is scaling invariant. To reduce it with respect to scaling, we proceed as follows. Besides η, one more scaling invariant is

$$\zeta = \xi(\eta - 1)\eta_\xi. \tag{25}$$

Although dispensable, the factor $\eta - 1$ simplifies the computations to follow.

2.1. If $\eta' = 0$, i.e., $\rho'' = 0$, then (10) reduces to $\rho' = c$, where c is either of $-1, 0, 1$. The corresponding surfaces are, respectively, the constant mean curvature surfaces (a subclass of linear Weingarten surfaces), the tubular surfaces (surfaces swept by spheres of constant radius moving along a space curve) and once more the constant astigmatism surfaces.

2.2. Otherwise $\rho'' \neq 0$ and we have

$$\zeta_\eta = \frac{\rho'''}{\rho''}(\rho - \sigma) + \rho' - 1.$$

In terms of η, ζ, the reduced governing equation (24) becomes the Bernoulli equation

$$\zeta_\eta = \frac{3}{2}\frac{\zeta}{\eta} + 2\frac{\eta^3 - \eta}{\zeta}$$

with the general solution $\zeta^2 = 4(\eta^2 + 2c_0\eta + 1)\eta^2$, where c_0 is the integration constant. Substituting from eq. (25) yields the separable first-order equation

$$\xi\frac{d\eta}{d\xi} = \pm 2\frac{\eta}{\eta - 1}\sqrt{\eta^2 + 2c_0\eta + 1} \tag{26}$$

containing the parameter c_0. Being written in terms of the scaling and translation invariants, this equation determines the integrable Weingarten surfaces up to rescaling and offsetting. Depending on the value of the parameter c_0 and on the choice of the '±' sign, we obtain the following cases.

2.2.1 Let $c_0 = 1$. Equation (26) becomes

$$\xi\frac{d\eta}{d\xi} = \pm 2\frac{(\eta+1)\eta}{\eta-1}. \tag{27}$$

2.2.1.1. With the choice of the plus sign in (27), the general solution is $(\eta+1)^2 = c_1\eta\xi^2$. Substituting from eq. (22), we obtain

$$(\rho'+1)^2 = c_1(\rho-\sigma)^2\rho'.$$

If $c_1 = 0$, the general solution is $\rho+\sigma = \text{const}$. Otherwise, we apply the transformation

$$\kappa = \rho+\sigma, \quad \xi = \rho-\sigma \tag{28}$$

to get

$$(c_1\xi^2-4)\Big(\frac{d\kappa}{d\xi}\Big)^2 = c_1\xi^2.$$

The equation is separable with a general solution $(\kappa-c_2)^2-\xi^2+4/c_1 = 0$, i.e.,

$$4\rho\sigma - 2c_2(\rho+\sigma) + \frac{4+c_1c_2^2}{c_1} = 0.$$

In both cases, $c_1 = 0$ and $c_1 \neq 0$, solutions correspond to the linear Weingarten surfaces.

2.2.1.2. With the choice of the minus sign in (27), the general solution is $(\eta+1)^2\xi^2 = c_1\eta$. Substituting from eq. (22), we obtain $(\rho'+1)^2(\rho-\sigma)^2 = c_1\rho'$. For $c_1 = 0$ we have the special linear Weingarten surfaces $\rho+\sigma = \text{const}$ again. Otherwise, we apply the transformation (28) to get

$$(4\xi^2-c_1)\Big(\frac{d\kappa}{d\xi}\Big)^2 + c_1 = 0.$$

The solutions are

$$\kappa = \pm\tfrac{1}{2}\sqrt{-c_1}\,\ln\big(2\sqrt{-c_1}\,\xi + \sqrt{c_1^2-4c_1\xi^2}\,\big) + c_2,$$

where c_2 is the integration constant.

2.2.1.2.1. For $c_1 < 0$ we can write

$$\xi = \frac{\sqrt{-c_1}}{2}\sinh\Big(\pm\frac{2}{\sqrt{-c_1}}(\kappa-c_2) - \ln(-c_1)\Big)$$

or

$$\frac{\rho-\sigma}{C_1} = \pm\sinh\Big(\frac{\rho+\sigma}{C_1} + C_0\Big).$$

2.2.1.2.2. Similarly, solutions corresponding to positive c_1 are

$$\frac{\rho-\sigma}{C_1} = \sin\Big(\frac{\rho+\sigma}{C_1} + C_0\Big). \tag{29}$$

2.2.2 Let $c = -1$. Equation (26) becomes

$$(\eta-1)^2\Big(\xi\frac{d\eta}{d\xi} - 2\eta\Big)\Big(\xi\frac{d\eta}{d\xi} + 2\eta\Big) = 0. \tag{30}$$

Solutions corresponding to $\eta = 1$ belong to Case 1 (constant astigmatism surfaces).

2.2.2.1. The general solution of $\xi(d\eta/d\xi) = 2\eta$ is $\eta = c_1\xi^2$. Substituting from eq. (22), we obtain the Riccati equation $\rho' = c_1(\rho-\sigma)^2$.

2.2.2.1.1. For $c_1 > 0$ we get

$$\rho = \sigma - \frac{\tanh(\sqrt{c_1}\,\sigma + c_2)}{\sqrt{c_1}} \quad \text{or} \quad \rho = \sigma - \frac{\coth(\sqrt{c_1}\,\sigma + c_2)}{\sqrt{c_1}} \tag{31}$$

according to whether the integration constant is positive or negative.

2.2.2.1.2. Similarly, for $c_1 < 0$ we get

$$\rho = \sigma - \frac{\tan(\sqrt{-c_1}\,\sigma + c_2)}{\sqrt{-c_1}} \quad \text{or} \quad \rho = \sigma + \frac{\cot(\sqrt{-c_1}\,\sigma + c_2)}{\sqrt{-c_1}}. \tag{32}$$

2.2.2.2. When solving $\xi(d\eta/d\xi) = -2\eta$, we get (31) and (32) with ρ, σ interchanged.

2.2.3. We are left with the generic case $c_0 \notin \{-1, 1\}$. Equation (26) has the general solution

$$\big(\eta + c_0 + \sqrt{\eta^2 + 2c_0\eta + 1}\big)\big(c_0\eta + 1 + \sqrt{\eta^2 + 2c_0\eta + 1}\big) = c_1\xi^{\pm 2}\eta. \tag{33}$$

If $c_1 = 0$, then $\eta = 0$ in view of $c_0 \notin \{-1, 1\}$, which yields the tubular surfaces $\rho = \text{const}$. Let us, therefore, assume that $c_1 \neq 0$. Upon substituting from (22), equation (33) becomes a first-order ODE, separable in terms of variables (28) and having the elliptic integral

$$\kappa = \int^{\xi} \frac{-c_1 t^{\pm 2} + c_0^2 - 1}{\sqrt{c_1^2 t^{\pm 4} - 2(c_0+1)(c_0+3)c_1 t^{\pm 2} + (c_0^2-1)^2}}\,\mathrm{d}t$$

as the general solution. The two cases the '$\pm$' symbol refers to can be converted one into another by the substitution $c_1 \to (c_0^2-1)^2/c_1$. Therefore, we can safely choose the sign to be '+', which we do in the sequel. Moreover, if κ is a solution, then so is $-\kappa$ (as a combination of the $\rho \leftrightarrow \sigma$ switch and a scaling by factor of -1). This is why we often ignore the sign of κ in what follows.

Substituting $t \to s/m$, $m = \sqrt{|c_1/(1-c_0^2)|}$, we simplify the integral above to

$$\kappa = \frac{1}{m} I_\pm(m\xi, c), \qquad I_\pm(\xi, c) = \int^\xi \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}} \, \mathrm{d}s, \tag{34}$$

where '$\pm$' refers to the signum of $c_1/(1-c_0^2)$; in particular, is unrelated to the '$\pm$' sign in (33). The real parameter c is related to c_0 by $c = \pm(c_0+3)/(c_0-1)$.

Formula (34) describes possible dependences $\rho(\sigma)$ via the substitution $\kappa = \rho + \sigma$, $\xi = \rho - \sigma$. Three independent parameters are involved: m, c and the integration constant (the lower limit of the integral). Obviously, m plays the role of the scaling parameter. The integration constant can be easily identified with the offsetting parameter T from (21).

Each dependence between κ and ξ has a unique representative modulo scaling and offsetting, obtainable by fixing the lower limit of the integral $I_\pm(\xi, c)$ in (34). This is straightforward when $c > -1$; we simply redefine $I_\pm(\xi, c)$ to be

$$I_\pm(\xi, c) = \int_0^\xi \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}} \, \mathrm{d}s. \tag{35}$$

If, however, $c < -1$, then the integrand in (34) is real in three separate intervals $(-\infty, -\sqrt{\gamma_+})$, $(-\sqrt{\gamma_-}, \sqrt{\gamma_-})$, and $(\sqrt{\gamma_+}, \infty)$, where

$$\gamma_\pm = -c \pm \sqrt{c^2 - 1} > 0. \tag{36}$$

We choose the representatives $-\tilde{I}_\pm(-\xi, c)$, $I_\pm(\xi, c)$, and $\tilde{I}_\pm(\xi, c)$, respectively, where $I_\pm(\xi, c)$ is given by (35) in the interval $-\gamma_- \le \xi \le \gamma_-$, while

$$\tilde{I}_\pm(\xi, c) = \int_{\gamma_+}^\xi \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}} \, \mathrm{d}s, \quad \gamma_+ \le \xi. \tag{37}$$

5. Summary of the solutions

As demonstrated in the preceding section, each integrable class is determined by certain relation between the radii of curvature, which can be subject to rescaling $\rho \to c_1\rho$, $\sigma \to c_1\sigma$, offsetting $\rho \to \rho + c_0$, $\sigma \to \sigma + c_0$ and the twist $\rho \leftrightarrow \sigma$.

With the help of Proposition 1, we can find the corresponding integrable Gauss equation. To start with, we investigate the generic class determined by formula (34); we fix the scaling for simplicity.

Proposition 4. *Assuming*

$$\rho + \sigma = I_\pm(\rho - \sigma, c), \quad I_\pm(\xi, c) = \int^\xi \frac{1 \pm s^2}{\sqrt{1 + 2cs^2 + s^4}} \, \mathrm{d}s, \tag{38}$$

the Gauss equation (4) *for* $\xi = \rho - \sigma$ *reads*

$$R'\xi_{yy} + R''\xi_y^2 + S'\xi_{xx} + S''\xi_x^2 + T = 0, \tag{39}$$

where

$$R' = \frac{1 + c\xi^2 + \Delta(\xi, c)}{\xi^2 \Delta(\xi, c)}, \quad S' = \frac{c \mp 1}{2} \frac{\xi^2}{(1 + c\xi^2 + \Delta(\xi, c))\Delta(\xi, c)},$$
$$\Delta(\xi, c) = \sqrt{1 + 2c\xi^2 + \xi^4}, \quad T = -\frac{1}{\xi}.$$

The metric coefficients u, v *in* (1) *are*

$$u = \frac{\xi + I_\pm(\xi, c)}{2\xi} \sqrt{1 \mp \xi^2 + \Delta(\xi, c)}, \quad v = \frac{\xi - I_\pm(\xi, c)}{2\xi} \sqrt{\frac{1 \mp \xi^2 - \Delta(\xi, c)}{2c \pm 2}}.$$

Proof. We parameterize ρ and σ by ξ, i.e., we resolve (38) as

$$\rho = \frac{I_\pm(\xi, c) + \xi}{2}, \quad \sigma = \frac{I_\pm(\xi, c) - \xi}{2}.$$

The general form of the Gauss equation, along with the last term $T = 1/(\sigma - \rho) = -1/\xi$, follow from Proposition 1. To find R', S', we compute

$$(\ln R')' = \frac{R''}{R'} = \frac{(\rho - \sigma)\rho'' - 2\rho'^2}{(\rho - \sigma)\rho'} = -\frac{2}{\xi} \frac{c\xi^2 + \xi^4 + \sqrt{1 + 2c\xi^2 + \xi^4}}{1 + 2c\xi^2 + \xi^4},$$
$$(\ln S')' = \frac{S''}{S'} = \frac{(\rho - \sigma)\sigma'' + 2\sigma'^2}{(\rho - \sigma)\sigma'} = -\frac{2}{\xi} \frac{c\xi^2 + \xi^4 - \sqrt{1 + 2c\xi^2 + \xi^4}}{1 + 2c\xi^2 + \xi^4}$$

from (9) under the constraint (8). These equations need to be integrated once, which is easy; the integration constants have been chosen to match equations (8) and (9). Finally, from (9) one easily computes the coefficients u, v as $u = \sqrt{R'\rho^2/\rho'}$, $v = \sqrt{-S'\sigma^2/\sigma'}$. □

Apart from the generic class we also obtained a number of special solutions, listed in Table 1 (omitting the tubular surfaces). Rows 5b and 6b differ only by translation (offsetting) and can be identified one with another.

The first column contains a determining relation (up to a scaling), while the second harbours the corresponding integrable equation in the compact form (7). Table 2 gives the principal radii of curvature ρ, σ, metric coefficients u, v, and the variable z (see Table 1) in terms of a suitably chosen parameterizing variable w.

Neither of the special cases is new to differential geometry. Row 1 reflects that, in terms of the curvature line coordinates, minimal surfaces correspond to solutions of the Liouville equation [5, §351]. Similarly, row 2a reproduces the relation between surfaces of negative constant Gaussian curvature and solutions of the elliptic sinh-Gordon equation. Row 2b does the same for the hyperbolic sine-Gordon equation and surfaces of positive constant Gaussian curvature (or constant mean curvature, by the theorem of Bonnet on parallel surfaces). Nowadays, surfaces of constant mean or Gaussian curvature are undoubtedly the best understood classes of surfaces integrable in the sense of soliton theory (see, e.g., [6, 7, 13, 19, 27, 29] and references therein).

	relation	integrable equation
1.	$\rho+\sigma=0$	$z_{xx}+z_{yy}+\mathrm{e}^z=0$
2a.	$\rho\sigma=1$	$z_{xx}+z_{yy}-\sinh z=0$
2b.	$\rho\sigma=-1$	$z_{xx}-z_{yy}+\sin z=0$
3a.	$\rho-\sigma=\sinh(\rho+\sigma)$	$(\tanh z-z)_{xx}+(\coth z-z)_{yy}+\operatorname{csch}2z=0$
3b.	$\rho-\sigma=\sin(\rho+\sigma)$	$(\tan z-z)_{xx}+(\cot z+z)_{yy}+\csc 2z=0$
4.	$\rho-\sigma=1$	$z_{xx}+(1/z)_{yy}+2=0$
5a.	$\rho-\sigma=\tanh\rho$	$\frac14(\sinh z-z)_{xx}+(\coth\frac12 z)_{yy}+\coth\frac12 z=0$
5b.	$\rho-\sigma=\tan\rho$	$\frac14(\sin z-z)_{xx}+(\cot\frac12 z)_{yy}+\cot\frac12 z=0$
6a.	$\rho-\sigma=\coth\rho$	$\frac14(\sinh z+z)_{xx}-(\tanh\frac12 z)_{yy}+\tanh\frac12 z=0$
6b.	$\rho-\sigma=-\cot\rho$	$\frac14(\sin z+z)_{xx}+(\tan\frac12 z)_{yy}+\tan\frac12 z=0$

Table 1. Special integrable cases and the associated integrable Gauss equations

	ρ	σ	u	v	z
1.	w	$-w$	$\sqrt{w/2}$	$\sqrt{w/2}$	$-\ln w$
2a.	w	$\dfrac{1}{w}$	$\dfrac{w}{\sqrt{w^2-1}}$	$\dfrac{-1}{\sqrt{w^2-1}}$	$2\operatorname{arctanh} w$
2b.	w	$-\dfrac{1}{w}$	$\dfrac{w}{\sqrt{w^2+1}}$	$\dfrac{1}{\sqrt{w^2+1}}$	$2\arctan w$
3a.	$\dfrac{w+\sinh w}{2}$	$\dfrac{w-\sinh w}{2}$	$\dfrac{w+\sinh w}{2\sqrt{\cosh w-1}}$	$\dfrac{w-\sinh w}{2\sqrt{\cosh w+1}}$	$\frac12 w$
3b.	$\dfrac{w+\sin w}{2}$	$\dfrac{w-\sin w}{2}$	$\dfrac{w+\sin w}{2\sqrt{1-\cos w}}$	$\dfrac{w-\sin w}{2\sqrt{1+\cos w}}$	$\frac12 w$
4.	w	$w-1$	$\dfrac{w}{\mathrm{e}^w}$	$(1-w)\mathrm{e}^w$	e^{2w}
5a.	w	$w-\tanh w$	$\dfrac{w}{\sinh w}$	$\sinh w-w\cosh w$	$2w$
5b.	w	$w-\tan w$	$\dfrac{w}{\sin w}$	$\sin w-w\cos w$	$2w$
6a.	w	$w-\coth w$	$\dfrac{w}{\cosh w}$	$\cosh w-w\sinh w$	$2w$
6b.	w	$w+\cot w$	$\dfrac{w}{\cos w}$	$\cos w+w\sin w$	$2w$

Table 2. Special integrable cases. The radii of curvature ρ, σ, the metric coefficients u, v, and the unknown z of the integrable Gauss equation in terms of a variable w.

It may come as a surprise that the other cases are classical as well. Introduced by Weingarten [36, §4] ('eine neue Flächenklasse'), surfaces satisfying the relation $\rho-\sigma=\sin(\rho+\sigma)$ (row 4b) are covered in Darboux [12, §§745, 746, 766, 769, 770] ('une classe nouvelle de surfaces découverte par M. Weingarten') and Bianchi [4, §135], [5, §245]. Darboux [12, §746] gave a general solution of an equation equivalent to our $(\tan z-z)_{xx}+(\cot z+z)_{yy}+\csc 2z=0$. He also provided a remarkable geometric construction in [12, §770], further developed by Bianchi [5, §245]. In a nutshell: the middle evolutes are translation surfaces generated by curves of opposite constant nonzero

	relation	limit
1.	$\kappa = 0$	$I_\pm(\xi, \infty)$
2a.	$\kappa^2 = \xi^2 + 4$	$\lim\limits_{m=\infty} I_\pm(m\xi, 2m^2)/m$
2b.	$\kappa^2 = \xi^2 - 4$	$\lim\limits_{m=\infty} I_\pm(m\xi, -2m^2)/m$
3a.	$\kappa = \operatorname{arcsinh} \xi$	$\lim\limits_{m=0} I_\pm(m\xi, 1/2m^2)/m$
3b.	$\kappa = \arcsin \xi$	$\lim\limits_{m=0} I_\pm(m\xi, -1/2m^2)/m$
4.	$\xi = 1$	$\lim\limits_{m=\infty} \tilde{I}_\pm(m\xi, -m^2/2)/m$
5a.	$\kappa = -\xi + 2 \operatorname{arctanh} \xi$	$I_+(\xi, -1)$, $\lvert\xi\rvert < 1$
5b.	$\kappa = -\xi + 2 \arctan \xi$	$I_-(\xi, 1)$
6a.	$\kappa = -\xi + 2 \operatorname{arccoth} \xi$	$I_+(\xi, -1)$, $\lvert\xi\rvert > 1$
6b.	$\kappa = -\xi - 2 \operatorname{arccot} \xi$	$I_-(\xi, 1)$

Table 3. Special integrable cases as limits of $I_\pm(\xi, c)$

torsion; conversely the Weingarten surfaces are orthogonal to the osculation planes of the generating curves. Bianchi's research extends to the complementary relation $\rho - \sigma = \sinh(\rho + \sigma)$ (row 3a) as well [5, §246]. The remaining rows (from 4 to 6b) correspond to involutes of surfaces of constant Gaussian curvature studied by Beltrami [3, Ch. 9, §20]. Row 4 (surfaces of constant astigmatism) has been addressed in Part I; we have nothing to add except the Beltrami's work as the earliest reference we know of.

Table 3 demonstrates how the cases expressible in terms of elementary functions arise as limits of the generic integral (34) for c approaching ± 1 or $\pm\infty$ along a suitable curve in the (c, m) space. The tubular surfaces $\sigma = \text{const}$, which are omitted, correspond to $\kappa = I_+(\xi, 1) = \xi + \text{const}$.

6. Curvature diagrams

To exemplify the wealth of classes of integrable surfaces, we plot the representative solutions of the governing equation (10) in Figures 1 and 2. We call them curvature diagrams, even though the radii of curvature ρ, σ, rather than the curvatures $1/\rho, 1/\sigma$, are plotted, contrary to the customary practice [21, Ch. 5]. The benefit is that diagrams can be not only scaled arbitrarily, but also freely translated along the dashed line $\rho = \sigma$; the translation corresponds to offsetting. For clarity, we adjusted the offsetting so that the diagrams are symmetric about the origin, i.e., $\rho(\sigma) = -\rho(-\sigma)$.

The diagrams contain plots of functions $\mathcal{I}_\text{A}(\xi, k)$, $\mathcal{I}_\text{B}(\xi, k)$, $\mathcal{I}_{\text{C}\pm}(\xi, k)$, and $k\tilde{\mathcal{I}}_\text{A}(\xi/k, k)$. All special cases are explicitly included as limits, except the surfaces of constant positive curvature (row 2a). These could be obtained as the limit of $k\mathcal{I}_\text{B}(\xi/k, k)$ as k approaches zero.

The plots have been calculated using the Legendre normal form [15, 30] of the

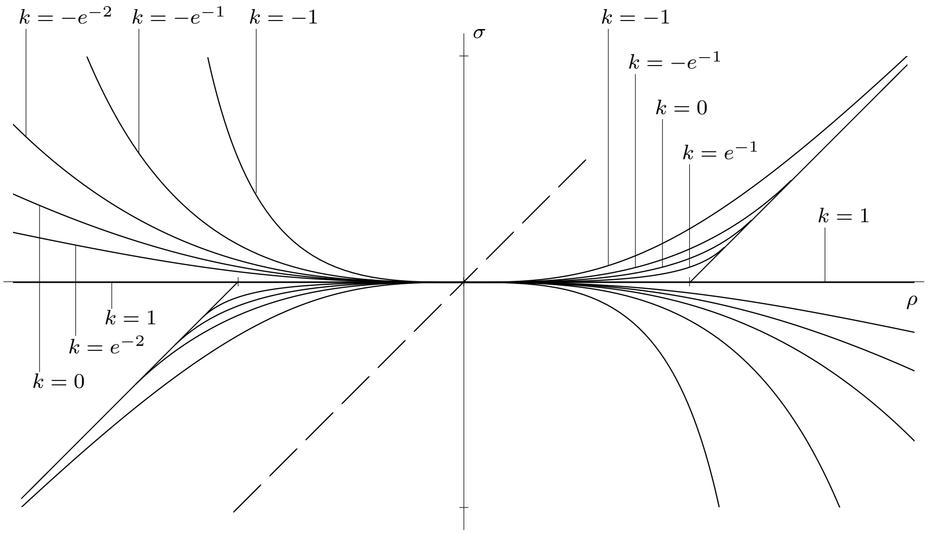

Figure 1. Curvature diagrams $\kappa = \mathcal{I}_{\mathrm{B}}(\xi, k)$ (the left-hand legend) and $\kappa = \mathcal{I}_{\mathrm{A}}(\xi, k)$, $|\xi| < 1$ (the right-hand legend), where $\kappa = \rho + \sigma$, $\xi = \rho - \sigma$. More can be obtained by rescaling and translating along the dashed line $\rho = \sigma$, the axis κ. Here $\mathcal{I}_{\mathrm{A}}(\xi, -1) = -\xi + 2\arctan\xi$ (row 5b), $\mathcal{I}_{\mathrm{A}}(\xi, 0) = \arcsin\xi$ (row 3b), $\mathcal{I}_{\mathrm{A}}(\xi, 1) = \xi$; $\mathcal{I}_{\mathrm{B}}(\xi, -1) = -\xi + 2\,\mathrm{arctanh}\,\xi$ (row 5a), $\mathcal{I}_{\mathrm{B}}(\xi, 0) = \mathrm{arcsinh}\,\xi$ (row 3a), $\mathcal{I}_{\mathrm{B}}(\xi, 1) = \xi$. Graphs of $\kappa = \mathcal{I}_{\mathrm{A}}(\xi, k)$ end on the solid lines $|\xi| = 1$.

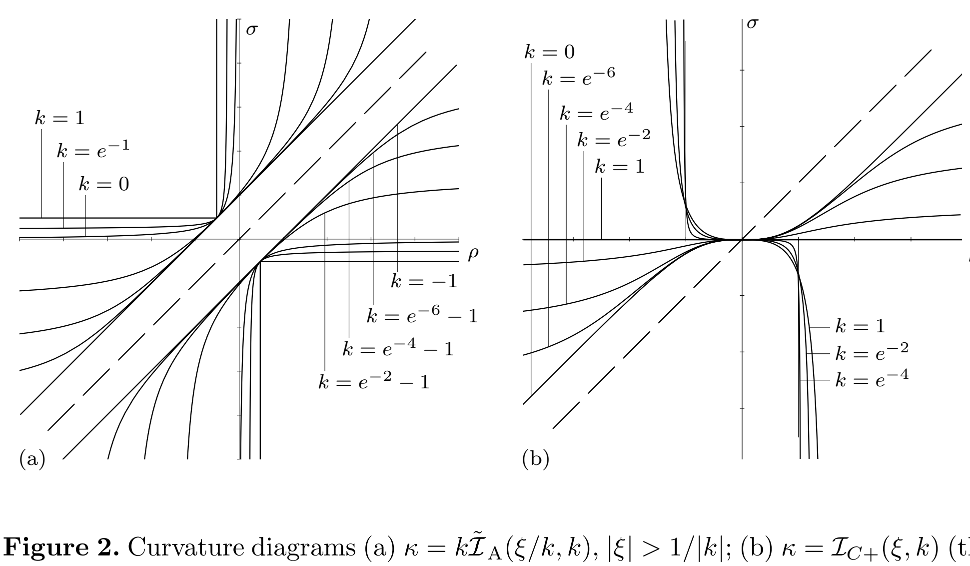

Figure 2. Curvature diagrams (a) $\kappa = k\tilde{\mathcal{I}}_{\mathrm{A}}(\xi/k, k)$, $|\xi| > 1/|k|$; (b) $\kappa = \mathcal{I}_{C+}(\xi, k)$ (the top left-hand legend) and $\kappa = -\mathcal{I}_{C-}(\xi, k)$ (the bottom right-hand legend), where $\kappa = \rho + \sigma$, $\xi = \rho - \sigma$. More can be obtained by rescaling and translating along the dashed line $\rho = \sigma$, the axis κ.
In (a), the line $k = 1$ corresponds to tubular surfaces, $k = 0$ to surfaces of negative constant curvature (row 2b), and $k = -1$ to the constant astigmatism surfaces (row 4). In (b), $\mathcal{I}_{C+}(\xi, 0) = -\xi + 2\arctan\xi$ (row 5b) $\mathcal{I}_{C-}(\xi, 1) = \xi - 2\arctan((\xi - 1)/(\xi + 1))$ (row 5a after reparameterization).

elliptic integrals (35) and (37), which could be of independent interest. As is well known, the Legendre normal form depends on the configuration of roots of the quartic polynomial $\Pi = s^4 + 2cs^2 + 1$.

A) If $c < -1$, then $\Pi = (s^2 - \gamma_+)(s^2 - \gamma_-)$ has four real roots $\sqrt{\gamma_\pm}$ and $-\sqrt{\gamma_\pm}$ given by formula (36). By using the substitution $s = \sqrt{k}\, r$, where $k = \gamma_-$, we easily obtain the Legendre normal form

$$\frac{1}{\sqrt{k}} I_\pm\Big(\xi\sqrt{k}, -\frac{k^2+1}{2k}\Big) = \int_0^\xi \frac{1 \pm kr^2}{\sqrt{(1-r^2)(1-k^2r^2)}}\,\mathrm{d}r, \quad 0 < k < 1.$$

On the right-hand side, we can remove the $\pm$ sign from the numerator by allowing k to range between -1 and 1. For $-1 \le \xi \le 1$, $-1 < k < 1$, we have a unified representative given by $\kappa = \mathcal{I}_\mathrm{A}(\xi, k)$, where

$$\mathcal{I}_\mathrm{A}(\xi, k) = \int_0^\xi \frac{1 - kr^2}{\sqrt{(1-r^2)(1-k^2r^2)}}\,\mathrm{d}r = \frac{1}{k} E(\xi; k) + \frac{k-1}{k} F(\xi; k)$$

in terms of the Legendre elliptic integrals E, F.

For real ξ such that $|\xi| > 1$, the function $\mathcal{I}_\mathrm{A}(\xi, k)$ is complex valued. Yet we obtain a real function for $1/|k| \le \xi$ by choosing the lower limit of the integral to be $1/k$, $-1 < k < 1$. Thus,

$$\tilde{\mathcal{I}}_\mathrm{A}(\xi, k) = \begin{cases} \displaystyle\int_{1/|k|}^\xi \frac{1 - kr^2}{\sqrt{(1-r^2)(1-k^2r^2)}}\,\mathrm{d}r = \mathcal{I}_\mathrm{A}(\xi, k) - \mathcal{I}_\mathrm{A}\Big(\frac{1}{|k|}, k\Big), & \xi > \dfrac{1}{|k|}, \\ -\tilde{\mathcal{I}}_\mathrm{A}(-\xi, k), & \xi < -\dfrac{1}{|k|}. \end{cases}$$

B) Similarly, when $c > 1$, then $\gamma_\pm < 0$, the roots $\sqrt{\gamma_\pm}$, $-\sqrt{\gamma_\pm}$ of Π are purely imaginary, and

$$\frac{1}{\sqrt{k}} I_\pm\Big(\xi\sqrt{k}, \frac{k^2+1}{2k}\Big) = \int_0^\xi \frac{1 \pm kr^2}{\sqrt{(1+r^2)(1+k^2r^2)}}\,\mathrm{d}r, \quad 0 < k < 1.$$

The two representatives can be unified into $\kappa = \mathcal{I}_\mathrm{B}(\xi, k)$, where

$$\mathcal{I}_\mathrm{B}(\xi, k) = \int_0^\xi \frac{1 - kr^2}{\sqrt{(1+r^2)(1+k^2r^2)}}\,\mathrm{d}r = \frac{1}{k\mathrm{i}} E(\xi\mathrm{i}; k) + \frac{k-1}{k\mathrm{i}} F(\xi\mathrm{i}; k)$$

for $-1 < k < 1$.

C) When $-1 < c < 1$ (four distinct complex roots), we substituted

$$s = \frac{1 + \sqrt{k}\, r}{1 - \sqrt{k}\, r}, \quad 0 < k < 1,$$

to obtain two more representatives $\kappa = \mathcal{I}_{\mathrm{C}+}$ and $\kappa = \mathcal{I}_{\mathrm{C}-}$, where

$$\mathcal{I}_{\mathrm{C}\pm} = \begin{cases} J_{\mathrm{C}\pm}(\xi, k) - J_{\mathrm{C}\pm}(0, k), & \xi \geq 0, \\ -\mathcal{I}_{\mathrm{C}\pm}(-\xi, k), & \xi < 0, \end{cases}$$

$$J_{\mathrm{C}\pm}(\xi, k) = \frac{\sqrt{1 + 2c\xi^2 + \xi^4}}{1 + \xi} + \frac{2}{(k+1)\mathrm{i}} E\left(\frac{\xi - 1}{\xi + 1} \frac{\mathrm{i}}{\sqrt{k}}, k\right) + \frac{\varepsilon_\pm}{\mathrm{i}} F\left(\frac{\xi - 1}{\xi + 1} \frac{\mathrm{i}}{\sqrt{k}}, k\right),$$

$$\varepsilon_\pm = \frac{(1 \pm 1)k - 3 \pm 1}{2} = \begin{cases} k - 1, \\ -2, \end{cases}$$

$$c = -\frac{k^2 - 6k + 1}{(k+1)^2}.$$

7. Normal congruences and their focal surfaces

The fact that the governing equation (10) has the offsetting symmetry (21) is not a pure coincidence. Being invertible, the offsetting transformation $\mathbf{r} \mapsto \mathbf{r} + T\mathbf{n}$ preserves integrability in every reasonable sense of the word. Surfaces related by the offsetting transformation are said to be parallel and either all are integrable or none is. However, parallel surfaces can be alternatively described as normal surfaces to the same line congruence. Consequently, integrability is a property of this congruence and, therefore, must have an expression in terms of congruence invariants.

Normal congruences of Weingarten surfaces, also known as W-congruences, are rather special with regard to properties of their focal surfaces. It is therefore natural to look for characterization of the former in terms of the latter. Naturally, we expect the focal surfaces of integrable W-congruences to be integrable as well.

Recall that a generic surface has two focal surfaces (often considered as two sheets of a single surface),

$$\mathbf{r}^{(1)} = \mathbf{r} + \sigma\mathbf{n}, \qquad \mathbf{r}^{(2)} = \mathbf{r} + \rho\mathbf{n}.$$

each of which is formed by the evolutes of one family of the curvature lines. Focal surfaces can degenerate into a line or even a point. In the case of a Weingarten surface $\mathbf{r}$ with fundamental forms (1), one of the focal surfaces degenerates into a line if $\sigma_y = \sigma' w_y = 0$ or $\rho_x = \rho' w_x = 0$; both degenerate into a point if the surface is a sphere (already excluded from consideration); otherwise they are regular surfaces. Therefore, we assume $\rho'\sigma' \neq 0$ in what follows.

To compute the respective first and second fundamental forms $\mathrm{I}^{(i)}$ and $\mathrm{II}^{(i)}$, $i = 1, 2$, we proceed as follows. In view of the Gauss–Mainardi–Codazzi equations (4) and (5), the Gauss–Weingarten (3) equations can be written as

$$\begin{aligned}
\mathbf{r}_{xx} &= \frac{u_x}{u}\mathbf{r}_x + \frac{\sigma\rho' u^2 w_y}{\rho(\rho-\sigma)v^2}\mathbf{r}_y + \frac{u^2}{\rho}\mathbf{n}, \qquad \mathbf{n}_x = -\frac{1}{\rho}\mathbf{r}_x,\\
\mathbf{r}_{xy} &= \frac{\sigma\rho' w_y}{\rho(\sigma-\rho)}\mathbf{r}_x + \frac{\rho\sigma' w_x}{\sigma(\rho-\sigma)}\mathbf{r}_y,\\
\mathbf{r}_{yy} &= \frac{\rho\sigma' v^2 w_x}{\sigma(\sigma-\rho)u^2}\mathbf{r}_x + \frac{v_y}{v}\mathbf{r}_y + \frac{v^2}{\sigma}\mathbf{n}, \qquad \mathbf{n}_y = -\frac{1}{\sigma}\mathbf{r}_y.
\end{aligned} \tag{40}$$

One easily finds

$$\begin{aligned}
&\mathbf{r}^{(1)}_x = \frac{\rho-\sigma}{\rho}\mathbf{r}_x + \sigma' w_x\mathbf{n}, \quad \mathbf{r}^{(1)}_y = \sigma' w_y\mathbf{n}, \quad \mathbf{n}^{(1)} = \frac{\mathbf{r}_y}{v},\\
&\mathbf{r}^{(2)}_x = \rho' w_x\mathbf{n}, \quad \mathbf{r}^{(2)}_y = \frac{\sigma-\rho}{\sigma}\mathbf{r}_y + \rho' w_y\mathbf{n}, \quad \mathbf{n}^{(2)} = \frac{\mathbf{r}_x}{u}.
\end{aligned}$$

Using the equations (40) and (1), we get

$$\mathrm{I}^{(1)} = \frac{(\rho-\sigma)^2u^2}{\rho^2}\mathrm{d}x^2 + \mathrm{d}\sigma^2, \qquad \mathrm{I}^{(2)} = d\rho^2 + \frac{(\rho-\sigma)^2v^2}{\sigma^2}\mathrm{d}y^2, \tag{41}$$

where $\mathrm{d}\rho = \rho'\,\mathrm{d}w = \rho'(w_x\,\mathrm{d}x + w_y\,\mathrm{d}y)$, $\mathrm{d}\sigma = \sigma'\,\mathrm{d}w = \sigma'(w_x\,\mathrm{d}x + w_y\,\mathrm{d}y)$.

With u, v determined from Proposition 1, we can write

$$\mathrm{I}^{(1)} = (f^{(1)}(\sigma)\,\mathrm{d}x)^2 + \mathrm{d}\sigma^2, \qquad \mathrm{I}^{(2)} = (f^{(2)}(\rho)\,\mathrm{d}y)^2 + \mathrm{d}\rho^2.$$

Hence, all focal surfaces $\mathbf{r}^{(i)}$ corresponding to a given dependence $\rho(\sigma)$ are isometric. Moreover, the first fundamental forms (41) are typical of surfaces of revolution. These are among the classical results by Weingarten [36].

Omitting details, we further compute the second fundamental forms

$$\mathrm{II}^{(1)} = \frac{\sigma w_y}{v}\Big(\frac{\rho' u^2}{\rho^2}\mathrm{d}x^2 - \frac{\sigma' v^2}{\sigma^2}\mathrm{d}y^2\Big), \quad \mathrm{II}^{(2)} = -\frac{\rho w_x}{u}\Big(\frac{\rho' u^2}{\rho^2}\,\mathrm{d}x^2 - \frac{\sigma' v^2}{\sigma^2}\mathrm{d}y^2\Big) \tag{42}$$

and note that they are conformally related, which is another way to express Ribaucour's classical result [32] that asymptotic coordinates on $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$ correspond. The Gaussian curvatures are

$$K^{(1)} = \frac{\det \mathrm{II}^{(1)}}{\det \mathrm{I}^{(1)}} = -\frac{\rho'}{(\rho-\sigma)^2\sigma'}, \qquad K^{(2)} = \frac{\det \mathrm{II}^{(2)}}{\det \mathrm{I}^{(2)}} = -\frac{\sigma'}{(\rho-\sigma)^2\rho'}. \tag{43}$$

Consequently, the focal surfaces have one and the same sign of the Gaussian curvature, which we denote as ε. We have $\varepsilon = -1$ (both focal surfaces are hyperbolic) if and only if $\mathrm{d}\rho/\mathrm{d}\sigma = \rho'/\sigma' > 0$ (if ρ increases as σ increases), and $+1$ if $\mathrm{d}\rho/\mathrm{d}\sigma < 0$. The relation

$$K^{(1)}K^{(2)} = \frac{1}{(\rho-\sigma)^4} \tag{44}$$

away of umbilic points is known as the Halphen theorem (see [4, §129]).

As we have already explained, to every particular relation $\rho(\sigma)$ of curvatures there corresponds an isometry class of focal surfaces, which contains a unique rotational representative (which is the way the classes have been characterized in the classical literature). However, we believe that a description in terms of metric invariants is more appropriate. It is convenient to choose

$$\kappa^{(i)} = \frac{1}{\sqrt{\varepsilon K^{(i)}}},$$

where $\varepsilon K^{(i)} = |K^{(i)}|$ is the absolute value of the Gaussian curvature of the ith focal surface.

Further, let $\gamma^{(i)}$ be defined by

$$\begin{aligned} \gamma^{(1)} &= \frac{(\rho - \sigma)(\rho''\sigma' - \sigma''\rho') - 2\rho'\sigma'(\rho' - \sigma')}{2\,(-\varepsilon\rho'\sigma')^{3/2}}, \\ \gamma^{(2)} &= \frac{(\rho - \sigma)(\rho''\sigma' - \sigma''\rho') + 2\rho'\sigma'(\rho' - \sigma')}{2\,(-\varepsilon\rho'\sigma')^{3/2}}. \end{aligned} \tag{45}$$

One can directly check that $|\gamma^{(i)}|$ equals the norm of the gradient of $\kappa^{(i)}$ with respect to $\mathrm{I}^{(i)}$,

$$|\gamma^{(i)}| = \|\mathrm{grad}^{(i)}\,\kappa^{(i)}\|^{(i)} = \sqrt{I^{(i)}(\mathrm{grad}^{(i)}\,\kappa^{(i)}, \mathrm{grad}^{(i)}\,\kappa^{(i)})}\,.$$

Hence, $\gamma^{(i)}$ is a metric invariant of the respective focal surface. It is sometimes more convenient to use invariants

$$\begin{aligned} G^{(1)} &= \frac{[(\rho - \sigma)(\rho''\sigma' - \sigma''\rho') - 2\rho'\sigma'(\rho' - \sigma')]^2}{16\,(\rho'\sigma')^3}, \\ G^{(2)} &= \frac{[(\rho - \sigma)(\rho''\sigma' - \sigma''\rho') + 2\rho'\sigma'(\rho' - \sigma')]^2}{16\,(\rho'\sigma')^3}, \end{aligned} \tag{46}$$

satisfying

$$\gamma^{(i)2} = -4\varepsilon G^{(i)}, \qquad -\,16\,G^{(i)}K^{(i)3} = I^{(i)}(\mathrm{grad}^{(i)}\,K^{(i)}, \mathrm{grad}^{(i)}\,K^{(i)}).$$

Clearly, both $\kappa^{(i)}$ and $G^{(i)}$ are functions of w. Consequently, $G^{(i)}$ can be considered as a function of $\kappa^{(i)}$ unless $\kappa^{(i)}$ is a constant. Our nearest aim is to establish the dependence between $\kappa^{(i)}$ and $G^{(i)}$ in terms of the dependence between ρ and σ.

Proposition 5. *Let the principal radii of curvature ρ, σ of an integrable surface satisfy the generic relation* (34)*. Then the metric invariants $G^{(i)}$ and $\kappa^{(i)}$ satisfy the relations*

$$\begin{aligned} G^{(1)} &= \left(-1 \pm \sqrt{\frac{2}{c \mp 1}\,\frac{\kappa^{(1)}}{m}}\right)\left(1 + \sqrt{\frac{2}{c \mp 1}\,\frac{m}{\kappa^{(1)}}}\right), \\ G^{(2)} &= \left(1 \pm \sqrt{\frac{2}{c \mp 1}\,\frac{\kappa^{(2)}}{m}}\right)\left(-1 + \sqrt{\frac{2}{c \mp 1}\,\frac{1}{m}\kappa^{(2)}}\right). \end{aligned} \tag{47}$$

Furthermore,

$$G^{(1)}G^{(2)} = \Big(\frac{c \pm 1}{c \mp 1}\Big)^2$$

is constant (hence, so is the product $\gamma^{(1)}\gamma^{(2)}$).

Table 5 lists the product $G^{(1)}G^{(2)}$ and the algebraic relations between $G^{(i)}$ and $\kappa^{(i)}$ in the special cases.

Proof. For simplicity, we start assuming a fixed scaling, i.e., we depart from formula (38). We routinely compute

$$K^{(1)} = \frac{(1 \pm w^2 + \sqrt{1 + 2cw^2 + w^4}\,)^2}{2(c \mp 1)w^4}, \quad K^{(2)} = \frac{(1 \pm w^2 - \sqrt{1 + 2cw^2 + w^4}\,)^2}{2(c \mp 1)w^4}.$$

Consequently, $\varepsilon = \operatorname{sgn}(c \mp 1))$, and

$$\kappa^{(1)} = \frac{1 \pm w^2 - \sqrt{1 + 2cw^2 + w^4}}{\sqrt{2|c \mp 1|}}, \quad \kappa^{(2)} = \frac{1 \pm w^2 + \sqrt{1 + 2cw^2 + w^4}}{\sqrt{2|c \mp 1|}}.$$

Furthermore,

$$G^{(1)} = -\frac{(1 \mp w^2 + \sqrt{1 + 2cw^2 + w^4}\,)^2}{2(c \mp 1)w^2}, \quad G^{(2)} = -\frac{(1 \mp w^2 - \sqrt{1 + 2cw^2 + w^4}\,)^2}{2(c \mp 1)w^2}.$$

Under the scaling by factor of m, the metric invariants $K^{(i)}$ and $\kappa^{(i)}$ become $K^{(i)}/m^2$ and $m\kappa^{(i)}$, respectively, while $G^{(i)}$ remains invariant. Formulas (47) are then easily checked. Moreover, all three metric invariants are invariant under the offsetting (21).

Formulas for $G^{(i)}$ and $\kappa^{(i)}$ in the special cases are given in Table 4 along with the sign ε of the Gaussian curvatures. □

Summarizing, focal surfaces of integrable Weingarten surfaces belong to the isometry classes specified in Proposition 5.

A natural question is whether the condition $G^{(1)}G^{(2)} = \text{const}$ or, equivalently, $\gamma^{(1)}\gamma^{(2)} = \text{const}$, is not only necessary, but also sufficient for the condition (10) to hold.

Proposition 6. *Under the condition $\gamma^{(1)} + \gamma^{(2)} \neq 0$, a surface satisfies the governing equation* (10) *if and only if the product*

$$\gamma^{(1)}\gamma^{(2)} = \pm\|\operatorname{grad}^{(1)} \kappa^{(1)}\|^{(1)}\, \|\operatorname{grad}^{(2)} \kappa^{(2)}\|^{(2)} \tag{48}$$

is constant.

Proof. Assuming the $\rho(\sigma)$ dependence, $\gamma^{(1)} + \gamma^{(2)}$ simplifies to $(\rho - \sigma)\rho''/\sqrt{|\rho'|^3}$ and the product in question to

$$\gamma^{(1)}\gamma^{(2)} = \frac{(\rho' - 1)^2}{\varepsilon\rho'} - \frac{(\rho - \sigma)^2\rho''^2}{4\,\varepsilon\rho'^3}.$$

	ε	$\kappa^{(1)}$	$\kappa^{(2)}$	$G^{(1)}$	$G^{(2)}$
1.	1	$2\,\lvert w\rvert$	$2\,\lvert w\rvert$	-1	-1
2a.	1	$\left\lvert \frac{1}{w^2}-1\right\rvert$	$\lvert w^2-1\rvert$	$-\frac{1}{w^2}$	$-w^2$
2b.	-1	$\frac{1}{w^2}+1$	w^2+1	$\frac{1}{w^2}$	w^2
3a.	1	$-1+\cosh w$	$1+\cosh w$	$\frac{1+\cosh w}{1-\cosh w}$	$\frac{1-\cosh w}{1+\cosh w}$
3b.	-1	$1-\cos w$	$1+\cos w$	$\frac{1+\cos w}{1-\cos w}$	$\frac{1-\cos w}{1+\cos w}$
4.	-1	1	1	0	0
5a.	-1	$\tanh^2 w$	1	$\frac{1}{\sinh^2 w\cosh^2 w}$	0
5b.	1	$\tan^2 w$	1	$-\frac{1}{\sin^2 w\cos^2 w}$	0
6a.	-1	$\coth^2 w$	1	$\frac{1}{\sinh^2 w\cosh^2 w}$	0
6b.	1	$\cot^2 w$	1	$-\frac{1}{\sin^2 w\cos^2 w}$	0

Table 4. Special integrable cases. Metric invariants of focal surfaces in terms of w.

	ε	$G^{(1)}G^{(2)}$	$G^{(1)}(\kappa^{(1)})$	$G^{(2)}(\kappa^{(2)})$
1.	1	-1	-1	-1
2a.	1	1	$-1\pm\kappa^{(1)}$	$-1\pm\kappa^{(2)}$
2b.	-1	1	$-1+\kappa^{(1)}$	$-1+\kappa^{(2)}$
3a.	1	-1	$-1-\frac{2}{\kappa^{(1)}}$	$-1-\frac{2}{\kappa^{(2)}}$
3b.	-1	1	$-1+\frac{2}{\kappa^{(1)}}$	$-1+\frac{2}{\kappa^{(2)}}$
4.	-1	0	0	0
5a.	-1	0	$\left(\sqrt{\kappa^{(1)}}-\frac{1}{\sqrt{\kappa^{(1)}}}\right)^2$	0
5b.	1	0	$-\left(\sqrt{\kappa^{(1)}}+\frac{1}{\sqrt{\kappa^{(1)}}}\right)^2$	0
6a.	-1	0	$\left(\sqrt{\kappa^{(1)}}-\frac{1}{\sqrt{\kappa^{(1)}}}\right)^2$	0
6b.	1	0	$-\left(\sqrt{\kappa^{(1)}}+\frac{1}{\sqrt{\kappa^{(1)}}}\right)^2$	0

Table 5. Special integrable cases. Relations between metric invariants of focal surfaces.

Factorizing the σ-derivative of this expression as

$$\pm\frac{(\rho-\sigma)^2}{2\,\varepsilon\rho'^3}\Big(\rho'''-\frac{3}{2\rho'}\rho''^2+\frac{\rho'-1}{\rho-\sigma}\rho''-2\frac{(\rho'-1)\rho'(\rho'+1)}{(\rho-\sigma)^2}\Big)\rho''$$

and comparing to the governing equation (10) proves the proposition. □

It follows from the proof that condition (48) also holds when $\rho''=0$, i.e., if there is a linear relation between the radii of curvature. As of now, there seems to be no indication towards integrability of the latter class (except when $\rho\pm\sigma=\text{const}$, which satisfies (10) as well).

8. Conclusions and future work

In this work we singled out a class of Weingarten surfaces on the basis of its solitonic integrability. Although special cases were not unknown to nineteenth century geometers, the overall result appears to be new. We also characterized integrability in terms of metric invariants of the focal surfaces.

For time reasons, many questions had to be left for further research. We do not know the Bäcklund transformation, recursion operator, bi-Hamiltonian structure and other attributes of integrability. We did not provide any solutions to the Gauss equation (39). We do not know what is the true geometric meaning of the spectral parameter. Even the task of computing third order symmetries of the Gauss equation proved to be too complex.

We have seen in Part I that integrability of surfaces of constant astigmatism is attributable to the fact that their focal surfaces are pseudospherical. In the general case, the existence of an integrability-preserving relation to previously known integrable surfaces is an open problem.

Our nearest goals include exploring the induced Bianchi type transformation between surfaces satisfying relations (47) as well as investigating the extended symmetries of the class in the sense of Cieśliński [10, 11].

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References

[1] H. Baran and M. Marvan, On integrability of Weingarten surfaces: a forgotten class *J. Phys. A: Math. Theor.* **42** (2009) 404007.

[2] H. Baran and M. Marvan, Jets – a Maple package for differential calculus on jet spaces and diffieties, `jets.math.slu.cz` and `www.diffiety.org`.

[3] E. Beltrami, *Opere Mathematiche,* Vol. 1, Ulrico Hoepli, Milano, 1902.

[4] L. Bianchi, *Lezioni di Geometria Differenziale*, Vol. I (E. Spoerri, Pisa, 1902).

[5] L. Bianchi, *Lezioni di Geometria Differenziale*, Vol. II (E. Spoerri, Pisa, 1903).

[6] A.I. Bobenko, All constant mean curvature tori in R^3, S^3, H^3 in terms of theta-functions, *Math. Ann.* **290** (1991) 209–245.

[7] A.I. Bobenko, Surfaces in terms of 2 by 2 matrices. Old and new integrable cases, in: A.P. Fordy and J.C. Wood, eds., *Harmonic Maps and Integrable Systems*, Aspects Math. E23 (Vieweg, Braunschweig, 1994), 83–127.

[8] E.D. Belokolos, A.I. Bobenko, V.Z. Enolski, A.R. Its and V.B. Matveev, *Algebro-Geometric Approach in the Theory of Integrable Equations,* Springer Series in Nonlinear Dynamics, Springer, Berlin, 1994.

[9] A.V. Bocharov, V.N. Chetverikov, S.V. Duzhin, N.G. Khor'kova, I.S. Krasil'shchik, A.V. Samokhin, Yu.N. Torkhov, A.M. Verbovetsky and A.M. Vinogradov, *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics,* Transl. Math. Monographs 182 (Amer. Math. Soc., Providence, 1999).

[10] J. Cieśliński, Non-local symmetries and a working algorithm to isolate integrable symmetries, *J. Phys. A: Math. Gen.* **26** (1993) L267–L271.

[11] J. Cieśliński, P. Goldstein and A. Sym, On integrability of the inhomogeneous Heisenberg ferromagnet model: examination of a new test, *J. Phys. A: Math. Gen.* **27** (1994) 1645–1664.

[12] G. Darboux, *Leçons sur la théorie générale des surface et les applications géométriques du calcul infinitésimal*, Vol. III (Chelsea, Bronx, NY, 1972).

[13] B.A. Dubrovin and S.M. Natanzon, Real two-zone solutions of the sine-Gordon equation, *Funct. Anal. Appl.* **16** (1982) 21–33.

[14] B.A. Dubrovin, I.M. Krichever and S.P. Novikov, *Integrable systems I*, Encyclopaedia Math. Sci. 4, Springer, Berlin, 2001.

[15] A. Erdelyi, ed., *Higher Transcendental Functions,* Vol. 3, Mc Graw-Hill, New York, 1955.

[16] F. Finkel, On the integrability of Weingarten surfaces, in: A. Coley et al., ed., *Bäcklund and Darboux Transformations. The Geometry of Solitons*, AARMS-CRM Workshop, June 4-9, 1999, Halifax, N.S., Canada, (Amer. Math. Soc., Providence, 2001) 199–205.

[17] J.A. Gálvez, A. Martínez and F. Milán, Linear Weingarten surfaces in $\mathbb{R}^3$, *Monatsh. Math.* **138** (2003) 133–144.

[18] P. Hartman and A. Wintner, Umbilical points and W-surfaces, *Amer. J. Math.* **76** (1954) 502–508.

[19] F. Hélein, *Constant Mean Curvature Surfaces, Harmonic Maps and Integrable Systems,* Birkhäuser, Basel, 2001.

[20] H. Hopf, Über Flächen mit einer Relation zwischen den Hauptkrümmungen, *Math. Nachr.* **4** (1951) 232–249.

[21] H. Hopf and S.S. Chern, *Differential Geometry in the Large: Seminar Lectures, New York University*, Lect. Notes Math. 1000, Springer, Berlin, 2008.

[22] W. Kühnel and M. Steller, On closed Weingarten surfaces, *Monatsh. Math.* **146** (2005) 113–126.

[23] I.S. Krasil'shchik and A.M. Vinogradov, Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations, *Acta Appl. Math.* **15** (1989) 161–209.

[24] M. Marvan, A direct method to compute zero curvature representations. The case sl_2, in: *Proc. Conf. Secondary Calculus and Cohomological Physics*, Moscow, Russia, Aug. 24-31, 1997; http://www.emis.de/proceedings/SCCP97.

[25] M. Marvan, On the spectral parameter problem, *Acta Appl. Math.* **109** (2010) 239-255.

[26] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin, 1991.

[27] M. Melko and I. Sterling, Integrable systerms, harmonic maps and the classical theory of surfaces, in: A.P. Fordy and J.C. Wood, eds., *Harmonic Maps and Integrable Systems*, Vieweg,

Braunschweig, 1994, 129–144.

[28] A.V. Mikhailov and V.V. Sokolov, Symmetries of differential equations and the problem of integrability, in: A.V. Mikhailov, ed., *Integrability*, Lect. Notes Phys. 767 (Springer, 2009).

[29] U. Pinkall and I. Sterling, On the classification of constant mean curvature tori, *Ann. of Math.* **130** (1989) 407–451.

[30] V.V. Prasolov and Y.P. Solovyev, *Elliptic Functions and Elliptic Integrals*, Amer. Math. Soc., Providence, 1997.

[31] R. Prus and A. Sym, Rectilinear congruences and Bäcklund transformations: roots of the soliton theory, in: D. Wójcik and J. Cieśliński, *Nonlinearity & Geometry, Luigi Bianchi Days*, Proc. 1st Non-Orthodox School, Warsaw, September 21–28, 1995 (Polish Scientific, Warsaw, 1998) 25–36.

[32] A. Ribaucour, Note sur les développées des surfaces, *C. R. Acad. Sci. Paris* **74** (1872) 1399–1403.

[33] C. Rogers and W.K. Schief, *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory* (Cambridge Univ. Press, Cambridge, 2002).

[34] A. Sym, Soliton surfaces and their applications. Soliton geometry from spectral problems, in: R. Martini, ed., *Geometric Aspects of the Einstein Equations and Integrable Systems,* Lecture Notes in Physics 239 (Springer, Berlin, 1985) 154–231.

[35] K. Voss, Über geschlossene Weingartensche Flächen, *Math. Ann.* **138** (1959) 42–54.

[36] J. Weingarten, Über die Oberflächen, für welche einer der beiden Hauptkrümmungshalbmesser eine function des anderen ist, *J. Reine Angew. Math.* **62** (1863) 160–173.

[37] J.A. Wolf, Exotic metrics on immersed surfaces, *Proc. Amer. Math. Soc.* **17** (1966) 871–877.

[38] Hongyou Wu, Weingarten surfaces and nonlinear partial differential equations, *Ann. Global Anal. Geom.* **11** (1993) 49–64.

[39] V.E. Zakharov, S.V. Manakov, S.P. Novikov and L.P. Pitaevskii, *Theory of Solitons. The Inverse Problem Method,* Plenum, New York, 1984.