Effective phase dynamics of noise-induced oscillations in excitable systems

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We develop an effective description of noise-induced oscillations based on deterministic phase dynamics. The phase equation is constructed to exhibit correct frequency and distribution density of noise-induced oscillations. In the simplest one-dimensional case the effective phase equation is obtained analytically, whereas for more complex situations a simple method of data processing is suggested. As an application an effective coupling function is constructed that quantitatively describes periodically forced noise-induced oscillations.

PACS numbers: 05.40.Ca, 05.45.Xt

Complex dynamics of self-sustained oscillating systems lies in the focus of nonlinear science. Prominent physical examples include lasers, electronic circuits, chemical autokatalitic reactions, but also many biological processes, like firing neurons, oscillating genetic networks, rhythmic heartbeats, and circadian rhythms, can be attributed to this class although one can hardly derive the corresponding mathematical models from the first principles. Many phenomena characteristic for oscillatory systems, such as synchronization [1, 2] are common for all these examples. The theoretical and experimental description of oscillatory dynamics relies heavily on the notion of *phase*, which is a starting point for the treatment of deterministic and noisy dynamics [1–3].

Many oscillating systems (the best example are neurons) are not autonomous, but excitable: they posses a stable steady state, but being adequately perturbed they perform a stereotypical large-amplitude oscillation before they relax back to the stable state. In the presence of an appropriate periodic or noisy perturbation such a system may demonstrate persistent oscillations, as it never stays long enough close to the stable steady state. If the perturbation is noisy, the observed dynamics is termed *noise-induced oscillations* (see review [4] and [5]). In some situations noise-induced oscillations can be rather coherent, this is often called *coherence resonance* [6]. In many aspects noise-induced oscillations behave similar to the self-sustained ones: they can demonstrate synchrony when coupled in ensembles [7] and can be controlled by a time-delayed feedback [8]. While a qualitative similarity between noise-induced and self-sustained oscillations is quite obvious, an extension of theoretical and analytical tools suitable for self-sustained dynamics on the excitable case is problematic. Indeed, the basic tool in the study of self-sustained noisy oscillators, the introduction of the phase, can not even perturbatively be applied to excitable oscillators, because phase cannot be defined for a system residing on a stable steady state.

In this Letter we propose to describe noise-induced oscillations via an *effective phase* dynamics, where we define an invariant phase in a non-perturbative way (as opposed to typical perturbative approaches to the noisy dynamics of self-sustained oscillators [3]). Therefore, our definition of the phase inherently depends on the noise intensity, and correspondingly all derived characteristics like coupling functions as well. We present the theoretical framework by the example of noise-induced oscillations in one dimension, for which we also construct an effective coupling function describing a periodic forcing. Finally, we consider periodically driven noise-induced oscillations in a prototypic example of excitable dynamics, the FitzHugh-Nagumo system, and construct its effective phase description.

Our basic model is a noise-driven oscillator described by a 2π-periodic variable θ called hereafter *proto-phase* [9] governed by the Langevin equation

$$\dot{\theta} = h(\theta) + g(\theta)\xi(t) \;, \qquad \langle \xi(t)\xi(t')\rangle = 2\delta(t-t') \;. \quad (1)$$

In the excitable case the deterministic system $\dot{\theta} = h(\theta)$ has two steady states, one stable and one unstable, but in the presence of noise one observes nearly monotonic growth of θ with a mean frequency ω, and a smooth probability density $P(\theta)$. Therefore, we model the dynamics as that of an "effective" autonomous oscillator, by approximating the equation for the proto-phase as

$$\dot{\theta} = H(\theta) \;. \quad (2)$$

We impose following conditions on the *effective velocity* H: (i) the oscillation frequency should coincide with ω, and (ii) the distribution density of the proto-phase should be equal to $P(\theta)$. To meet these requirements we draw a correspondence between the Fokker-Planck equation to the oscillator (1) given by

$$\partial_t P = -\partial_\theta h P + \partial_\theta g \partial_\theta g P \;, \quad (3)$$

and the Liouville equation to the model (2) given by $\partial_t P = -\partial_\theta H P = -\partial_\theta J$. In the stationary case the flux J is related to the frequency by $\omega = 2\pi J$. Thus, the effective velocity can be expressed in terms of its frequency and distribution density by

$$\dot{\theta} = H(\theta) = \frac{\omega}{2\pi P(\theta)} \;. \quad (4)$$

Conditions (i) and (ii) are fulfilled exactly if ω and $P(\theta)$ are given by the corresponding stationary solutions of (3), which are well-known [2, 10, 11]. Then, using Eq. (3) the effective velocity can be written as $H = h(\theta) - \tilde{h}(\theta) = h(\theta) - gg' - g^2\partial_\theta \ln P$. Additionally to the deterministic velocity h there appears a noise-induced velocity $\tilde{h}$ which can be called *osmotic* [12].

From the proto-phase of the effective model we define the *phase* φ by the transformation

$$\varphi = S(\theta) = 2\pi \int_0^\theta P(\eta)d\eta. \tag{5}$$

The phase satisfies the properties $P(\varphi) = 1/2\pi$ and $\dot{\varphi} = \omega$. Thus, we have constructed an invariant effective phase dynamics of noise-induced oscillations.

By a simple modification we may extend the effective model to account for the random component of noise-induced oscillations. We introduce an *effective fluctuating force* to the dynamics of φ:

$$\dot{\varphi} = \omega + \sqrt{D}\eta(t)\ , \qquad \langle \eta(t)\eta(t')\rangle = 2\delta(t-t')\ . \tag{6}$$

For any coefficient D of the noise term, the distribution of φ is uniform and the mean frequency is ω, thus the conditions (i) and (ii) above remain fulfilled. Therefore, we are free to choose D, and we choose it from the condition: (iii) The diffusion constant of the effective phase (mapped on the real line), which is D, should be the same as in the original oscillator (1). It is given by $D = \lim_{t\to\infty}\langle[\theta(t)-\omega t]^2\rangle/2t$. Fortunately, one can get an exact expression for D following [13]:

$$D = \frac{\frac{1}{2\pi}\int_0^{2\pi}\frac{d\psi}{g(\psi)}\left[\int_{\psi-2\pi}^{\psi}\frac{d\phi}{g(\phi)}r(\phi,\psi)\right]^2\int_{\psi}^{\psi+2\pi}\frac{d\phi}{g(\phi)}r(\psi,\phi)}{\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{d\psi}{g(\psi)}\int_{\psi-2\pi}^{\psi}\frac{2d\phi}{g(\phi)}r(\phi,\psi)\right]^3},$$

where $r(\theta,\phi) = \exp\left[-\int_\theta^\phi \frac{h(\eta)}{g^2(\eta)}d\eta\right]$. Inverting the transformation to the phase φ, we obtain the effective model with noise (6) in terms of the proto-phase θ:

$$\dot{\theta} = H(\theta) + \frac{\sqrt{D}}{\omega}H(\theta)\eta(t)\ . \tag{7}$$

We see that the effective model is fully determined by the distribution density $P(\theta)$ and the mean frequency ω, and knowing the diffusion constant D also random effects can be taken into account, effectively. These quantities can be estimated from synthetic (numerical) or experimental observations $\theta(n\Delta t) = \theta_n$ by a straightforward analysis. Alternatively, $H(\theta)$ can be estimated via averaging of central differences as

$$H(\theta) \approx \left.\frac{\langle\theta_{n+1}-\theta_{n-1}\rangle}{2\Delta t}\right|_{\theta_n=\theta}\ , \tag{8}$$

(while forward differences $\theta_{n+1} - \theta_n$ provide the deterministic part $h(\theta)$ only, see [14] for details).

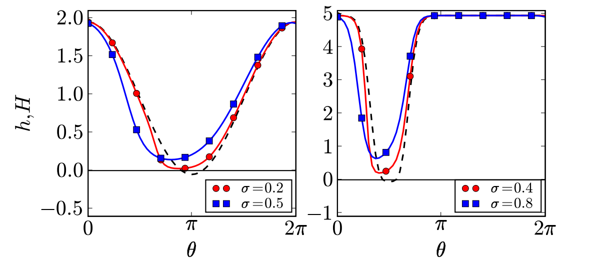

FIG. 1. (color online). Functions h (dashed) together with effective velocities H (solid) for Model A (left) at $a = 0.95$, and Model B (right) at $c = -0.05$. Noise intensities are as indicated.

Although model (7) captures many essential properties of noise-induced oscillations, it fails to describe the Lyapunov exponent properly. The exponent vanishes in the effective model with noise (7), while in the original system (1) it is generally negative, corresponding to synchronization of oscillators by a common external noise (see [15] and references therein).

We illustrate the above theory in Fig. 1 with two examples, both with an additive noise $g(\theta) = \sigma$. Model A is a simplified theta-model (cf. [16]) used in the description of excitable neurons: $h(\theta) = a + \cos\theta$. Model B is constructed to mimic an excitable oscillator demonstrating a pronounced coherence resonance: $h(\theta) = 5\tanh^2(5(1-\sin\theta)) + c$. The effective velocities H heavily depend on the noise intensity σ, especially at the region around the stable equilibrium: For large σ the effective velocity converges to the the constant function $H(\theta) = \omega$.

Next, we extend the effective model (4) to describe *periodically driven* noise-induced oscillations described by

$$\dot{\theta} = h(\theta) + g(\theta)\xi(t) + f(\psi(t),\theta), \tag{9}$$

with a 2π-periodic driving phase $\psi = \Omega t$. We want to obtain an effective phase description including an effective coupling. As above, the principle of correspondence between the flux of the Liouville equation and the θ-component of the probability flux

$$J = [h(\theta) + f(\psi,\theta) - g\partial_\theta g]\,P(\theta,\psi) \tag{10}$$

is applied, that yields the driven effective dynamics

$$\dot{\theta} = H(\theta,\psi) = \frac{J}{P} = h - gg' - g^2\partial_\theta \ln P + f\ . \tag{11}$$

It is essential to rewrite H as a sum of a ψ-independent *marginal effective velocity* $H_m(\theta)$ and an *effective coupling* $F(\theta,\psi)$. The former is obtained in terms of the marginal probability density $P_m(\theta) = \int_0^{2\pi}P(\theta,\psi)d\psi$ by integrating Eq. (10) over ψ:

$$H_m(\theta) = \frac{\omega}{2\pi P_m(\theta)} = h - gg' - g^2\partial_\theta \ln P_m + \int_0^{2\pi} f\frac{P}{P_m}d\psi\ .$$

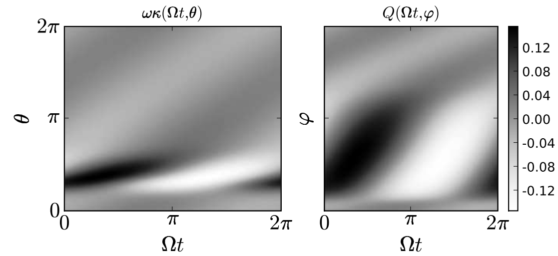

FIG. 2. A comparison of effective coupling functions for the proto-phase (left panel) and the phase (right panel) for the model B with $c = -0.05$, $\sigma = 0.8$ and $\Omega = 3.4$.

Rearranging $H = H_m + F$, we find the effective coupling

$$F(\psi, \theta) = f - \int_0^{2\pi} f \frac{P}{P_m} d\psi - g^2(\theta)\partial_\theta \ln \frac{P(\theta, \psi)}{P_m(\theta)} \, . \quad (12)$$

As for the effective velocity, the first two terms represent the deterministic part of the coupling, while the last term, proportional to the noise intensity, represents the osmotic part. The local effect of the coupling on the proto-phase is naturally described by the smooth quotient $\kappa(\psi, \theta) = F(\psi, \theta)/H_m(\theta)$.

For the driven effective model, we introduce a phase variable φ by transformation (5) using the marginal density. Then we have $P_m(\varphi) = 1/2\pi$ (this definition of phase slightly differs from the one presented in [9]). Transforming Eq. (11) in this way we get

$$\dot{\varphi} = \omega + 2\pi P_m(S^{-1}(\varphi))F(\psi, S^{-1}(\varphi)) = \omega + Q(\psi, \varphi) \, . \quad (13)$$

Equation (13) provides the effective phase dynamics of the periodically driven noise-induced oscillations in a standard form, with an effective coupling function Q that heavily depends on the noise intensity.

In the following examples we use the periodic force $f(\psi(t), \theta) = 0.1 \sin(\Omega t - \theta)$. First, we illustrate the difference between the coupling in terms of the proto-phase $\omega\kappa$ and the phase coupling function Q in Fig. 2 for the model B. The function $\kappa(\psi, \theta)$ is concentrated around a vicinity of the stable steady state $\theta_s \approx \pi/2$, as this value is apparently most sensitive to external forces. However, around θ_s the evolution of θ is slow, and thus this region is significantly extended when transformed to the phase φ. Correspondingly, the sensitive region of θ transformed to the phase φ is stretched.

Second, we consider the important case of weak coupling. Here, an averaged (over period of forcing) coupling function provides an adequate description of the dynamics. By averaging Eq. (13) over the period 2π of the external phase ψ, the equation for the phase difference $\Delta\varphi = \varphi - \Omega t$ is obtained in the standard Adler form [2]

$$\begin{aligned} \frac{d\Delta\varphi}{dt} &= \omega - \Omega + q(\Delta\varphi) \, , \\ q(\Delta\varphi) &= \frac{1}{2\pi} \int_0^{2\pi} Q(\Delta\varphi + \psi, \psi) d\psi \, . \end{aligned} \quad (14)$$

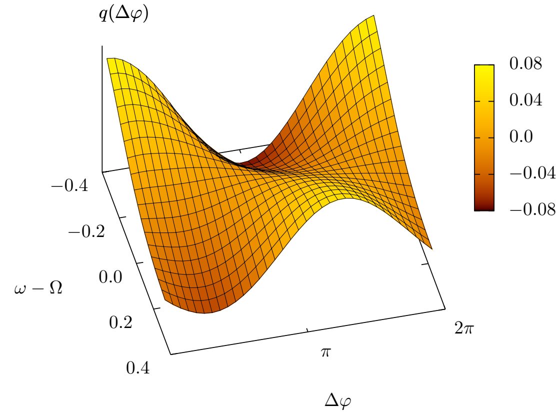

FIG. 3. (color online). Averaged coupling function $q(\Delta\varphi)$ (Eq. 14) for model A at parameters $a = 0.95$ and $\sigma = 0.5$, drawn for several values of Ω shifted by $\omega \approx 0.436$.

Again, there is a deterministic and an osmotic contribution to q, and they are in general of the same order of magnitude, but typically have opposite signs.

The case where the external frequency Ω is close to the natural frequency ω of noise-induced oscillations is of special interest. From equation (14) and the form of Q as shown in the example it could be expected that the oscillator would enter a synchronization regime where the phase is completely locked and $\Delta\varphi(t)$ remains bounded. However, for a stochastic oscillator (9) with $g \neq 0$, such a perfect synchronization with the external forcing is in general impossible. In the effective model (14) the riddle is resolved by the fact, that as Ω approaches ω, the deterministic and the osmotic parts of the averaged coupling function cancel so that the oscillator does not "feel" the coupling on average. We illustrate this phenomenon in Fig. 3, where the averaged coupling function $q(\Delta\varphi)$ is shown for different values of Ω. As $\omega - \Omega$ approaches zero, the function flattens.

After a throughout treatment of one-dimensional oscillators, we demonstrate how to construct an effective phase model for a general noise-driven excitable system, which, contrary to the one-dimensional example above, does not allow an analytic treatment. To illustrate this construction, based on the observations of the oscillations, we take a noise-driven FitzHugh-Nagumo model as a paradigmatic example of an excitable system:

$$\begin{aligned} \epsilon \frac{dx}{dt} &= x - \frac{x^3}{3} - y \\ \frac{dy}{dt} &= x + a + \sigma\xi(t) + b\cos\Omega t \, . \end{aligned} \quad (15)$$

Together with a noisy force $\sigma\xi(t)$, that in the chosen excitable case $a = 1.1$, $\epsilon = 0.05$ induces oscillations, we have incorporated a periodic force for which we determine the effective phase coupling.

Although we do not have analytical expressions for the mean frequency and the probability density, these charac-

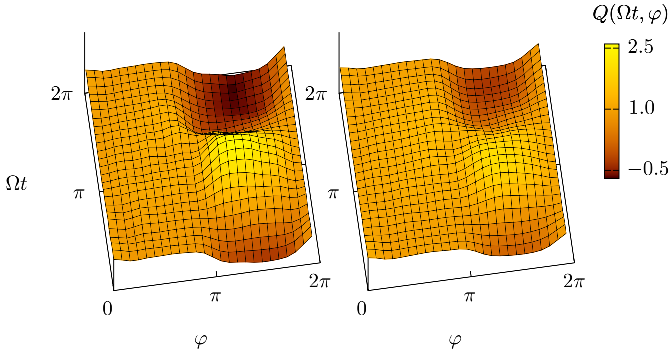

FIG. 4. (color online). Coupling functions $Q(\psi, \varphi)$ for the noise-induced oscillations in the FitzHugh-Nagumo model (15), for $b = 0.1$ and two different values of noise: at $\sigma = 0.08$ (left panel, here the mean frequency is $\omega \approx 0.62$), and at $\sigma = 0.11$ (right panel, $\omega \approx 0.95$).

teristics can be straightforwardly obtained from the numerical simulations. Adopting the simplest choice for the protophase $\theta = \arctan(y/x)$, and calculating ω and $P_m(\theta)$, we perform a transformation to the phase φ according to (5). With long enough time series ϕ_n and ψ_n at hand, we determine the effective coupling function $Q(\psi, \varphi)$. For this we use a least square fit to approximate the dependence of the central difference (8) on ψ and φ with a double Fourier series (see [9] for details). Our numerical result $Q(\varphi, \psi)$ is shown in Fig. 4 for two representative noise intensities. For the chosen driving frequency the amplitude of coupling decreases with increasing noise intensity, due to more pronounced cancellation of the deterministic and osmotic contributions to the coupling. This is related to a frequency shift as has been discussed with Eq. (14) and Fig. 3.

In summary, we have presented an effective phase dynamics description of autonomous and driven noise-induced oscillations. For oscillators based on one-dimensional dynamics many features of the effective dynamics can be found analytically. For complex oscillating processes, where an analytical treatment is not possible, we propose to determine an effective phase dynamics from synthetic or experimental observations of the system under analysis, this method is exemplified with the FitzHugh-Nagumo system. The main feature of the effective phase dynamics is that it intrinsically depends on the noise intensity and on the regime observed. Thus, the effective dynamics obtained from one observation generally cannot be used for a prediction of the dynamics at other noise intensities, forcing amplitudes, or driving frequencies. In a general context of noisy oscillating systems, the effective phase approach gives a novel tool of reductional analysis where noise is not treated perturbatively. Thereby it can be applied to systems where the noise is not just an additional small factor but changes the dynamics qualitatively, such as excitable systems. In this article we restricted our attention to single and periodically driven noise-induced oscillators. The approach can be extended to the case of several coupled oscillators; this study will be presented elsewhere.

The authors would like to thank M. Rosenblum and B. Kralemann for useful discussions.

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