Capacities of lossy bosonic channel with correlated noise

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Abstract.

We evaluate the information capacities of a lossy bosonic channel with correlated noise. The model generalizes the one recently discussed in [Phys. Rev. A **77**, 052324 (2008)], where memory effects come from the interaction with correlated environments. Environmental correlations are quantified by a multimode squeezing parameter, which vanishes in the memoryless limit. We show that a global encoding/decoding scheme, which involves input entangled states among different channel uses, is always preferable with respect to a local one in the presence of memory. Moreover, in a certain range of the parameters, we provide an analytical expression for the classical capacity of the channel showing that a global encoding/decoding scheme allows to attain it. All the results can be applied to a broad class of bosonic Gaussian channels.

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1. Introduction

One of the main tasks of quantum information theory is the evaluation of the capacities of quantum channels for the transmission of classical or quantum information. Recently, a growing attention has been devoted to the study of quantum channels with memory. Coding theorems were provided for a subset of memory channels [1], the so-called 'forgetful channels'. One can distinguishes the cases in which the output at the kth use of the channel is influenced by the input at the k'th use, with $k' < k$, as the models studied in [2]; and those in which memory effects come from correlations among subsequent channel uses, as the ones considered in [3, 4, 5, 6]. Here we consider the second case, which is also referred to as 'channel with correlated noise'. A correlation-free channel can be considered as an ideal limit since correlations are unavoidable in physical realizations. Another motivation for studying channels with correlated noise is the possibility of enhancing the information capacities. There are indeed evidences of the possibility of amplifying the classical capacity in both the cases of discrete [3] and continuous [4] variables quantum channels.

Here we consider a model of bosonic Gaussian channel in which memory effects come from the interaction with a bosonic Gaussian environment. The model is a generalization of the one discussed in [5] and it belongs to a family of channels presented in [6]. Even though each channel belonging to this family is unitary equivalent to a memoryless one (in the sense specified in [6]), the presence of energy constraints can break the unitary symmetry, leaving the problem of capacities evaluation open. An instance of a channel belonging to that family is obtained by specifying the state of the environment. Here we consider a multimode squeezed thermal state, determined by two parameters. The first parameter expresses the degree of squeezing, which in turn determines the amount of correlations in the channel; the second one is a temperature parameter expressing the mixedness of the state. It is clear that at zero temperature the correlations in the multimode squeezed state are quantum, on the other hand above a certain temperature the states becomes separable and the correlations are classical.

The choice of a Gaussian state for the environment makes in turns the channel Gaussian. In this way, using [7, 8, 9, 10], we are able to evaluate, analytically or numerically, the classical and quantum capacities of the memory channel. To emphasize the role of correlations, we compare two different scenarios for encoding and decoding classical and quantum information: in the first one, which we refer to as the *global scenario*, we allow preparation of states at the input field which are entangled among different channel uses; in the second one, called the *local scenario*, we only allow preparation of simply separable states (i.e. uncorrelated) at the input field, moreover we do not allow the receiver to access the correlations among the output modes.

The paper develops along the following lines. In section 2 we present the model and define the global and the local encoding/decoding scenarios. In section 3 we present analytical and numerical results for the classical, entanglement-assisted, and quantum capacity. Conclusions and comments are drawn in section 4.

2. A model of lossy bosonic Gaussian channel with correlated noise

We consider an instance of the general model for a bosonic channel with correlated noise presented in [6]. For any integer n, its action is defined over a set of n input bosonic oscillators, with canonical variables $\{q_k, p_k\}_{k=1,\dots n}$. A collection of ancillary modes $\{Q_k, P_k\}_{k=1,\dots n}$, which play the role of the environment, is also needed. In the following we refer to this set of oscillators as the input and environment *local modes*. All the frequencies are assumed to be degenerate and equal to one, together with $\hbar = 1$. The integer k labels the sequential uses of the channel. At the kth use, the kth input mode is linearly mixed with the kth mode of the environment at a beam splitter with given transmissivity η (see figure 1). In the Heisenberg picture the channel transforms the input field variables as

$$\begin{aligned} q_k' &= \sqrt{\eta}\, q_k + \sqrt{1-\eta}\, Q_k \,, \\ p_k' &= \sqrt{\eta}\, p_k + \sqrt{1-\eta}\, P_k \,. \end{aligned} \tag{1}$$

A constraint on the energy is required to avoid infinite capacities. We constraint the average number of photons at the input field; for a given N we require:

$$\frac{1}{2n}\sum_{k=1}^{n}\langle q_k^2 + p_k^2\rangle_{\rm in} \leq N + \frac{1}{2}. \tag{2}$$

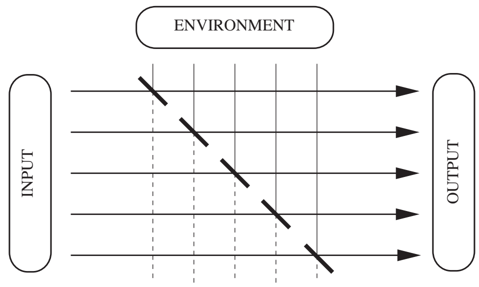

Figure 1. A schematic picture of the model of lossy bosonic channel. Each input mode (left-right line), representing one use of the channel, interacts with the corresponding environment mode (top-bottom line) through a beam-splitter. To introduce correlations effects, environment modes are considered in a correlated state.

Memory effects appear in the presence of correlations among the local modes of the environment. For a given integer n, a channel $\mathfrak{L}^{(n)}$ over the n input modes is defined. In the Schroedinger picture its action is

$$\mathfrak{L}^{(n)}(\rho_{\rm in}) = {\rm tr}_{\rm env}\left(\mathcal{U}\ \rho_{\rm in}\otimes\rho_{\rm env}\ \mathcal{U}^{\dagger}\right) \tag{3}$$

where $\rho_{\rm in}$ indicates the state of the input field, $\mathcal{U}$ is the unitary transformation at the n beam splitters, $\rho_{\rm env}$ indicates the state of the environment field and ${\rm tr}_{\rm env}$ the partial trace over the environment variables. The channel is correlation-free if the state of the environment is simply separable (i.e. uncorrelated) in the basis of the local mode.

We assume the environment to be in a Gaussian state, which in turns makes the channel Gaussian. Here we consider an environment covariance matrix of the following *block-diagonal* form:

$$V = \begin{pmatrix} \langle \mathbf{Q}\mathbf{Q}^{\mathsf{T}}\rangle & \langle \frac{\mathbf{Q}\mathbf{P}^{\mathsf{T}}+\mathbf{P}\mathbf{Q}^{\mathsf{T}}}{2}\rangle \\ \langle \frac{\mathbf{Q}\mathbf{P}^{\mathsf{T}}+\mathbf{P}\mathbf{Q}^{\mathsf{T}}}{2}\rangle & \langle \mathbf{P}\mathbf{P}^{\mathsf{T}}\rangle \end{pmatrix} := \left(T+\frac{1}{2}\right)\begin{pmatrix} e^{s\Omega} & \mathbb{O} \\ \mathbb{O} & e^{-s\Omega} \end{pmatrix}, \tag{4}$$

where $\mathbf{Q} := (Q_1, Q_2, \dots Q_n)^{\mathsf{T}}$ and $\mathbf{P} := (P_1, P_2, \dots P_n)^{\mathsf{T}}$. This is a *bona fide* covariance matrix as long as the matrix Ω is symmetric and $T \geq 0$. To fix the ideas we chose a $n\times n$ matrix Ω of the following form:

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \ddots & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{5}$$

In this way, for $T=0$, we recover the model analyzed in [5]. The physical interpretation is that of a multimode squeezed thermal state, that can be obtained by applying the multimode squeezing operator (see [11])

$$U(s) = \exp\left[-i\frac{s}{2}\sum_{k=1}^{n-1}\left(Q_k P_{k+1} + P_k Q_{k+1}\right)\right] \tag{6}$$

to a thermal state with T average excitations per mode. Since the memory of the channel comes from squeezing in the environment we refer to the parameter s (or $|s|$) as the *memory parameter*; for $s = 0$ the environment is a correlation-free thermal state. The integer n can be interpreted as the characteristic length of correlations. To evaluate lower bounds we need the eigenvalues and eigenvectors of the matrix Ω which were presented in [5]. The eigenvalues are

$$\lambda_j = 2\cos\left(\frac{\pi j}{n+1}\right) \text{ for } j = 1, \dots n, \tag{7}$$

where the corresponding eigenvectors have components

$$v_{j,k} = \sqrt{\frac{2}{n+1}} \sin\left(\frac{jk\pi}{n+1}\right) \text{ for } k = 1, \dots n. \tag{8}$$

Let us introduce the global variables $\{\tilde{Q}_j, \tilde{P}_j\}_{j=1,\dots n}$, defined from the eigenvectors of Ω as:

$$\begin{array}{rcl} \tilde{Q}_j & := & \sum_k v_{j,k}\, Q_k \,, \\ \tilde{P}_j & := & \sum_k v_{j,k}\, P_k \,. \end{array} \tag{9}$$

It follows that the environment covariance matrix is diagonal in this basis; it can be written as a direct sum

$$\tilde{V} = \bigoplus_{j=1}^{n} \tilde{V}_j \tag{10}$$

of single-mode covariance matrices of the form:

$$\tilde{V}_j = \begin{pmatrix} \langle \tilde{Q}_j^2 \rangle & \langle \frac{\tilde{Q}_j \tilde{P}_j + \tilde{P}_j \tilde{Q}_j}{2} \rangle \\ \langle \frac{\tilde{Q}_j \tilde{P}_j + \tilde{P}_j \tilde{Q}_j}{2} \rangle & \langle \tilde{P}_j^2 \rangle \end{pmatrix} = \left(T + \frac{1}{2}\right) \begin{pmatrix} e^{s_j} & 0 \\ 0 & e^{-s_j} \end{pmatrix}, \tag{11}$$

with $s_j := s\lambda_j$. Hence, moving to the global variables, the state of the environment is the direct product of n modes, each being in a squeezed thermal state, with squeezing parameter s_j and T thermal photons.

2.1. Global versus local scenario

In the following sections we estimate the quantum and classical capacities for a given value of the correlation length n. To emphasize the role of correlations we compare two different encoding/decoding scenarios.

The first scenario is a global one, in the sense that it involves preparation of the input field in states which are (in general) entangled among different uses of the channel (i.e. among the local modes). The second scenario is a local one, involving preparation of the input field in states which are simply separable among the channel uses; moreover we do not allow the receiver to access the correlations among different local modes at the output field.

As to the global scenario, we introduce the following global variables at the input field

$$\begin{array}{rcl} \tilde{q}_j & := & \sum_k v_{j,k}\, q_k \,, \\ \tilde{p}_j & := & \sum_k v_{j,k}\, p_k \,, \end{array} \tag{12}$$

(notice that this set of variables is 'parallel' to the environment global variables defined in (9)) and consider states of the input field which are factorized in the basis of these global modes. These states are in general entangled among channel uses (i.e. among local modes). In terms of the global variables, the channel factorizes as

$$\mathfrak{L}^{(n)} = \bigotimes_j \tilde{\mathfrak{L}}_j^{(1)} \,, \tag{13}$$

where the channel $\tilde{\mathfrak{L}}_j^{(1)}$ acts on the jth global mode of the input field. In the Heisenberg picture $\tilde{\mathfrak{L}}_j^{(1)}$ transforms the jth global input variables as:

$$\begin{aligned} \tilde{q}'_j &= \sqrt{\eta}\,\tilde{q}_j + \sqrt{1-\eta}\,\tilde{Q}_j \,, \\ \tilde{p}'_j &= \sqrt{\eta}\,\tilde{p}_j + \sqrt{1-\eta}\,\tilde{P}_j \,. \end{aligned} \tag{14}$$

Furthermore, due to the form of the transformation (12) (a linear passive one, see Appendix A), the energy constraint is preserved in the basis of global variables:

$$\frac{1}{2n}\sum_{j=1}^{n}\langle \tilde{q}_j^2 + \tilde{p}_j^2\rangle_{\rm in} \leq N + \frac{1}{2}. \tag{15}$$

It is worth noticing that the channel $\tilde{\mathfrak{L}}_j^{(1)}$ is a lossy bosonic channel in which one (global) mode is mixed with an environment mode which is squeezed (it is described by the covariance matrix (11)). Hence the n-use channel is unitary equivalent to a correlation-free channel, which is the product of n single-mode channels.

Concerning the local scenario let us say that, as long as the correlations among the local modes at the output field are neglected, the environment can be effectively described by a state which is factorized in the local modes. For each k, by integrating the environment Wigner function over the local variables $\{Q_h, P_h\}$ for $h \neq k$, we obtain the Wigner function of the kth local mode of the environment. The corresponding state is thermal-like, the average number of photon can be computed from (9), yielding:

$$T_{\rm eff}(k) = \left(T + \frac{1}{2}\right)\left[\sum_{j=1}^{n} |v_{j,k}|^2 e^{s_j}\right] - \frac{1}{2}. \tag{16}$$

By the symmetries of $v_{j,k}$ and s_j, this can be rewritten as follows:

$$T_{\rm eff}(k) = \begin{cases} (2T+1)\left(\sum_{j=1}^{n/2} |v_{j,k}|^2 \cosh s_j\right) - \frac{1}{2} & \text{if } n \text{ is even,} \\ (2T+1)\left(\sum_{j=1}^{(n-1)/2} |v_{j,k}|^2 \cosh s_j\right) + \frac{1}{2}|v_{(n+1)/2,k}|^2 - \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases} \tag{17}$$

Hence in the local scenario the state of the environment can be substituted with a thermal state with k-dependent temperature. Notice that, as one can expect, the *local temperature* $T_{\rm eff}(k)$ monotonically increases with $|s|$.

3. Evaluating capacities

This section is devoted to the evaluation of classical and quantum capacities of the channel $\mathfrak{L}^{(n)}$. This requires the constrained optimization of several entropic quantities, as the Holevo information, the quantum mutual information, the coherent information.

Let us recall that the von Neumann entropy $S(\rho)$ of a n mode Gaussian state ρ can be computed as follows:

$$S(\rho) = \sum_{a=1}^{n} g(\nu_a - 1/2) \tag{18}$$

where

$$g(x) := (x+1)\log_2(x+1) - x\log_2 x \tag{19}$$

and ν_a are the n symplectic invariants, i.e. the symplectic eigenvalues of the covariance matrix (see e.g. [12]). In the case of a single mode with covariance matrix σ the only symplectic invariant is $\det(\sigma)$, and the symplectic eigenvalue is $\nu = \sqrt{\det(\sigma)}$. It is worth remarking that both in the global and local scenarios introduced above the evaluation of the entropy of the n-mode fields reduces to the single-mode case.

As to the evaluation of the classical capacity of a quantum channel $\mathfrak{L}$, one is led to the Holevo information, defined as

$$\chi(\mathfrak{L}, \{\rho_\alpha, dp_\alpha\}) := S(\mathfrak{L}(\rho)) - \int dp_\alpha S(\mathfrak{L}(\rho_\alpha)), \tag{20}$$

where ρ_α denotes the quantum state encoding the classical variable α, with probability density dp_α, and ρ is the ensemble state, $\rho = \int dp_\alpha \rho_\alpha$.

The evaluation of quantum capacity and entanglement assisted classical capacity involves the coherent information, defined as

$$J(\mathfrak{L}, \rho) := S(\mathfrak{L}(\rho)) - S(\mathfrak{L}, \rho). \tag{21}$$

The quantity denoted $S(\mathfrak{L}, \rho)$ is the entropy exchange, defined as

$$S(\mathfrak{L}, \rho) := S(\mathfrak{L} \otimes \mathfrak{I}(\tilde{\rho})), \tag{22}$$

where $\tilde{\rho}$ is a purification of the state ρ involving an ancillary system, and $\mathfrak{I}$ is the identical channel acting on the ancilla.

It is worth remarking that Gaussian encoding it is known to be optimal for classical [8, 9] and quantum capacities [10] in the memoryless case. Motivated by these results we are going to estimate the capacities of the memory channel using Gaussian encoding.

Concerning the global scenario, the maximization of the entropic functions is performed over a set of Gaussian states which are simply separable with respect to the global modes. Notice that these states are in general entangled in the local basis, i.e. entangled among channel uses. In particular, we consider covariance matrices of the form

$$\sigma = \bigoplus_{j=1}^{n} \sigma_j, \tag{23}$$

where σ_j is the covariance matrix of the jth global input mode. For real r_j and $t_j \geq 0$, the jth covariance matrix is chosen as follows:

$$\sigma_j = \begin{pmatrix} \langle \tilde{q}_j^2 \rangle & \langle \frac{\tilde{q}_j\tilde{p}_j + \tilde{p}_j\tilde{q}_j}{2} \rangle \\ \langle \frac{\tilde{q}_j\tilde{p}_j + \tilde{p}_j\tilde{q}_j}{2} \rangle & \langle \tilde{p}_j^2 \rangle \end{pmatrix} := \left(t_j + \frac{1}{2} \right) \begin{pmatrix} e^{r_j} & 0 \\ 0 & e^{-r_j} \end{pmatrix}. \tag{24}$$

Under the action of the channel $\tilde{\mathfrak{L}}_j^{(1)}$ this matrix is mapped into the covariance matrix

$$\sigma_j' = \eta\sigma_j + (1-\eta)V_j. \tag{25}$$

The energy constraint can be written in terms of the input covariance matrix as

$$\frac{1}{2n}\sum_{j=1}^{n} \mathrm{tr}\,(\sigma_j) \leq N + \frac{1}{2} \tag{26}$$

that is

$$\frac{1}{n}\sum_{j=1}^{n}\left(t_j + \frac{1}{2}\right)\cosh r_j \leq N + \frac{1}{2}. \tag{27}$$

It can be useful to write the energy constraints in two steps, namely

$$\left(t_j + \frac{1}{2}\right)\cosh r_j = N_j + \frac{1}{2} \tag{28}$$

$$\frac{1}{n}\sum_{j=1}^{n} N_j \leq N. \tag{29}$$

Concerning the local scenario, as noticed above, the channel reduces to a correlation-free channel with thermal environment, with a k-dependent effective temperature. For this kind of channel, expressions for the (one-shot) classical and quantum capacities are available in literature [7, 10] and will be used as a term of comparison.

3.1. Holevo information

First, let us compute an upper bound for the classical capacity of the memory channel. This can be obtained by the maximal output entropy. For given n, we have

$$C \leq \frac{1}{n}\sup S(\mathfrak{L}^{(n)}(\rho)) \leq \frac{1}{n}\sum_{j=1}^{n}\sup S(\tilde{\mathfrak{L}}_j^{(1)}(\rho_j)) =: C^{>}, \tag{30}$$

where the superior is over all input states ρ (ρ_j) satisfying the energy constraints, which is reached by Gaussian input states. The maximum output entropy is reached by input states as in (24). The contribution of the jth global mode to the output entropy reads

$$S(\tilde{\mathfrak{L}}_j^{(1)}(\rho_j)) = g\left(\sqrt{\det(\sigma_j')} - 1/2\right), \tag{31}$$

where

$$\begin{aligned}\det(\sigma_j') = & \left(\eta(t_j+1/2)e^{r_j} + (1-\eta)(T+1/2)e^{s_j}\right)\\ \times & \left(\eta(t_j+1/2)e^{-r_j} + (1-\eta)(T+1/2)e^{-s_j}\right).\end{aligned}$$

For sufficiently small values of s, it is possible to write an explicit solution. The maximum output entropy is reached in correspondence of the optimal values satisfying

$$\left(t_j^{\mathrm{opt}} + \frac{1}{2}\right)e^{\pm r_j^{\mathrm{opt}}} = N_j^{\mathrm{opt}} + \frac{1}{2} \mp \frac{1-\eta}{\eta}\left(T + \frac{1}{2}\right)\sinh s_j, \tag{32}$$

with

$$N_j^{\mathrm{opt}} = N - \frac{1-\eta}{\eta}\left(T + \frac{1}{2}\right)\cosh s_j + \frac{1-\eta}{\eta}\left(M(s,T) + \frac{1}{2}\right), \tag{33}$$

and

$$M(s,T) := \frac{1}{n}\left(T+\frac{1}{2}\right)\left(\sum_{k=1}^{n}\cosh\left(s_k\right)\right)-\frac{1}{2}. \tag{34}$$

The range in which these values are optimal is determined by the relations $t_j^{\mathrm{opt}} \geq 0$, namely:

$$\left(N_j^{\mathrm{opt}}+\frac{1}{2}\right)^2-\left(\frac{1-\eta}{\eta}\right)^2\left(T+\frac{1}{2}\right)^2\sinh^2 s_j \geq \frac{1}{4}. \tag{35}$$

In this range the upper bound can be computed analytically, yielding

$$C^{>} = g\left[\eta N+(1-\eta)M(s,T)\right]. \tag{36}$$

Let us now compute a lower bound for the classical capacity. Motivated by the fact that it is optimal for the memoryless channel [8] we consider encoding in displaced states:

$$\varrho_\alpha = \mathcal{D}(\alpha)\varrho\mathcal{D}^\dagger(\alpha). \tag{37}$$

Here ϱ denotes a Gaussian seed state, not necessarily the vacuum, of the n mode input field. As to the global scenario, its covariance matrix is assumed to be as in equations (23), (24). The displacement operator can be decomposed in the basis of global modes as

$$\mathcal{D}(\alpha) = \bigoplus_{j=1}^{n}\mathcal{D}_j(\alpha_j), \tag{38}$$

where $\alpha_j = (y_{q,j}+iy_{p,j})/\sqrt{2}$ is the displacement amplitude of the jth global input mode, and $\alpha := (\alpha_1,\ldots\alpha_n)$. The classical noise is assumed to be Gaussian with zero mean and covariance matrix Y of the following form:

$$Y = \bigoplus_{j=1}^{n} Y_j\,, \tag{39}$$

where, for $c_{q,j}, c_{p,j} \geq 0$,

$$Y_j = \begin{pmatrix} \langle y_{q,j}^2\rangle & \langle y_{q,j}y_{p,j}\rangle \\ \langle y_{q,j}y_{p,j}\rangle & \langle y_{p,j}^2\rangle \end{pmatrix} := \begin{pmatrix} c_{q,j} & 0 \\ 0 & c_{p,j} \end{pmatrix}. \tag{40}$$

At the jth global mode the classical noise induces an ensemble state described by the covariance matrix

$$\bar{\sigma}_j := \sigma_j + Y_j, \tag{41}$$

that, under the action of the channel $\tilde{\mathfrak{L}}_j^{(1)}$, is mapped into

$$\bar{\sigma}'_j = \eta\bar{\sigma}_j + (1-\eta)V_j. \tag{42}$$

In our setting, the lower bound for the classical capacity is computed by maximizing the Holevo information

$$\chi = \sum_{j=1}^{n}\chi_j = \sum_{j=1}^{n} g\left(\sqrt{\det(\bar{\sigma}'_j)}-1/2\right) - g\left(\sqrt{\det(\sigma'_j)}-1/2\right) \tag{43}$$

over the $4n$ parameters $\{t_j, r_j, c_{q,j}, c_{p,j}\}$ under the energy constraints

$$\frac{1}{2}(c_{q,j} + c_{p,j}) + (t_j + 1/2)\cosh r_j = N_j + \frac{1}{2} \tag{44}$$

$$\frac{1}{n}\sum_{j=1}^{n} N_j \leq N. \tag{45}$$

For given values of N, $\eta \in (0,1)$, T and for sufficiently small values of s we can write an explicit solution. The maximum Holevo information is reached in correspondence of the following optimal values of the parameters: $t_j^{\mathrm{opt}} = 0$, $r_j^{\mathrm{opt}} = s_j$,

$$c_{q,j}^{\mathrm{opt}} = N_j^{\mathrm{opt}} + \frac{1}{2} - \frac{e^{s_j}}{2} - \frac{1-\eta}{\eta}\left(T + \frac{1}{2}\right)\sinh s_j \tag{46}$$

and

$$c_{p,j}^{\mathrm{opt}} = N_j^{\mathrm{opt}} + \frac{1}{2} - \frac{e^{-s_j}}{2} + \frac{1-\eta}{\eta}\left(T + \frac{1}{2}\right)\sinh s_j, \tag{47}$$

where N_j^{opt} is as in equation (33).

The range of s for which these values are optimal is defined by the conditions $c_{q,j}^{\mathrm{opt}} \geq 0$, $c_{p,j}^{\mathrm{opt}} \geq 0$. In that range, we are able to provide the following analytical lower bound for the classical capacity per channel use:

$$C^{<} = g\left[\eta N + (1-\eta)M(s,T)\right] - g\left[(1-\eta)T\right]. \tag{48}$$

For $T = 0$, this improves the lower bound computed in [5]. From the lower bound we can deduce that, besides the trivial cases $\eta(1-\eta) = 0$, the classical capacity monotonically increases with $|s|$. Moreover, it is worth noticing that, at $T = 0$, the lower bound (48) coincides with the upper bound (36). Hence, in the intersection of their ranges of validity the analytical upper and lower bounds are strict and the expression in (48) is the capacity of the memory channel at zero environment temperature. It is easy to see that the range of validity of the upper bound (35) contains the one of the lower bound, thus the lower bound in (48) is strict in the whole range of its validity at $T = 0$. Notice that the results can be extended to the limit $n \to \infty$ as in [5]. For higher values of $|s|$ one can look for a numerical solution. Figure 2 shows the analytical and numerical lower bound as function of the memory parameter for several values of η and T.

It is interesting to consider the limit $|s| \to \infty$ corresponding to 'infinite correlations'. Let us consider the term χ_j in (43) coming from the contribution of the jth global mode. Without loss of generality, we can assume $s_j > 0$. In the limit $s_j \gg 1$ we can write the following asymptotic expressions:

$$\det(\sigma_j') \simeq \eta(1-\eta)\left(T + \frac{1}{2}\right)\left(t_j + \frac{1}{2}\right)e^{s_j - r_j} + O(e^{-s_j}), \tag{49}$$

$$\det(\bar{\sigma}_j') \simeq \eta(1-\eta)\left(T + \frac{1}{2}\right)\left[\left(t_j + \frac{1}{2}\right)e^{-r_j} + c_{p,j}\right]e^{s_j} + O(e^{-s_j}). \tag{50}$$

By noticing that

$$\lim_{x\to\infty}\left[g(x) - (\log_2 x - \log_2 e)\right] = 0, \tag{51}$$

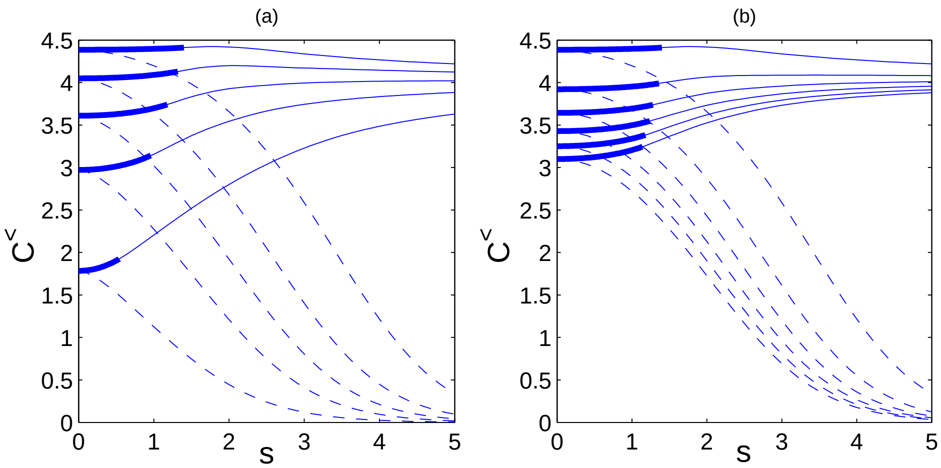

Figure 2. The plots show the lower bounds for the classical capacity, for $n = 10$, as function of the memory parameter $|s|$. In (a) at $T = 0$, where the lower bound is strict, for different values of η, from bottom to top η varies from 0.1 to 0.9 by steps of 0.2. In (b) at $\eta = 0.9$ for different values of T, from top to bottom T varies from 0 to 5 by steps of 1. The solid lines refer to the global scenario, the tick ones refer to the analytical solution in the region in which it is available. The lower bounds for the local scenario are plotted in dashed lines. The maximum average number of excitations per mode in the input field is $N = 8$.

we obtain the following asymptotic expression for the jth term in the Holevo information:

$$\chi_{\infty j} = \lim_{s\to\infty} \chi_j = \frac{1}{2}\log_2\left(1 + \frac{c_{p,j}e^{r_j}}{t_j + 1/2}\right). \tag{52}$$

For given value of N_j, from the last expression one obtains that the maximum of the mutual information is reached for $t_j^{\text{opt}} = 0$, $r_j^{\text{opt}} = \ln(2N_j + 1)$, $c_{q,j}^{\text{opt}} = 0$, and $c_{p,j}^{\text{opt}} = \sinh r_j^{\text{opt}}$, yielding the following value for the jth contribution to the classical capacity:

$$C^{<}_{\infty j} = \max_{\{t_j, r_j, c_{q,j}, c_{p,j}\}} \chi_j^{\infty} = \log_2(2N_j + 1). \tag{53}$$

Summing over j we obtain the following expression for the capacity per channel use:

$$C^{<}_{\infty} = \begin{cases} \log_2(2N+1) & \text{if } n \text{ is even,} \\ \frac{n-1}{n}\log_2(2N+1) + \frac{1}{n}\{g[\eta N + (1-\eta)T] - g[(1-\eta)T]\} & \text{if } n \text{ is odd.} \end{cases} \tag{54}$$

The presence of an extra term for odd n comes from the contribution of the $s_j = 0$ term and it leads to the oscillations of the Holevo information with the number of uses already observed in [5]. However, the relative amplitude of these oscillations becomes negligible as the number of uses increases. Interestingly enough, in the limit of perfect memory the maximal Holevo information is determined solely by the value of N, i.e. by the energy constraints. The asymptotic lower bound can be reached by homodyne detection (see [13]).

To conclude this section, let us mention that a lower bound concerning the local scenario can be obtained by the following expression (see [6]):

$$C^{<} = \frac{1}{n}\max_{\{N_k\}}\left\{\sum_{k=1}^{n} g[\eta N_k + (1-\eta)T_{\text{eff}}(k)] - g[(1-\eta)T_{\text{eff}}(k)] \;\middle|\; \frac{1}{n}\sum_{k=1}^{n} N_k = N\right\}. \tag{55}$$

This lower bound saturates the channel capacity for $T_{\rm eff} = 0$, see [8], which is obtained for $s = 0$, $T = 0$. This bound is plotted in figure 2 together with the lower bound computed in the global scenario.

3.2. Coherent information, quantum mutual information

The problem of evaluating the quantum capacity is greatly simplified by the fact that the channel $\mathfrak{L}^{(n)}$ is degradable for $\eta \in [1/2, 1]$ and anti-degradable for $\eta \in [0, 1/2)$. It follows that the coherent information is additive for $\eta \in [1/2, 1]$ and the quantum capacity vanishes for $\eta \in [0, 1/2[$ (see [14, 15, 10]). It is easy to recognize that this property is shared by all the Gaussian memory channels of the kind presented in [6]. For the same reason, in the global scenario, the n-mode channel reduces to the single-mode case as

$$\sup_{\rho} J(\mathfrak{L}^{(n)}, \rho) = \sup_{\{\rho_j\}} \sum_{j=1}^{n} J(\tilde{\mathfrak{L}}_j^{(1)}, \rho_j), \tag{56}$$

with the proper energy constraint.

For the quantum capacity, and for the entanglement-assisted classical capacity, we need to evaluate the entropy exchange of the channel $\tilde{\mathfrak{L}}_j^{(1)}$. It follows from [10], and from [7, 9] that it is sufficient to consider Gaussian input states. Numerical analysis shows that the choice of input state with covariance matrix of the form (24) is optimal. The input state at the jth global mode, with covariance matrix as in (24), can be purified into a two mode Gaussian state with covariance matrix:

$$\tau_j = \begin{pmatrix} a_j & 0 & x_j & 0 \\ 0 & b_j & 0 & -x_j \\ x_j & 0 & b_j & 0 \\ 0 & -x_j & 0 & a_j \end{pmatrix} \tag{57}$$

where

$$a_j := (t_j + 1/2)e^{r_j}, \quad b_j := (t_j + 1/2)e^{-r_j}, \quad x_j := \sqrt{a_j b_j - 1/4}. \tag{58}$$

The action of the channel $\tilde{\mathfrak{L}}_j^{(1)} \otimes \mathfrak{I}$ leads to the output covariance matrix:

$$\tau'_j = \begin{pmatrix} A_j & C_j^{\mathsf{T}} \\ C_j & B_j \end{pmatrix} := \begin{pmatrix} \eta a_j + (1-\eta)c_j & 0 & \sqrt{\eta}x_j & 0 \\ 0 & \eta b_j + (1-\eta)d_j & 0 & -\sqrt{\eta}x_j \\ \sqrt{\eta}x_j & 0 & b_j & 0 \\ 0 & -\sqrt{\eta}x_j & 0 & a_j \end{pmatrix} \tag{59}$$

where

$$c_j := (T + 1/2)e^{s_j}, \quad d_j = (T + 1/2)e^{-s_j}. \tag{60}$$

The symplectic eigenvalues of the covariance matrix in (59) are:

$$\nu_{j,\pm} = \frac{1}{\sqrt{2}} \sqrt{I_j \pm \sqrt{I_j^2 - 4\det(\tau'_j)}} \tag{61}$$

where

$$I_j := \det(A_j) + \det(B_j) + 2\det(C_j). \tag{62}$$

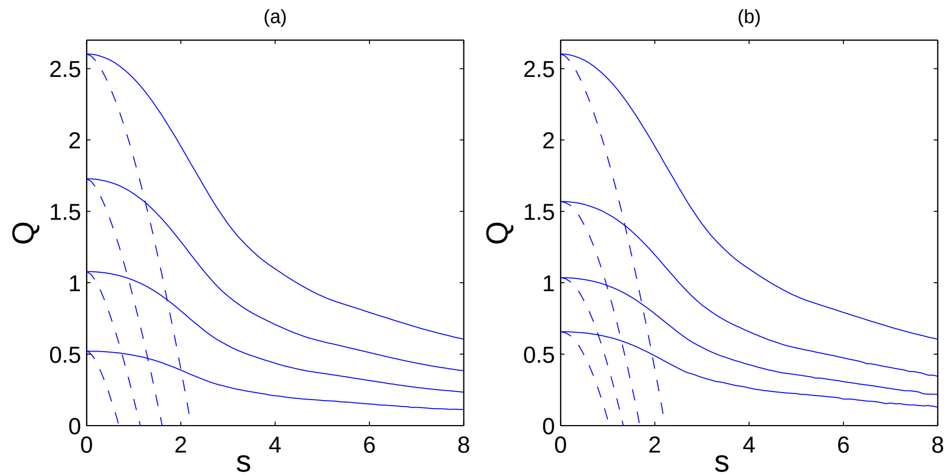

Figure 3. The plots show the numerically evaluated quantum capacity, for $n = 10$, as function of the memory parameter $|s|$. In (a) at $T = 0$ for different values of η, from bottom to top η varies from 0.6 to 0.9 by steps of 0.1. In (b) at $\eta = 0.9$ for different values of T, from top to bottom T varies from 0 to 1.5 by steps of 0.5. The solid lines refer to the global scenario, the dashed lines to the local one. The maximum average number of excitations per mode in the input field is $N = 8$.

Hence, the contribution of the jth global mode to the coherent information reads

$$J_j = g\left(\sqrt{\det(\sigma_j')} - 1/2\right) - g(\nu_{j,+} - 1/2) - g(\nu_{j,-} - 1/2). \tag{63}$$

The constrained maximization of the total coherent information gives the quantum capacity of the memory channel per channel use:

$$Q = \frac{1}{n} \max \left\{ \sum_{j=1}^{n} J_j \right\}, \tag{64}$$

where the maximum is over the parameters $\{r_j, t_j\}$ under the energy constraints (27). The results of numerical optimization are plotted in figure 3.

A simple expression can be written in the limit of infinite correlations $|s| \gg 1$. For $s_j > 0$, we can write the following asymptotic expressions for the symplectic eigenvalues:

$$\nu_{j,+} \simeq \sqrt{\eta(1-\eta)\left(t_j + \frac{1}{2}\right)\left(T + \frac{1}{2}\right)e^{s_j - r_j}} + O(e^{-s_j/2}) \tag{65}$$

and

$$\nu_{j,-} \simeq \frac{1}{2} + O(e^{-s_j/2}). \tag{66}$$

Analogous expressions can be obtained for $s_j < 0$. Taking in account the asymptotic expression in (49), it follows that the coherent information vanishes in the limit $|s_j| \to \infty$:

$$J_{\infty j} = \lim_{s \to \infty} J_j = 0. \tag{67}$$

Hence, we can write the following expression in the limit of infinite correlations:

$$Q_\infty = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{\delta}{n} & \text{if } n \text{ is odd.} \end{cases} \tag{68}$$

The finite term

$$\delta = g(N') - g\left(\frac{D + N' - N - 1}{2}\right) - g\left(\frac{D - N' + N - 1}{2}\right), \tag{69}$$

where

$$N' := \eta N + (1 - \eta)T \tag{70}$$

and

$$D := \sqrt{(N + N' + 1)^2 - 4\eta N(N + 1)}, \tag{71}$$

comes from the contribution of the global mode with $s_j = 0$ (see [7]); however, this contribution becomes negligible if $n \gg 1$.

The entanglement-assisted classical capacity is obtained maximizing the quantum mutual information:

$$C_e = \frac{1}{n} \max \left\{ \sum_{j=1}^{n} I_j \right\} \tag{72}$$

where

$$I_j = g(t_j) + J_j. \tag{73}$$

The results of numerical maximization are plotted in figure 4. In the limit of infinite memory, and for $s_j \neq 0$, the contribution of the channel $\tilde{\mathfrak{L}}_j^{(1)}$ to the quantum mutual information is

$$I_{\infty j} = \lim_{|s_j| \to \infty} I_j = g(t_j). \tag{74}$$

Notice that this asymptotic expression is independent of the transmissivity η and the temperature parameter T. Summing over j we obtain:

$$C_{e\infty} = \begin{cases} g(N) & \text{if } n \text{ is even,} \\ g(N) + \frac{\delta}{n} & \text{if } n \text{ is odd.} \end{cases} \tag{75}$$

As for the quantum capacity the term δ comes from the contribution of the global mode with $s_j = 0$, this contribution becomes negligible if $n \gg 1$.

Concerning the local scenario, figures 3, 4 show in dashed lines the quantum and assisted capacity computed applying the formulas in [7].

4. Conclusion and comments

We have presented analytical and numerical results for the capacities of a lossy bosonic Gaussian channel with correlated noise. To emphasize the role of correlations, we have compared two different scenarios. The global one allows preparation of states at the input field which are entangled among different channel uses.

For our channel model we have shown that the global scenario is optimal in the presence of memory. In particular, we have shown that, in a certain range of the parameters, it allows to enhance the classical capacity over the memoryless (correlation-free) channel. The optimal seed state of equation (37) turns out to be entangled as shown by figure 5 where the von

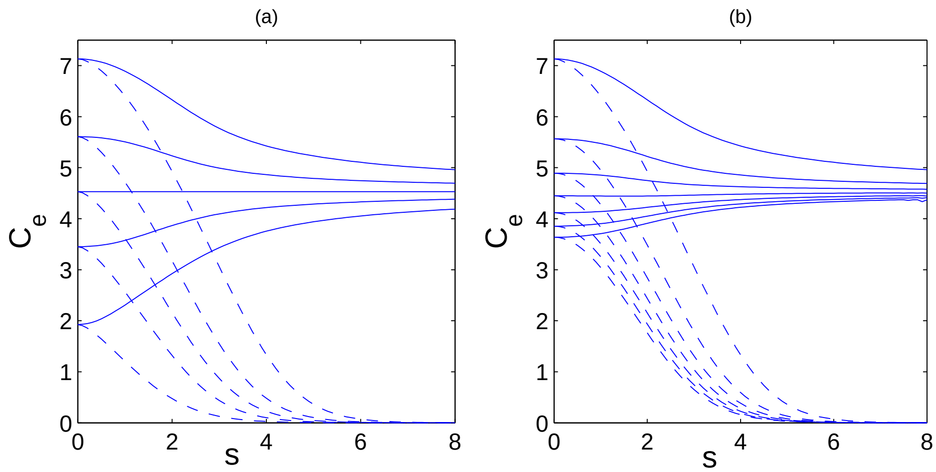

Figure 4. The plots show the numerically evaluated entanglement-assisted classical capacity, for $n = 10$, as function of the memory parameter $|s|$. In (a) at $T = 0$ for different values of η, from bottom to top η varies from 0.1 to 0.9 by steps of 0.2. In (b) at $\eta = 0.9$ for different values of T, from top to bottom T varies from 0 to 6 by steps of 1. The solid lines refer to the global scenario, the dashed lines to the local one. The maximum average number of excitations per mode in the input field is $N = 8$.

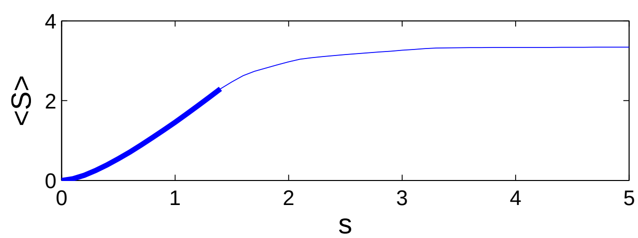

Figure 5. The plot shows the von Neumann entropy of the single-mode reduced state, obtained from the optimal seed state (see (37)), averaged over all the $1 : (n-1)$ partitions. This is for $n = 10$, at $T = 0$ and $\eta = 0.9$. The tick line refers to the analytical solution. The maximum average number of excitations per mode in the input field is $N = 8$.

Neumann entropy of the reduced state, averaged over all the $1 : (n-1)$ partitions, is plotted. The global scenario also allows to enhance the entanglement-assisted classical capacity, at least for $\eta < 0.5$. Moreover, it slows down the decrement of the quantum capacity, being the latter a decreasing function of the memory parameter. It is worth noticing that all the results can be generalized to a broad class of bosonic Gaussian channels with correlated noise. This channels are those defined by an environment covariance matrix that can be diagonalized by a transformation which is symplectic and orthogonal (see Appendix A).

Finally, we comment on the role of the environment temperature. From the analytical and numerical results we can deduce that increasing the temperature of the environment is qualitatively equivalent to decreasing the beam splitter transmissivity. Hence, by increasing the environment temperature more noise is injected in the channel without any qualitative change in the behavior of its capacities. It is also worth noticing that, for fixed nonvanishing value of the squeezing parameter s, by increasing the temperature the environment state makes

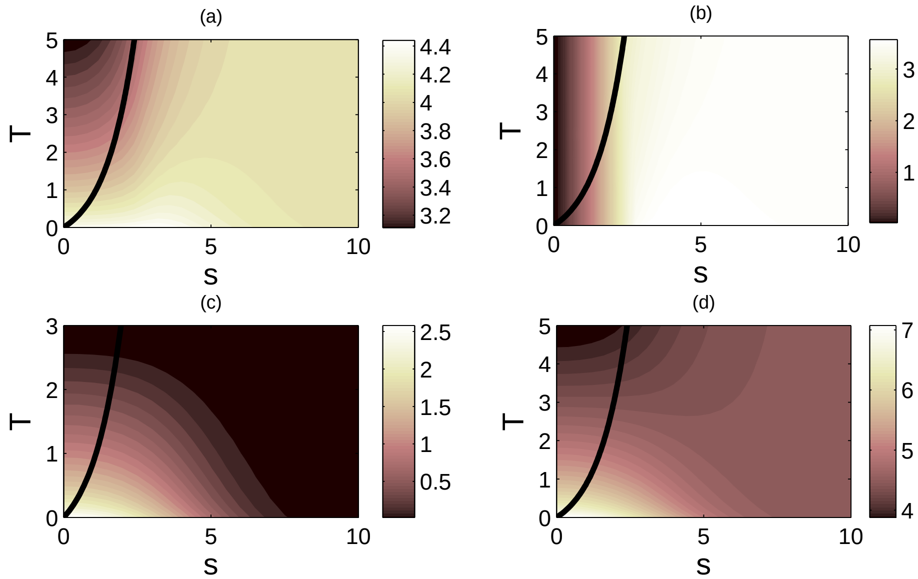

Figure 6. For $n = 2$, the contour plots show: (a) the lower bound for the classical capacity, (b) the von Neumann entropy of the reduced state of the optimal seed state, (c) the quantum capacity and (d) the entanglement-assisted classical capacity, as function of the parameters determining the state of the two-mode environment: the memory parameter and the temperature parameter. The value of the transmissivity is $\eta = 0.9$, the maximum average number of excitations per mode in the input field is $N = 8$. The black line indicates the boundary between the region in which the state of the environment is separable (on the left) and entangled (on the right).

a transition from entangled to separable. However, from the point of view of the channel capacities we do not find any evidence of this transition. In particular, all the (analytical and numerical) results are smooth functions of the environment parameters, moreover no qualitative difference is found in the pattern of the capacities at the transition form classical to quantum correlations. As an illustrative example we present in figure 6 some plots for the case of two channel uses.

Acknowledgments

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Appendix A. Capacities of a broad class of bosonic Gaussian channels with correlated noise

In the global scenario the channel $\mathfrak{L}^{(n)}$ is unitary equivalent to a correlation-free channel which is the product of n single-mode channels. This equivalence was already discussed in [6], however here the unitary equivalence preserves the form of the energy constraints. This property belongs to a large class of Gaussian memory channels. All the qualitative features regarding capacities are shared by all the channels belonging to this class. These channels are those defined by an environment covariance matrix which is diagonalized by an orthogonal transformation which is also symplectic (in optics this is called passive transformation). Let us recall that the action of such a transformation on the phase space coordinates $(Q_1, Q_2, \ldots Q_n, P_1, P_2, \ldots P_n)^\mathsf{T}$ is by a matrix of the form (see e.g. [12])

$$O = \begin{pmatrix} \mathbb{X} & \mathbb{Y} \\ -\mathbb{Y} & \mathbb{X} \end{pmatrix}, \tag{A.1}$$

with

$$\mathbb{X}\mathbb{X}^\mathsf{T} + \mathbb{Y}\mathbb{Y}^\mathsf{T} = \mathbb{I}, \tag{A.2}$$

$$\mathbb{X}\mathbb{Y}^\mathsf{T} - \mathbb{Y}\mathbb{X}^\mathsf{T} = \mathbb{O}. \tag{A.3}$$

It follows that the global scenario discussed here for the model defined by the environment covariance matrix in (4) can be equally introduced for all the covariance matrices of the following form

$$V = \begin{pmatrix} \mathbb{X}D_Q\mathbb{X}^T + \mathbb{Y}D_P\mathbb{Y}^T & \mathbb{Y}D_P\mathbb{X}^T - \mathbb{X}D_Q\mathbb{Y}^T \\ \mathbb{X}D_P\mathbb{Y}^T - \mathbb{Y}D_Q\mathbb{X}^T & \mathbb{X}D_P\mathbb{X}^T + \mathbb{Y}D_Q\mathbb{Y}^T \end{pmatrix}, \tag{A.4}$$

where the diagonal matrices D_Q, D_P satisfy the Heisenberg principle, namely

$$D_Q D_P \geq \frac{\mathbb{I}}{4}. \tag{A.5}$$

The class of covariance matrices of the form (A.4) contains all the pure state covariance matrices and all the mixed states which are obtained by applying a squeezing transformation to a thermal state. For these states the global scenario can be defined as in section 2.1; all the results concerning the global scenario can be straightforwardly extended, including its optimality to achieve the classical and quantum capacity.

References

[1] Bowen G, Devetak I and Mancini S 2005 Phys. Rev. A **71** 034310
Kretschmann D and Werner R F 2005 Phys. Rev. A **72** 062323
[2] Lupo C, Giovannetti V, Mancini S 2009 arXiv:0903.2764
Benenti G, D'Arrigo A, Falci G 2009 arXiv:0903.1424
[3] Macchiavello C and Palma G M 2002 Phys. Rev. A **65** 050301(R)
Macchiavello C, Palma G M and Virmani S 2004 Phys. Rev. A **69** 010303(R)
Daems D 2007 Phys. Rev. A **76** 012310
[4] Ruggeri G, Soliani G, Giovannetti V and Mancini S 2005 Europhys. Lett. **70** 719
Cerf N J, Clavareau J, Macchiavello C and Roland J 2005 Phys. Rev. A **72** 042330
[5] Pilyavets O V, Zborovskii V G and Mancini S 2008 Phys. Rev. A **77** 052324
[6] Giovannetti V and Mancini S 2005 Phys. Rev. A **71** 062304
[7] Holevo A S and Werner R F 2001 Phys. Rev. A **63** 032312
[8] Giovannetti V, Guha S, Lloyd S, Maccone L, Shapiro J H and Yuen H P 2004 Phys. Rev. Lett. **92** 027902
[9] Giovannetti V, Lloyd S, Maccone L and Shor P W 2003 Phys. Rev. Lett. **91** 047901
[10] Wolf M M, Perez-Garcia D and Giedke G 2007 Phys. Rev. Lett. **98** 130501
[11] Lo C F and Sollie R 1993 Phys. Rev. A **47** 733
[12] Ferraro A, Olivares S and Paris M G A, *Gaussian states in quantum information* (Bibliopolis, Napoli, 2005)
[13] Lupo C and Mancini S 2009 arXiv:0901.4966v1

[14] Devetak I, Shor P W 2003 quant-ph/0311131
[15] Caruso F, Giovannetti V 2006 Phys. Rev. A **74** 062307