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Charged Particle–Image Interaction Near a Conducting Surface

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Abstract

The interaction of a bulk electron with conducting surfaces is studied by means of the Bohm-Pines transformation in the second quantization formalism. The effective interaction potentials are obtained for the case of one plane and two plane configurations in the form of electron-image electron scattering.

Key Words: Bohm-Pines Transformation, Image Charge, Plasmons

1. Introduction

The electron interactions in an electron gas yield collective plasmon modes. Bohm and Pines [1] developed a collective description of electrons for a three-dimensional degenerate electron gas placed in a uniform background of positive charge. In order to obtain a plasmon description of electron interactions, a set of supplementary field coordinates are introduced, N' in number, which describe the collective motion of the system. Hence, the model possesses a total of $3N + N'$ degrees of freedom corresponding to electrons and plasma oscillations, respectively. This extended system of electrons and plasma waves have the same physical properties as the original system after imposing a set of N' subsidiary conditions [2]. Grecu [3] calculated the plasma frequency of the electron gas by treating it as a layered structure proposed by Visscher and Falicov [4] by the method of equation of motion in the RPA. Apostol [5] pointed out that the plasmons propagating in different layers of a solid are significantly coupled together via the electric field created by in-plane charge for finite values of wave vector $\vec{k}$. In other words, when we deal with a system of two dimensional electron gas, the electron-plasmon coupling may not be negligible. Therefore, the quantum system consists of electrons plus the plasmon field width the additional plasmon-plasmon coupling. These requirements are already implemented by the canonical transformation method of Bohm and Pines. An attempt employing self-energy approach to obtain the image potential near a surface is given in [6, 7].

In this paper the problem of a single bulk electron in interaction with an infinite conducting plane has been solved for the cases of one and two planes, respectively, where the bulk electron is placed between the planes for the latter case. A two dimensional quantum version of Bohm-Pines canonical transformation is used in second quantization formalism (natural units are used).

The bulk electron-plane interaction effectively becomes electron-electron scattering mediated via plasmon exchanges, where the second electron is in fact the image of the first one with respect to the conducting plane. The effective potential evaluated for this interaction has both static and dynamic components. In the static limit the effective potential reduces as expected to the classical potential obtained by the image method of classical electrodynamics. Similarly, the image method result for two planes with an electron in between is also obtained.

2. 2D Quantum Version of Bohm-Pines Transformation

The basic Hamiltonian for a two dimensional electron gas in the second quantized form is

$$H = \int d^2\rho\, \psi^\dagger(\vec\rho)\, \frac{p^2}{2m}\, \psi(\vec\rho) + \frac{1}{2}\int d^2\rho\, d^2\rho'\, \psi^\dagger(\vec\rho)\, \psi^\dagger(\vec\rho\,')\, V(\vec\rho - \vec\rho\,')\, \psi(\vec\rho\,')\, \psi(\vec\rho), \tag{1}$$

where ψ is the electron field computed as

$$\psi(\vec\rho) = \int \frac{d^2k}{(2\pi)^2}\, e^{i\vec k\cdot\vec\rho}\, a_{\vec k} \tag{2}$$

and V is the interparticle interaction potential. In the second quantized form H becomes

$$\begin{aligned} H &= \int \frac{d^2\rho\, d^2k_1\, d^2k_2}{(2\pi)^4}\frac{\hbar^2 k_2^2}{2m} e^{i(\vec k_2 - \vec k_1)\cdot\vec\rho} a^\dagger_{\vec k_1} a_{\vec k_2} + \frac{1}{2}\int \frac{d^2\rho\, d^2\rho'}{(2\pi)^8} d^2k_1\, d^2k_2\, d^2k_3\, d^2k_4 \\ &\quad \times e^{-i\vec k_1\cdot\vec\rho} e^{-i\vec k_2\cdot\vec\rho\,'} V(\vec\rho - \vec\rho\,') e^{i\vec k_3\cdot\vec\rho\,'} e^{i\vec k_4\cdot\vec\rho}\, a^\dagger_{\vec k_1} a^\dagger_{\vec k_2} a_{\vec k_3} a_{\vec k_4}. \end{aligned} \tag{3}$$

After making the transformation $\vec r_1 = \vec\rho\,'$, $\vec r_2 = \vec\rho - \vec\rho\,'$, we have

$$\begin{aligned} H &= \int \frac{d^2k_1\, d^2k_2}{(2\pi)^4}\frac{\hbar^2 k_2^2}{2m} a^\dagger_{\vec k_1} a_{\vec k_2} \delta(\vec k_2 - \vec k_1) + \frac{1}{2}\int \frac{d^2r_1\, d^2r_2}{(2\pi)^8} d^2k_1\, d^2k_2\, d^2k_3\, d^2k_4 \\ &\quad \times a^\dagger_{\vec k_1} a^\dagger_{\vec k_2} e^{-i\vec k_1\cdot(\vec r_2 + \vec r_1)} e^{-i\vec k_2\cdot\vec r_1} V(r_2) e^{i\vec k_3\cdot\vec r_1} e^{i\vec k_4\cdot(\vec r_2+\vec r_1)}\, a_{\vec k_3} a_{\vec k_4} \\ &= \int \frac{d^2k_1}{(2\pi)^4}\frac{\hbar^2 k_1^2}{2m} a^\dagger_{\vec k_1} a_{\vec k_1} + \frac{1}{2}\int \frac{d^2k_1\, d^2k_2\, d^2k_3\, d^2k_4}{(2\pi)^6}\, \delta[(\vec k_1 + \vec k_2) - (\vec k_3 + \vec k_4)] \\ &\quad \times \widetilde V(k_1 - k_4)\, a^\dagger_{\vec k_1} a^\dagger_{\vec k_2} a_{\vec k_3} a_{\vec k_4}. \end{aligned} \tag{4}$$

The substitutions $\vec k_1 = \vec k + \vec q$, $\vec k_2 = \vec p - \vec q$, $\vec k_3 = \vec p$ and $\vec k_4 = \vec k$ in the second term give

$$H = \int \frac{d^2k}{(2\pi)^2}\frac{\hbar^2 k^2}{2m}\, a^\dagger_{\vec k} a_{\vec k} + \int \frac{d^2k}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\, \widetilde V(\vec q)\, a^\dagger_{\vec k+\vec q} a^\dagger_{\vec p-\vec q} a_{\vec p} a_{\vec k} \tag{5}$$

in terms of single-particle operators. In order to obtain the collective motion of the planar electrons, charge density operators are introduced as

$$A_{\vec q} = \int \frac{d^2k}{(2\pi)^2}\, a^\dagger_{\vec k+\vec q} a_{\vec k}, \tag{6}$$

with the properties

$$[A_{\vec q}, A_{\vec q\,'}] = [A^\dagger_{\vec q}, A^\dagger_{\vec q\,'}] = [A_{\vec q}, A^\dagger_{\vec q\,'}] = 0\ , \quad [a_{\vec k}, A_{\vec q}] = a_{\vec k-\vec q}\ , \quad [a^\dagger_{\vec k}, A_{\vec q}] = -a_{\vec k+\vec q}\ .$$

In terms of the density operators A, the second term of the Hamiltonian (5) becomes,

$$\begin{aligned} H &= \int \frac{d^2k}{(2\pi)^2}\, \varepsilon_0(\vec k)\, a^\dagger_{\vec k} a_{\vec k} \\ &\quad + \int \frac{d^2k}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\, \widetilde V(q)\, (-\delta_{\vec p,\vec k+\vec q}\, a^\dagger_{\vec p-\vec q} a_{\vec k} + a^\dagger_{\vec p-\vec q} a_{\vec p} a^\dagger_{\vec k+\vec q} a_{\vec k}) \\ &= \int \frac{d^2k}{(2\pi)^2}\, \varepsilon_0(\vec k)\, a^\dagger_{\vec k} a_{\vec k} - \int \frac{d^2k}{(2\pi)^2} V(0)\, a^\dagger_{\vec k} a_{\vec k} + \int \frac{d^2k}{(2\pi)^2}\, \widetilde V(k) A^\dagger_{\vec k} A_{\vec k}, \end{aligned} \tag{7}$$

where

$$\varepsilon_0(\vec k) = \frac{\vec k^2}{2m}. \tag{8}$$

The potential part of the Hamiltonian (5) appears to be a purely kinetic term in terms of charge densities:

$$H = \int \frac{d^2k}{(2\pi)^2}\left[\varepsilon(\vec{k})a^\dagger_{\vec{k}}a_{\vec{k}} + \frac{1}{2}\widetilde{V}(\vec{k})A^\dagger_{\vec{k}}A_{\vec{k}}\right], \tag{9}$$

where

$$\varepsilon(\vec{k}) = \varepsilon_0(\vec{k}) + V(0). \tag{10}$$

The Hamiltonian in (9) possesses two kinds of purely kinetic terms, one for single electrons and one for electron densities. In order to reach the plasmon modes, Bohm and Pines [2] proposed to complete this Hamiltonian by introducing the conjugate momenta $P_{\vec{k}}$ such that

$$H = H_0 + H_c, \tag{11}$$

where

$$H_0 = \int \frac{d^2k}{(2\pi)^2}\,\epsilon(k)a^\dagger_{\vec{k}}a_{\vec{k}} \tag{12}$$

$$H_c = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\left[M_k^2 A^\dagger_{\vec{k}}A_{\vec{k}} + P^\dagger_{\vec{k}}P_{\vec{k}} + M_k\left(A^\dagger_{\vec{k}}P_{\vec{k}} + P^\dagger_{\vec{k}}A_{\vec{k}}\right)\right] \tag{13}$$

$$M_k = \sqrt{\widetilde{V}(\vec{k})} \tag{14}$$

and $P_{\vec{k}}$ satisfies

$$P_{\vec{k}}|\Psi\rangle = 0\,. \tag{15}$$

The additional terms clearly do not affect the space of physical states and the subsidiary conditions (15) are restrictions which turn out to be consistent with the physical properties of the system. The definition of M_k is signaling the onset of the medium-coupling region represented by plasmons. The plasmon modes are represented by the electron density Hamiltonian H_c whereas the single electron term H_0 is decoupled from it.

The k-integration in the expression for H_c is usually divided into two parts: the long range interaction defined by the integration domain $k < k_c$, where k_c is a cut-off for the collective behavior; and the shorter range screened electron interaction for $k > k_c$. However, we shall keep k_c as a cut-off parameter throughout the calculations to explicitly demonstrate plasmon contributions; but when evaluating the effective potential, shall let $k_c \to \infty$ to include both types of interactions.

In two dimensions the Bohm-Pines transformation reads

$$U = e^{iS}, \tag{16}$$

where

$$S = \int_{k<k_c} \frac{d^2k}{(2\pi)^2} M_k Q_{\vec{k}} A_{\vec{k}}, \tag{17}$$

k_c is a physically set cut-off value for momentum and $Q_{\vec{k}}$ are the conjugate collective coordinates defined through

$$\left[Q_{\vec{k}}, P_{\vec{k}'}\right] = i(2\pi)^2\delta(\vec{k} - \vec{k}'). \tag{18}$$

The transformation of H by U up to second order terms in S, expressed as

$$H' = U^\dagger H U = H - i[S,H] - \frac{1}{2}[S,[S,H]], \tag{19}$$

involves the following commutators:

$$[S, P_{\vec{k}}] = \int \frac{d^2k'}{(2\pi)^2} M_{k'} A_{\vec{k}'} [Q_{\vec{k}'}, P_{\vec{k}}] = i M_k A_{\vec{k}}$$

$$[S, [S, P_{\vec{k}}]] = [S, A_{\vec{k}}] = 0$$

$$\begin{aligned}[S, H_0] &= \int \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \varepsilon(\vec{k}) M_q Q_{\vec{q}} [A_{\vec{q}}, a^\dagger_{\vec{k}} a_{\vec{k}}] \\ &= \int \frac{d^2q}{(2\pi)^2} M_q Q_{\vec{q}} \int \frac{d^2k}{(2\pi)^2} \left(\frac{\vec{q}^2}{2m} + \frac{\vec{k}\cdot\vec{q}}{m} \right) a^\dagger_{\vec{k}+\vec{q}} a_{\vec{k}}\end{aligned}$$

$$\begin{aligned}[S, [S, H_0]] &= \int \frac{d^2q_1}{(2\pi)^2} \frac{d^2q_2}{(2\pi)^2} M_{q_1} M_{q_2} Q_{\vec{q}_1} Q_{\vec{q}_2} \\ &\quad \times \int \frac{d^2k}{(2\pi)^2} [\varepsilon(\vec{k}+\vec{q}_1) - \varepsilon(\vec{k})] \left[A_{\vec{q}_2}, a^\dagger_{\vec{k}+\vec{q}_1} a_{\vec{k}}\right] \\ &= \int \frac{d^2q_1}{(2\pi)^2} \frac{d^2q_2}{(2\pi)^2} M_{q_1} M_{q_2} Q_{\vec{q}_1} Q_{\vec{q}_2} \frac{\vec{q}_1\cdot\vec{q}_2}{m} A_{\vec{q}_1+\vec{q}_2} \,.\end{aligned}$$

In terms of the transformed conjugate momenta,

$$P'_{\vec{k}} = U^\dagger P_{\vec{k}} U = P_{\vec{k}} + M_k A_{\vec{k}} \tag{20}$$

and the transformed Hamiltonian H_c becomes

$$H'_c = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} P'^\dagger_{\vec{k}} P'_{\vec{k}} \tag{21}$$

$$\begin{aligned}H'_0 &= \int \frac{d^2k}{(2\pi)^2} \epsilon(\vec{k}) a^\dagger_{\vec{k}} a_{\vec{k}} + i \int \frac{d^2q}{(2\pi)^2} M_q Q_{\vec{q}} \int \frac{d^2k}{(2\pi)^2} \frac{\vec{q}\cdot\left(\vec{k}+\frac{\vec{q}}{2}\right)}{m} a^\dagger_{\vec{k}+\vec{q}} a_{\vec{k}} \\ &\quad - \int \frac{d^2q_1}{(2\pi)^2} \frac{d^2q_2}{(2\pi)^2} M_{q_1} M_{q_2} Q_{\vec{q}_1} Q_{\vec{q}_2} \frac{\vec{q}_1\cdot\vec{q}_2}{m} A_{\vec{q}_1+\vec{q}_2} \,.\end{aligned} \tag{22}$$

If the term with $\vec{q}_1 = -\vec{q}_2$ in the third integral is singled out and $Q_{-\vec{q}} = -Q^\dagger_{\vec{q}}$ is used, the Hamiltonian for the system turns out to be

$$\begin{aligned}H' &= \int \frac{d^2k}{(2\pi)^2} \varepsilon(\vec{k}) a^\dagger_{\vec{k}} a_{\vec{k}} + i \int \frac{d^2q}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} M_q Q_{\vec{q}} \frac{\vec{q}\cdot\left(\vec{k}+\frac{\vec{q}}{2}\right)}{m} a^\dagger_{\vec{k}+\vec{q}} a_{\vec{k}} \\ &\quad + \frac{1}{2} \int_{\vec{q}_1\neq\vec{q}_2} \frac{d^2k}{(2\pi)^2} \frac{d^2q_1}{(2\pi)^2} \frac{d^2q_2}{(2\pi)^2} M_{q_1} M_{q_2} Q^\dagger_{\vec{q}_1} Q_{\vec{q}_2} \frac{\vec{q}_1\cdot\vec{q}_2}{m} a^\dagger_{\vec{k}-\vec{q}_1+\vec{q}_2} a_{\vec{k}} \\ &\quad + \int \frac{d^2k}{(2\pi)^2} M_k^2 Q^\dagger_{\vec{k}} Q_{\vec{k}} \frac{k^2}{m} \int \frac{d^2k'}{(2\pi)^2} a^\dagger_{\vec{k}'} a_{\vec{k}'} + \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} P^\dagger_{\vec{k}} P_{\vec{k}} \,.\end{aligned} \tag{23}$$

Finally, the total Hamiltonian can be represented as

$$H' = H_0 + H_{\rm pl} + H_{\rm el-pl}, \tag{24}$$

where

$$H_0 = \int \frac{d^2k}{(2\pi)^2}\varepsilon(\vec{k})a^\dagger_{\vec{k}}a_{\vec{k}} \quad ; \quad \varepsilon(\vec{k}) = \frac{\vec{k}^{\,2}}{2m} - \frac{1}{2}V(0), \tag{25}$$

$$H_{\rm pl} = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\left[P^\dagger_{\vec{k}}P_{\vec{k}} + \frac{k^2M^2_{\vec{k}}}{m}Q^\dagger_{\vec{k}}Q_{\vec{k}}\hat{n}\right] \quad ; \quad \hat{n} = a^\dagger_{\vec{k}}a_{\vec{k}}, \tag{26}$$

$$\begin{aligned} H_{\rm el-pl} &= i\int \frac{d^2k}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}M_qQ_{\vec{q}}\frac{\vec{q}\cdot\left(\vec{k}+\frac{\vec{q}}{2}\right)}{m}a^\dagger_{\vec{k}}a_{\vec{k}} \\ &+\frac{1}{2}\int_{\vec{q}+1\neq\vec{q}_2}\frac{d^2k}{(2\pi)^2}\frac{d^2q_1}{(2\pi)^2}\frac{d^2q_2}{(2\pi)^2}M_{q_1}M_{q_2}Q^\dagger_{\vec{q}_1}Q_{\vec{q}_2}\frac{\vec{q}_1\cdot\vec{q}_2}{m}a^\dagger_{\vec{k}-\vec{q}_1+\vec{q}_2}a_{\vec{k}}. \end{aligned} \tag{27}$$

H_0 is the free energy of single electrons, $H_{\rm pl}$ represents the quantization of planar charge density oscillations (plasmons) with momenta P and positions Q. The electron-plasmon interaction, $H_{\rm el-pl}$ expresses the interaction of single electrons with plasmons.

A comment on the subsidiary conditions introduced in (15) is in order here. After the transformation, constraint (15) becomes

$$\left(P'_{\vec{k}} - M_kA_{\vec{k}}\right)|\Psi\rangle = 0, \tag{28}$$

where $A_{\vec{k}}$ corresponds to the surface charge density σ and M_k is given in Eq. 14. Equation 28 is actually Gauss' Law, namely

$$kP'_{\vec{k}} = 4\pi A_{\vec{k}} \qquad \Rightarrow \qquad \nabla\cdot\vec{E} = 4\pi\rho, \tag{29}$$

where $\vec{E}$ is the electric field due to planar charge and ρ is the charge density. Therefore, Bohm and Pines procedure incorporates new degrees of freedom without disturbing the known physics.

3. Effective Electron-Plane Interaction

The effective potential for an infinite conducting plane-single bulk charged particle interaction can be evaluated by using the two dimensional Bohm-Pines transformation given in the previous section. A single electron is located a distance $|\vec{z}| = d$ away from an infinite electrically neutral metallic sheet. In-plane electron gas is in a uniform background of positive charge; therefore, the electrons can be assumed to be quasi-free particles.

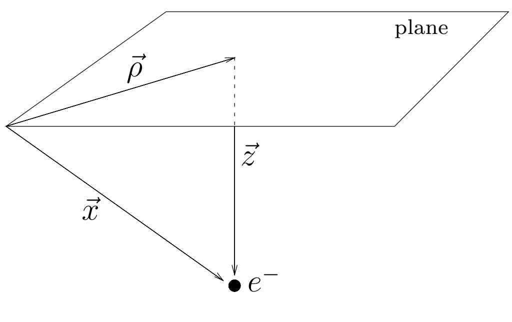

Figure 1. Bulk electron-plane configuration.

The interaction Hamiltonian for the system is

$$H_I = \int dz\, d^2\rho\, d^2\rho'\, V(\vec{\rho}-\vec{\rho}\,', z)\phi^\dagger(\vec{x})\psi^\dagger(\vec{\rho}\,')\psi(\vec{\rho}\,')\phi(\vec{x}), \tag{30}$$

where $\psi(\vec{\rho})$ corresponds to the electron field as given in Eq. 2 extending over the plane, and $\phi(\vec{x})$ is the electron field defined in the bulk defined as

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} b_{\vec{p}}\,. \tag{31}$$

We let momentum vector $\vec{k}$ be conjugate to the position vector $\vec{\rho}$ in two dimensions. Hence, in three dimensional space, momentum vector $\vec{p} = \vec{k} + \ell\hat{k}$ will be conjugate to position vector $\vec{x} = \vec{\rho} + z\hat{k}$.

Since Coulomb interaction depends on interparticle distances, it can be expressed in terms of momentum flows in momentum space:

$$\begin{aligned} H_I &= \int d^2\rho\, d^2\rho' \frac{d^3p_1}{(2\pi)^3}\frac{d^2k_1'}{(2\pi)^2}\frac{d^2k_2'}{(2\pi)^2}\frac{d^3p_2}{(2\pi)^3}\, e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{\rho}} e^{i(\vec{k}_2'-\vec{k}_1')\cdot\vec{\rho}\,'} \\ &\quad \times \int dz\, e^{i(\ell_2-\ell_1)z}\, V(\vec{\rho}-\vec{\rho}\,', z) b^\dagger_{\vec{p}_1} a^\dagger_{\vec{k}_1'} a_{\vec{k}_2'} b_{\vec{p}_2} \\ &= \int \frac{d^3p_1}{(2\pi)^3}\frac{d^2k_1'}{(2\pi)^2}\frac{d^2k_2'}{(2\pi)^2}\frac{d^3p_2}{(2\pi)^3} \\ &\quad \times \int d^2\rho\, d^2\rho'\, e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{\rho}} e^{i(\vec{k}_2'-\vec{k}_1')\cdot\vec{\rho}\,'}\, \widetilde{V}(\vec{\rho}-\vec{\rho}\,', \ell_2-\ell_1) b^\dagger_{\vec{p}_1} a^\dagger_{\vec{k}_1'} a_{\vec{k}_2'} b_{\vec{p}_2}\,. \end{aligned}$$

Substitution $\vec{r}_1 = \vec{\rho}\,'$ and $\vec{r}_2 = \vec{\rho} - \vec{\rho}\,'$ yields

$$\begin{aligned} &= \int \frac{d^3p_1}{(2\pi)^3}\frac{d^2k_1'}{(2\pi)^2}\frac{d^2k_2'}{(2\pi)^2}\frac{d^3p_2}{(2\pi)^3} \int d^2r_2\, \widetilde{V}(r_2, \ell_2-\ell_1) e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{r}_2} \\ &\quad \times \int d^2r_1\, e^{i[(\vec{k}_2-\vec{k}_1)+(\vec{k}_2'-\vec{k}_1')]\cdot\vec{r}_1} b^\dagger_{\vec{p}_1} a^\dagger_{\vec{k}_1'} a_{\vec{k}_2'} b_{\vec{p}_2} \\ &= \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^2k_1'}{(2\pi)^2}\, d^2k_2'\, U(\vec{p}_2-\vec{p}_1)\delta(\vec{k}_2-\vec{k}_1+\vec{k}\,'_2-\vec{k}\,'_1)\, b^\dagger_{\vec{p}_1} b_{\vec{p}_2} a^\dagger_{\vec{k}_1'} a_{\vec{k}_2'} \\ &= \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^2k_1'}{(2\pi)^2} U(\vec{p}_2-\vec{p}_1)\, b^\dagger_{\vec{p}_1} b_{\vec{p}_2} a^\dagger_{\vec{k}_1'} a_{\vec{k}_1+\vec{k}_1'-\vec{k}_2}\,. \end{aligned}$$

Another substitution $\vec{p}\,' = \vec{p}_2$ and $\vec{p} = \vec{p}_2 - \vec{p}_1$ gives the interaction term

$$= \int \frac{d^2k}{(2\pi)^2}\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\, U(\vec{p})\, b^\dagger_{\vec{p}\,'-\vec{p}}\, b_{\vec{p}\,'}\, a^\dagger_{\vec{k}+\vec{k}_2-\vec{k}_1}\, a_{\vec{k}}, \tag{32}$$

where

$$U(\vec{p}) = U(\vec{k}_2-\vec{k}_1, \ell_2-\ell_1) = (2\pi)^2 \int d^2\rho\, \widetilde{V}(\vec{\rho}, \ell_2-\ell_1)\, e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{\rho}} \tag{33}$$

and

$$\widetilde{V}(\vec{\rho}, \ell_2-\ell_1) = \int dz\, e^{i(k_{2_z}-k_{1_z})z}\, V(\vec{\rho}, z)\,. \tag{34}$$

Momentum space vector $\vec{k}_2 - \vec{k}_1$ conjugate to the position vector $\vec{\rho}$, which lies over the surface, can be considered as the parallel component of the vector $\vec{p}$: $\vec{k}_2 - \vec{k}_1 = \vec{p}_\parallel$. This interpretation is immediately apparent from Eq. 32, that there is no conservation for the perpendicular component of incoming momentum $\vec{p}$. This is because the metallic sheet is assumed to be rigid.

In terms of charge densities, the interaction Hamiltonian H_I can be written as

$$H_I = \int \frac{d^3p}{(2\pi)^3}\, U(\vec{p})\, B^\dagger_{\vec{p}} A_{\vec{p}_\parallel}\,, \tag{35}$$

where

$$B^{\dagger}_{\vec{p}} = \int \frac{d^3p'}{(2\pi)^3}\, b^{\dagger}_{\vec{p}-\vec{p}\,'} b_{\vec{p}\,'} \tag{36}$$

$$A_{\vec{p}_\parallel} = \int \frac{d^2k}{(2\pi)^2}\, a^{\dagger}_{\vec{k}+\vec{p}_\parallel} a_{\vec{k}}\ . \tag{37}$$

This interaction term can be rewritten by separating the momentum vector $\vec{p}$ to its parallel and perpendicular components:

$$H_I = \int \frac{d^2k}{(2\pi)^2}\, m^{\dagger}_{\vec{k}} A_{\vec{k}}, \tag{38}$$

where

$$m^{\dagger}_{\vec{k}} = \int \frac{dp_\perp}{2\pi}\, U(\vec{k},\vec{p}_\perp) B^{\dagger}_{\vec{k},\vec{p}_\perp}\ . \tag{39}$$

The new Hamiltonian for the system consists of H', the Hamiltonian in the absence of bulk electrons given in Eq. 9, the free energy of the bulk electrons, $H_{0,\mathrm{bulk}}$, and its interaction with the plane:

$$\begin{aligned} H_{\mathrm{new}} &= H' + H_{0,\mathrm{bulk}} + H_I \\ &= H_0 + H_{0,\mathrm{bulk}} + \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\left[M_k^2 A^{\dagger}_{\vec{k}} A_{\vec{k}} + 2m^{\dagger}_{\vec{k}} A_{\vec{k}}\right] \\ &= H_0 + H_{0,\mathrm{bulk}} + \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\left[M_k^2 A'^{\dagger}_{\vec{k}} A'_{\vec{k}} - \frac{m^{\dagger}_{\vec{k}} m_{\vec{k}}}{M_k^2}\right], \end{aligned} \tag{40}$$

where

$$\widetilde{A}_{\vec{k}} = A_{\vec{k}} + \frac{m_{\vec{k}}}{M_k^2}. \tag{41}$$

In order to employ the Bohm-Pines transformation we complete the Hamiltonian H by adding the plasmon-bulk coupling terms:

$$H_{c,\mathrm{new}} = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\left[M_k^2 \widetilde{A}^{\dagger}_{\vec{k}} \widetilde{A}_{\vec{k}} - \frac{m^{\dagger}_{\vec{k}} m_{\vec{k}}}{M_k^2} + P^{\dagger}_{\vec{k}} P_{\vec{k}} + M_k\left(\widetilde{A}^{\dagger}_{\vec{k}} P_{\vec{k}} + P^{\dagger}_{\vec{k}} \widetilde{A}_{\vec{k}}\right)\right]. \tag{42}$$

The Bohm-Pines transformation for $H_{c,\mathrm{new}}$ yields the familiar results,

$$P'_{\vec{k}} = P_{\vec{k}} + M_k \widetilde{A}_{\vec{k}} \tag{43}$$

$$H'_{c,\mathrm{new}} = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\left(P'^{\dagger}_{\vec{k}} P'_{\vec{k}} - \frac{m^{\dagger}_{\vec{k}} m_{\vec{k}}}{M_k^2}\right). \tag{44}$$

The second term in the transformed Hamiltonian $H'_{c,\mathrm{new}}$ is the effective interaction Hamiltonian for the bulk electron-plane system. With definitions (36) and (39), the effective Hamiltonian H_{eff} can be written as,

$$H_{\mathrm{eff}} = -\frac{1}{2}\int \frac{d^2k}{(2\pi)^2} \frac{m^{\dagger}_{\vec{k}} m_{\vec{k}}}{M_k^2} \tag{45a}$$

$$= -\frac{1}{2}\int \frac{d^2k}{(2\pi)^2} \frac{d\ell}{2\pi} \frac{d\ell'}{2\pi} \frac{U(\vec{k},\ell)U(\vec{k},\ell')}{\widetilde{V}(\vec{k})}\, B^{\dagger}_{\vec{k},\ell} B_{\vec{k},\ell'} \tag{45b}$$

$$= -\frac{1}{2}\int_{k<k_c} \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^2k}{(2\pi)^2} \frac{d\ell}{2\pi} \frac{d\ell'}{2\pi} \frac{U(\vec{k},\ell)U(\vec{k},\ell')}{\widetilde{V}(\vec{k})}\, b^{\dagger}_{\vec{p}_1-\vec{p}\,'} b_{\vec{p}_1} b^{\dagger}_{\vec{p}_2+\vec{p}} b_{\vec{p}_2}, \tag{45c}$$

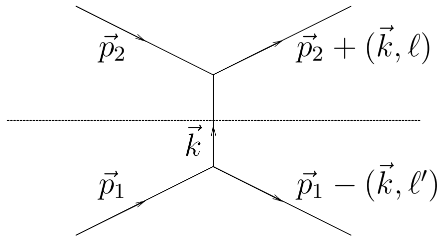

Figure 2. Electron scattering by the surface

where $\vec{p} = (\vec{k}, \ell)$, $\vec{p}\,' = (\vec{k}, \ell')$. This is nothing but the scattering of two electrons, effectively replacing the original electron-plane interaction. In terms of three dimensional particle momenta $\vec{p}_1$ and $\vec{p}_2$, the scattering diagram is shown in Figure 2.

In terms of field operators this term can be written as

$$\begin{aligned} H_{\text{eff}} &= \frac{-1}{2} \int d^3\vec{x}\, d^3\vec{x}\,'\, \phi^\dagger(\vec{x})\phi^\dagger(\vec{x}\,') \\ &\times \left[\int_{k<k_c} \frac{d^2k}{(2\pi)^2} \frac{d\ell}{2\pi} \frac{d\ell'}{2\pi} \frac{U(\vec{k},\ell)U(\vec{k},\ell')}{\widetilde{V}(\vec{k})} e^{i\vec{k}\cdot(\vec{\rho}-\vec{\rho}\,')+i\ell' z+i\ell z'} \right] \phi(\vec{x}\,')\phi(\vec{x}). \end{aligned} \tag{46}$$

The characterization of H_{eff} as an effective interaction in three dimensions by the three dimensional fields $\phi(\vec{x})$ is analogous to the replacement of classical "charge-conducting plane" problem to an equivalent "charge-image charge" problem.

The integral inside the curly brackets in Eq. (46) is an effective two-particle interaction potential. By introducing the Coulomb potential, $U(\vec{k}, \ell) = 4\pi e^2/(k^2 + \ell^2)$ and $\widetilde{V}(\vec{k}) = 2\pi e^2/k$, one can calculate the effective potential as

$$V_{\text{eff}}(\vec{x}, \vec{x}\,', k_c) = -\int \frac{d^2k}{(2\pi)^2} \frac{e^{i\vec{k}\cdot(\vec{\rho}-\vec{\rho}\,')}}{\widetilde{V}(\vec{k})} \int \frac{d\ell}{(2\pi)} \frac{d\ell'}{(2\pi)} U(\vec{k},\ell) e^{i\ell z'}\, U(\vec{k},\ell') e^{-i\ell' z} \tag{47a}$$

$$\begin{aligned} &= -\int \frac{d^2k}{(2\pi)^2} \frac{e^{i\vec{k}\cdot(\vec{\rho}-\vec{\rho}\,')}}{\widetilde{V}(\vec{k})} \left(16\pi^2 e^4 \int \frac{d\ell'}{2\pi} \frac{d\ell}{2\pi} \frac{e^{i\ell' z}}{k^2+\ell'^2} \frac{e^{-i\ell z'}}{k^2+\ell^2} \right) \\ &= -\int \frac{d^2k}{(2\pi)^2} e^{i\vec{k}\cdot(\vec{\rho}-\vec{\rho}\,')} e^{-k(z+z')} \widetilde{V}(\vec{k}) \\ &= -e^2 \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{k_c} dk\, e^{k(i|\vec{\rho}-\vec{\rho}\,'|\cos\theta - z - z')} \\ &= -\frac{ie^2}{\pi b} \int_0^{\pi} \frac{d\theta}{\cos\theta + i\frac{a}{b}} \left[1 - e^{k_c(ib\cos\theta - a)} \right], \end{aligned} \tag{47b}$$

where $a = z + z'$ and $b = |\vec{\rho} - \vec{\rho}\,'|$ for simplicity.

The first term of the integral in Eq. 47b is the dominant contribution to V_{eff}. The second term becomes negligible for large a or large b when k_c is fixed, or for large k_c with a, b fixed. In the $k_c \to \infty$ limit, the result becomes

$$\begin{aligned} V_{\text{eff}}(\vec{x},\vec{x}') &= -\frac{i}{\pi}\frac{e^2}{b}\int_0^{\pi}\frac{d\theta}{\cos\theta + i\frac{a}{b}} \\ &= -\frac{i}{\pi}\frac{e^2}{b}\left(\frac{-i\pi}{\sqrt{1+\frac{a^2}{b^2}}}\right) \\ &= \frac{-e^2}{\sqrt{|\vec{\rho}-\vec{\rho}\,'|^2+(z+z')^2}}. \end{aligned} \tag{48}$$

Contact with the classical image method result is made for $\vec{x}=\vec{x}\,'$, in which case $V_{\text{eff}} = -e^2/2z$.

4. Two Planes

In this case, a single bulk electron placed in between two infinite neutral conducting planes which have a separation d between them. The configuration of the problem is shown in Fig. 3. The planes act as 2 dimensional electron gas structures which interact with a bulk electron. $\vec{x} = \vec{r}_1 + \vec{z}\hat{k} = \vec{r}_2 - \vec{z} + d\hat{k}$ denotes the position of the bulk electron and is conjugate to $\vec{p} = \vec{p}_{\|} + \vec{p}_{\perp} = \vec{k} + \ell\hat{k}$.

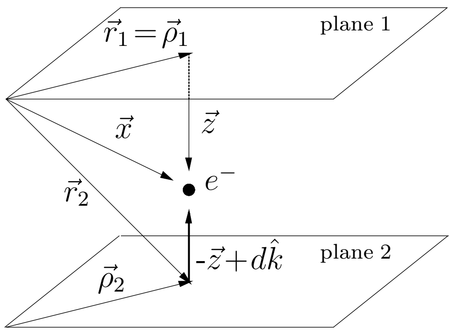

Figure 3. Plane-bulk electron-plane configuration.

The Hamiltonian for this system involves plane-plane and plane-bulk electron interactions as well as the free energies of surface charges and the bulk electron:

$$H = H_0^{(1)} + H_0^{(2)} + H_0^{(\text{v})} + H_{\text{ee}}^{(1)} + H_{\text{ee}}^{(2)} + H_{\text{pv}}^{(1)} + H_{\text{pv}}^{(2)} + H_{\text{pp}}, \tag{49}$$

where $H_0^{(1)}$, $H_0^{(2)}$ represents free energies of the plane electrons; $H_0^{(\text{v})}$ is the free energy of the bulk electron; H_{ee} is the interaction of electrons within the planes, or it comes out as a kinetic term of electron densities; H_{pv} and H_{pp} are plane-volume and plane-plane interactions and the superscripts (1),(2) and (v) stand for the first plane, the second plane and volume, respectively. H_0 and H_{ee} are as given in Eq. 9 in the first section. Remaining two interactions, H_{pv} and H_{pp}, must be calculated as follows:

$$\begin{aligned} H_{\text{pv}} &= \int dz \int d^2\rho\, d^2\rho'\, V(\vec{\rho}-\vec{\rho}\,', z)\phi^{\dagger}(\vec{x})\phi(\vec{x})\psi^{\dagger}(\vec{\rho}\,')\psi(\vec{\rho}\,') \\ &\quad + \int dz \int d^2\rho\, d^2\rho''\, V(\vec{\rho}-\vec{\rho}\,'', z-d)\phi^{\dagger}(\vec{x})\phi(\vec{x})\xi^{\dagger}(\vec{\rho}\,'')\xi(\vec{\rho}\,''), \end{aligned} \tag{50}$$

where ψ and ξ are the 2D electron field operators for the first and the second planes, respectively; and ϕ is the field operator for the bulk electron. Using the 2D electron field operators,

$$\psi(\vec{\rho}) = \int \frac{d^2k}{(2\pi)^2}\, e^{i\vec{k}\cdot\vec{\rho}}\, a_{\vec{k}} \tag{51a}$$

$$\xi(\vec{\rho}) = \int \frac{d^2k}{(2\pi)^2}\, e^{i\vec{k}\cdot\vec{\rho}}\, c_{\vec{k}}\,. \tag{51b}$$

One gets the plane-volume interaction term $H_{\rm pv}$ in terms of the single particle operators

$$H_{\rm pv} = \int \frac{d^2k}{(2\pi)^2}\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\left[U(\vec{p})b^{\dagger}_{\vec{p}'-\vec{p}}b_{\vec{p}'}a^{\dagger}_{\vec{k}+\vec{p}_\perp}a_{\vec{k}} + U(\vec{p})e^{ip_\perp d}b^{\dagger}_{\vec{p}'-\vec{p}}b_{\vec{p}'}c^{\dagger}_{\vec{k}+\vec{p}_\perp}c_{\vec{k}}\right]. \tag{52}$$

Derivation of Eq. 52 is given in Appendix A. Using the definitions Eq. 36 and Eq. 37, one obtains for $H_{\rm pv}$

$$H_{\rm pv} = \int \frac{d^3p}{(2\pi)^3}\left[U(\vec{p})\,B^{\dagger}_{\vec{p}}A_{p_\perp} + U(\vec{p})\,e^{ip_\perp d}B^{\dagger}_{\vec{p}}C_{p_\perp}\right]. \tag{53}$$

The expression in Eq. 53 is further reduced by integrating over the normal component $p_\perp$,

$$H_{\rm pv} = \int \frac{d^2k}{(2\pi)^2}\left[m^{\dagger}_{1\vec{k}}\,A_{\vec{k}} + m^{\dagger}_{2\vec{k}}\,C_{\vec{k}}\right], \tag{54}$$

where

$$m_{i\vec{k}} = \int \frac{dp_\perp}{2\pi}(\delta_{i1} + \delta_{i2}e^{ip_\perp d})U(\vec{k},p_\perp)\,B_{\vec{k},p_\perp}\,. \tag{55}$$

The plane-plane interaction term $H_{\rm pp}$, given by

$$H_{\rm pp} = \int d^2\rho_1\, d^2\rho_2\, V_{pp}(\vec{\rho}_1 - \vec{\rho}_2, d)\,\psi^{\dagger}(\vec{\rho}_1)\psi(\vec{\rho}_1)\xi^{\dagger}(\vec{\rho}_2)\xi(\vec{\rho}_2)\,, \tag{56}$$

similarly becomes

$$\begin{aligned} H_{\rm pp} &= \int \frac{d^2k_1}{(2\pi)^2}\frac{d^2k_2}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}V_{\rm pp}(\vec{q},d)\,a^{\dagger}_{\vec{k}_1+\vec{q}}a_{\vec{k}_1}c^{\dagger}_{\vec{k}_2-\vec{q}}c_{\vec{k}_2} \\ &= \int d^2q\, V_{\rm pp}(\vec{q},d)\,A_{\vec{q}}C^{\dagger}_{\vec{q}}, \end{aligned} \tag{57}$$

where the plane-plane interaction $V_{\rm pp}$

$$\begin{aligned} V_{\rm pp}(\vec{q},z) &= e^2\int_0^{k_c} d\rho\,\frac{\rho}{\sqrt{\rho^2+d^2}}\int_0^{2pi} d\theta\, e^{iq\rho\cos\theta} \\ &= 2\pi e^2\int_0^{k_c} d\rho\frac{\rho}{\sqrt{\rho^2+d^2}}\,J_0(q\rho). \end{aligned} \tag{58}$$

in the limit $k_c \to \infty$ turns out to be

$$V_{\rm pp}(q,d) = \frac{2\pi e^2}{q}e^{-qd} = T_q^2 = M_q^2 e^{-qd}\,. \tag{59}$$

Now, the total Hamiltonian is

$$\begin{aligned} H &= \int \frac{d^2k}{(2\pi)^2}\Big[\varepsilon(\vec{k})\Big(a^{\dagger}_{\vec{k}}a_{\vec{k}} + c^{\dagger}_{\vec{k}}c_{\vec{k}}\Big) + \frac{1}{2}M_k^2\Big(A^{\dagger}_{\vec{k}}A_{\vec{k}} + C^{\dagger}_{\vec{k}}C_{\vec{k}}\Big) \\ &\quad + \frac{1}{2}\Big(m_{1\vec{k}}A_{\vec{k}} + A^{\dagger}_{\vec{k}}m_{1\vec{k}} + m_{2\vec{k}}C_{\vec{k}} + C^{\dagger}_{\vec{k}}m_{2\vec{k}}\Big) \\ &\quad + \frac{1}{2}T_k^2\Big(A^{\dagger}_{\vec{k}}C_{\vec{k}} + C^{\dagger}_{\vec{k}}A_{\vec{k}}\Big)\Big] + \int \frac{d^3p}{(2\pi)^3}\,\varepsilon(\vec{p})\,b^{\dagger}_{\vec{p}}b_{\vec{p}} \\ &= H_0 + \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\Big(M_1^2A_1^{\dagger}A_1 + M_2^2A_2^{\dagger}A_2 - \frac{1}{M_1^2}W_1^{\dagger}W_1 - \frac{1}{M_2^2}W_2^{\dagger}W_2\Big), \end{aligned} \tag{60}$$

where H_0 represents the single electron free energy terms in the two planes and in the bulk,

$$H_0 = \int \frac{d^2k}{(2\pi)^2}\varepsilon(\vec{k})\Big(a^\dagger_{\vec{k}}a_{\vec{k}} + c^\dagger_{\vec{k}}c_{\vec{k}}\Big) + \int \frac{d^3p}{(2\pi)^3}\,\varepsilon(\vec{p})\, b^\dagger_{\vec{p}}b_{\vec{p}}$$

and

$$W_1 = \frac{1}{\sqrt{2}}(m_1+m_2) \qquad W_2 = \frac{1}{\sqrt{2}}(m_1-m_2)$$
$$A_1 = \frac{1}{\sqrt{2}}(A+C) + \frac{1}{M_1^2}W_1 \qquad A_2 = \frac{1}{\sqrt{2}}(A-C) + \frac{1}{M_2^2}W_2.$$

The above Hamiltonian is extended as before by introducing the conjugate momenta P_1 and P_2 for the two planes,

$$H = H_0 + H_c \tag{61}$$

where

$$\begin{aligned} H_c = \quad & \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\Big[M_1^2A_1^\dagger A_1 + M_2^2A_2^\dagger A_2 - \frac{1}{M_1^2}W_1^\dagger W_1 - \frac{1}{M_2^2}W_2^\dagger W_2 \\ & + P_1^\dagger P_1 + P_2^\dagger P_2 + M_1\Big(A_1^\dagger P_1 + P_1^\dagger A_1\Big) + M_2\Big(A_2^\dagger P_2 + P_2^\dagger A_2\Big)\Big]. \end{aligned} \tag{62}$$

After employing the Bohm-Pines transformation,

$$S = \int_{k<k_c} \frac{d^2k}{(2\pi)^2}\left(M_{1k}Q_{1k}A_1 + M_{2k}Q_{2k}A_2\right), \tag{63}$$

the transformed free Hamiltonian terms are obtained as

$$\begin{aligned} H_0^{(i)'} \quad = \quad & \int \frac{d^2k}{(2\pi)^2}\epsilon(\vec{k})(\delta_{i,1}a^\dagger_{\vec{k}}a_{\vec{k}} + \delta_{i,2}c^\dagger_{\vec{k}}c_{\vec{k}}) \\ & + i\int \frac{d^2q}{(2\pi)^2}M_qQ_{\vec{q}}\int \frac{d^2k}{(2\pi)^2}\frac{\vec{q}\cdot\left(\vec{k}+\frac{\vec{q}}{2}\right)}{m}(\delta_{i,1}a^\dagger_{\vec{k}+\vec{q}}a_{\vec{k}} + \delta_{i,2}c^\dagger_{\vec{k}}c_{\vec{k}}) \\ & - \int \frac{d^2q_1}{(2\pi)^2}\frac{d^2q_2}{(2\pi)^2}M_{q_1}M_{q_2}Q_{\vec{q}_1}Q_{\vec{q}_2}\frac{\vec{q}_1\cdot\vec{q}_2}{m}(\delta_{i,1}A_{\vec{q}_1+\vec{q}_2} + \delta_{i,2}C_{\vec{q}_1+\vec{q}_2}) \\ & \qquad (i=1,2) \end{aligned} \tag{64}$$

$$\begin{aligned} H_0^{(\mathrm{v})'} \quad \cong \quad & H_0^{(\mathrm{v})} + i\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3t}{(2\pi)^3}\frac{\vec{t}\cdot\left(\vec{p}+\frac{\vec{t}}{2}\right)}{\sqrt{2}m} \\ & \times U(\vec{t})\left[\frac{1+e^{-it_\perp d}}{M_{1t_\perp}}Q_{1t_\perp} + \frac{1-e^{-it_\perp d}}{M_{2t_\perp}}Q_{2t_\perp}\right]b^\dagger_{\vec{p}+\vec{t}}b_{\vec{p}}\,. \end{aligned} \tag{65}$$

(For the derivation of $H_0^{(\mathrm{v})'}$, see Appendix B.) However, we are interested in the transformed interaction Hamiltonian H_c':

$$H_c' = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\Big[P^\dagger_{1,\vec{k}}P_{1,\vec{k}} + P^\dagger_{2,\vec{k}}P_{2,\vec{k}} - \frac{1}{M_1^2}W^\dagger_{1,\vec{k}}W_{1,\vec{k}} - \frac{1}{M_2^2}W^\dagger_{2,\vec{k}}W_{2,\vec{k}}\Big]. \tag{66}$$

Since it provides the interaction Hamiltonian for the bulk electron-plane system,

$$\begin{aligned} H_{\mathrm{eff}} \quad = \quad & -\frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\left[\frac{1}{M_{1k}^2}W^\dagger_{1k}W_{1k} + \frac{1}{M_{2k}^2}W^\dagger_{2k}W_{2k}\right] \\ = \quad & -\frac{1}{4}\int \frac{d^2k}{(2\pi)^2}\left[\left(m^\dagger_{1k}m_{1k} + m^\dagger_{2k}m_{2k}\right)\left(\frac{1}{M_{1k}} + \frac{1}{M_{2k}}\right)\right. \\ & \qquad \left. + \left(m^\dagger_{1k}m_{2k} + m^\dagger_{2k}m_{1k}\right)\left(\frac{1}{M_{1k}} - \frac{1}{M_{2k}}\right)\right], \end{aligned} \tag{67}$$

and becomes

$$H_{\rm eff} = -\int \frac{d^2k}{2(2\pi)^2}\left\{\frac{1}{\widetilde{V}(k)\,(1-e^{-2kd})}\int\frac{d\ell}{2\pi}\frac{d\ell'}{2\pi}U(\vec{k},\ell)U(\vec{k},\ell')\left[1+e^{i(\ell'-\ell)d}\right]\right.$$
$$\left. -\frac{e^{-kd}}{\widetilde{V}(k)\,(1-e^{-2kd})}\int\frac{d\ell}{2\pi}\frac{d\ell'}{2\pi}U(\vec{k},\ell)U(\vec{k},\ell')\left[e^{i\ell'd}+e^{-i\ell'd}\right]\right\} B^{\dagger}_{\vec{k},\ell'}B_{\vec{k},\ell}. \tag{68}$$

The effective potential energy $V_{\rm eff}$ for this system is read out as done in Eq. 45b:

$$V_{\rm eff}(\vec{x},\vec{x}',k_c) = -\int\frac{d^2k}{(2\pi)^2}\frac{e^{i\vec{k}\cdot(\vec{\rho}-\vec{\rho}\,')}}{\widetilde{V}(k)(1-e^{-2kd})}\int\frac{d\ell}{2\pi}\frac{d\ell'}{2\pi}U(\vec{k},\ell)U(\vec{k},\ell')$$
$$\times\left\{\left[1+e^{i(\ell'-\ell)d}\right]-e^{-kd}\left[e^{i\ell'd}+e^{-i\ell'd}\right]\right\}e^{i(\ell z-\ell' z')}. \tag{69}$$

Evaluation of $V_{\rm eff}$ involves integrals of the form $\int\frac{d\ell}{2\pi}U(\vec{k},\ell)e^{i\alpha\ell}$ where $U(\vec{k},\ell)=4\pi e^2/(k^2+\ell^2)$. The result of ℓ-integration is $\widetilde{V}(k)e^{-|\alpha|k}$, where $\widetilde{V}(k)=2\pi e^2/k$. The remaining 2D integrals give:

$$V_{\rm eff}(\vec{x},\vec{x}',k_c) = -\int\frac{d^2k}{(2\pi)^2}\frac{V(k)e^{i\vec{k}\cdot(\vec{\rho}-\vec{\rho}\,')}}{1-e^{-2kd}}\left\{\left[e^{-k(|z|+|z'|)}+e^{-k(|z-d|+|z'-d|)}\right]\right.$$
$$\left.-e^{-kd}\left[e^{-k(|z|+|z'-d|)}+e^{-k(|z-d|+|z'|)}\right]\right\}. \tag{70}$$

The terms in Eq. 70 are evaluated by using the result

$$-\int\frac{d^2k}{(2\pi)^2}e^{i\vec{k}\cdot\Delta}\frac{\widetilde{V}(k)}{1-e^{-2kd}}e^{-sk} = -e^2\int_0^{\pi}\frac{d\theta}{2\pi}\int_0^{k_c}\frac{e^{k(-s+i\Delta\cos\theta)}}{1-e^{-2kd}}$$
$$= -e^2\int_0^{k_c}dk\frac{e^{-ks}}{1-e^{-2kd}}J_0(k\Delta)\,.$$

The $k_c\to\infty$ limit gives,

$$= \sum_{n=0}^{\infty}\int_0^{\infty}dk e^{-k(s+2nd)}J_0(k\Delta)$$
$$= \sum_0^{\infty}\left[\Delta^2+(s+2nd)^2\right]^{-1/2},$$

and one finally obtains the effective potential to be a sum of interactions between infinite number of images:

$$V_{\rm eff}(\vec{x},\vec{x}\,') = -e^2\sum_{n=0}^{\infty}\left\{\left[|\vec{\rho}-\vec{\rho}\,'|^2+(|z|+|z'|+2nd)^2\right]^{-1/2}\right.$$
$$+\left[|\vec{\rho}-\vec{\rho}\,'|^2+(|z-d|+|z'-d|+2nd)^2\right]^{-1/2}$$
$$-\left[|\vec{\rho}-\vec{\rho}\,'|^2+(|z|+|z'-d|+(2n+1)d)^2\right]^{-1/2}$$
$$\left.+\left[|\vec{\rho}-\vec{\rho}\,'|^2+(|z-d|+|z'|+(2n+1)d)^2\right]^{-1/2}\right\}. \tag{71}$$

For the case of a single bulk charge at $\vec{x} = \vec{x}\,'$ one obtains the result for the bulk point charge between two parallel planes in terms of infinite number of images:

$$
\begin{aligned}
V_{\rm eff}(z) &= -\frac{e^2}{2}\sum_n \left[\frac{1}{nd+z} + \frac{1}{nd+(d-z)} - \frac{2}{(n+1)d}\right] \\
&= -\frac{e^2}{2d}\sum_n \frac{1}{(n+\frac{z}{d})(n+1-\frac{z}{d})} \\
&\quad -\frac{e^2 z}{d^2}\left[\frac{z}{d}-1\right]\sum_n \frac{1}{(n+1)(n+\frac{z}{d})(n+1-\frac{z}{d})}.
\end{aligned}
$$

Setting the position to $z = d/2$, the result is simplified as:

$$
\begin{aligned}
V_{\rm eff} &= -\frac{e^2}{2d}\sum_n \frac{1}{(n+1)(n+\frac{1}{2})} \\
&= -\frac{e^2}{2d}\left[2 + 2\int_0^1 \frac{x^{1/2}-x}{1-x}dx\right] \\
&= -\frac{e^2}{d}\ln 2\,.
\end{aligned}
$$

5. Conclusion

The application of Bohm-Pines transformation to the electron-conducting surface problem has resulted in an effective two-particle interaction potential for the system. The transformed Hamiltonian contains the screened interaction ($k > k_c$) as well as the long range part of the Coulomb interaction ($k < k_c$). The two-point interaction term explicitly demonstrates the quantum dynamics of the electron with an image electron, which reduces to the well known classical image method results. The transformation appears to be a powerful method to study more complicated surface geometries.

A. Appendix A

$$
\begin{aligned}
H_{\rm pv} &= \int d^2\rho d^2\rho' \frac{d^3p_1}{(2\pi)^3}\frac{d^2k_1'}{(2\pi)^2}\frac{d^2k_2'}{(2\pi)^2}\frac{d^3p_2}{(2\pi)^3} e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{\rho}} e^{i(\vec{k}_2'-\vec{k}_1')\cdot\vec{\rho}\,'} b^\dagger_{\vec{p}_1} a^\dagger_{\vec{k}_1'} a_{\vec{k}_2'} b_{\vec{p}_2} \\
&\quad \times \int dz e^{i(\ell_2-\ell_1)z} V(\vec{\rho}-\vec{\rho}\,',z) \\
&\quad + \int d^2\rho d^2\rho'' \frac{d^3p_1}{(2\pi)^3}\frac{d^2k_1''}{(2\pi)^2}\frac{d^2k_2''}{(2\pi)^2}\frac{d^3p_2}{(2\pi)^3} e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{\rho}} e^{i(\vec{k}_2''-\vec{k}_1'')\cdot\vec{\rho}\,''} b^\dagger_{\vec{p}_1} a^\dagger_{\vec{k}_1''} a_{\vec{k}_2''} b_{\vec{p}_2} \\
&\quad \times \int dz e^{i(\ell_2-\ell_1)z} V(\vec{\rho}-\vec{\rho}\,'',z-d) \\
&= \int \frac{d^3p_1}{(2\pi)^3}\frac{d^2k_1'}{(2\pi)^2}\frac{d^2k_2'}{(2\pi)^2}\frac{d^3p_2}{(2\pi)^3} b^\dagger_{\vec{p}_1} a^\dagger_{\vec{k}_1'} a_{\vec{k}_2'} b_{\vec{p}_2} \\
&\quad \times \int d^2\rho d^2\rho' e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{\rho}} e^{i(\vec{k}_2'-\vec{k}_1')\cdot\vec{\rho}\,'} \widetilde{V}(\vec{\rho}-\vec{\rho}\,',\ell_2-\ell_1) \\
&\quad + \int \frac{d^3p_1}{(2\pi)^3}\frac{d^2k_1''}{(2\pi)^2}\frac{d^2k_2''}{(2\pi)^2}\frac{d^3p_2}{(2\pi)^3} b^\dagger_{\vec{p}_1} c^\dagger_{\vec{k}_1''} c_{\vec{k}_2''} b_{\vec{p}_2} \\
&\quad \times \int d^2\rho d^2\rho' e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{\rho}} e^{i(\vec{k}_2''-\vec{k}_1'')\cdot\vec{\rho}\,''} e^{i(\ell_2-\ell_1)d} \widetilde{V}(\vec{\rho}-\vec{\rho}\,'',\ell_2-\ell_1)\,. \qquad (72)
\end{aligned}
$$

The substitution $\vec{r}_1=\vec{\rho}\,'$, $\vec{r}_2=\vec{\rho}-\vec{\rho}\,'$ in the first integral and $\vec{r}_1=\vec{\rho}\,''$, $\vec{r}_2=\vec{\rho}-\vec{\rho}\,''$ in the second one gives

$$
\begin{aligned}
H_{\rm pv} &= \int \frac{d^3p_1}{(2\pi)^3}\frac{d^2k_1'}{(2\pi)^2}\frac{d^2k_2'}{(2\pi)^2}\frac{d^3p_2}{(2\pi)^3} b^\dagger_{\vec{p}_1} a^\dagger_{\vec{k}_1'} a_{\vec{k}_2'} b_{\vec{p}_2} \\
&\quad \times \int d^2r_2 \widetilde{V}(\vec{r}_2,\ell_2-\ell_1) e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{r}_2} \int d^2r_1 e^{i[(\vec{k}_2-\vec{k}_1)+(\vec{k}_2'-\vec{k}_1')]\cdot\vec{r}_1} \\
&\quad + \int \frac{d^3p_1}{(2\pi)^3}\frac{d^2k_1''}{(2\pi)^2}\frac{d^2k_2''}{(2\pi)^2}\frac{d^3p_2}{(2\pi)^3} b^\dagger_{\vec{p}_1} c^\dagger_{\vec{k}_1''} c_{\vec{k}_2''} b_{\vec{p}_2} e^{i(\ell_2-\ell_1)d} \\
&\quad \times \int d^2r_2 \widetilde{V}(\vec{r}_2,\ell_2-\ell_1) e^{i(\vec{k}_2-\vec{k}_1)\cdot\vec{r}_2} \int d^2r_1 e^{i[(\vec{k}_2-\vec{k}_1)+(\vec{k}_2''-\vec{k}_1'')]\cdot\vec{r}_1} \\
&= \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^2k_1'}{(2\pi)^2} U(\vec{p}_2-\vec{p}_1) b^\dagger_{\vec{p}_1} b_{\vec{p}_2} a^\dagger_{\vec{k}_1'} a_{\vec{k}_1+\vec{k}_1'-\vec{k}_2} \\
&\quad + \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^2k_1''}{(2\pi)^2} e^{i(\ell_2-\ell_1)d} U(\vec{p}_2-\vec{p}_1) b^\dagger_{\vec{p}_1} b_{\vec{p}_2} c^\dagger_{\vec{k}_1''} c_{\vec{k}_1+\vec{k}_1''-\vec{k}_2}\,. \qquad (73)
\end{aligned}
$$

We substitute $\vec{p}\,'=\vec{p}_2$ and $\vec{p}=\vec{p}_2-\vec{p}_1$ to get

$$
H_{\rm pv} = \int \frac{d^2k}{(2\pi)^2}\frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\left[U(\vec{p}) b^\dagger_{\vec{p}\,'-\vec{p}} b_{\vec{p}\,'} a^\dagger_{\vec{k}+\vec{p}_\perp} a_{\vec{k}} + U(\vec{p}) e^{ip_\perp d} b^\dagger_{\vec{p}\,'-\vec{p}} b_{\vec{p}\,'} c^\dagger_{\vec{k}+\vec{p}_\perp} c_{\vec{k}}\right]. \qquad (74)
$$

B. Appendix B

$$[S, H_0^{(\mathrm{v})}] = \int \frac{d^2k}{(2\pi)^2} \int \frac{d^3p}{(2\pi)^3}\, \varepsilon_p \Big\{ M_{1k} Q_{1,\vec{k}} [A_{1,\vec{k}}, b_{\vec{p}}^\dagger b_{\vec{p}}] + M_{2k} Q_{2,\vec{k}} [A_{2,\vec{k}}, b_{\vec{p}}^\dagger b_{\vec{p}}] \Big\}, \tag{75}$$

where

$$[A_{i,\vec{k}}, b_{\vec{p}}^\dagger b_{\vec{p}}] = \frac{1}{\sqrt{2} M_{ik}^2} \Big\{ [m_{1k}, b_{\vec{p}}^\dagger b_{\vec{p}}] \pm [m_{2k}, b_{\vec{p}}^\dagger b_{\vec{p}}] \Big\} \qquad i = 1, 2 \,, \tag{76}$$

and

$$\begin{aligned} [m_{ik}, b_{\vec{p}}^\dagger b_{\vec{p}}] &= \int \frac{dt_\perp}{2\pi} U(\vec{t}) \big(\delta_{i1} + \delta_{i2} e^{-ip_\perp d}\big) \big[B_t, b_{\vec{p}}^\dagger b_{\vec{p}}\big]_{t_\perp = k} \\ &= -\int \frac{dt_\perp}{2\pi} U(\vec{t}) \big(\delta_{i1} + \delta_{i2} e^{-ip_\perp d}\big) \Big[b_{\vec{p}-\vec{t}}^\dagger b_{\vec{p}} - b_{\vec{p}+\vec{t}}^\dagger b_{\vec{p}} \Big]_{t_\perp = k} . \end{aligned} \tag{77}$$

Then

$$\int \frac{d^3p}{(2\pi)^3} \varepsilon_p [A_{1,\vec{k}}, b_{\vec{p}}^\dagger b_{\vec{p}}] = -\frac{1}{\sqrt{2} M_{1k}^2} \int \frac{d^3p}{(2\pi)^3} \int \frac{dt_\perp}{2\pi} U(\vec{k}, t_\perp) \big(1 + e^{-it_\perp d}\big) \times (\varepsilon_{p+t} - \varepsilon_p) b_{\vec{p}+\vec{t}}^\dagger b_{\vec{p}} \Big|_{t_\perp = k} \tag{78}$$

$$\int \frac{d^3p}{(2\pi)^3} \varepsilon_p [A_{2,\vec{k}}, b_{\vec{p}}^\dagger b_{\vec{p}}] = -\frac{1}{\sqrt{2} M_{2k}^2} \int \frac{d^3p}{(2\pi)^3} \int \frac{dt_\perp}{2\pi} U(\vec{k}, t_\perp) \big(1 - e^{-it_\perp d}\big) \times (\varepsilon_{p+t} - \varepsilon_p) b_{\vec{p}+\vec{t}}^\dagger b_{\vec{p}} \Big|_{t_\perp = k} . \tag{79}$$

Therefore, the transformation of kinetic term for the volume can be written as:

$$\begin{aligned} H_0^{(\mathrm{v})\prime} &\cong H_0^{(\mathrm{v})} + i \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3t}{(2\pi)^3} \frac{\vec{t} \cdot \left(\vec{p} + \frac{\vec{t}}{2}\right)}{\sqrt{2} m} U(\vec{t}) \\ &\quad \times \left[\frac{1 + e^{-it_\perp d}}{M_{1t_\perp}} Q_{1t_\perp} + \frac{1 - e^{-it_\perp d}}{M_{2t_\perp}} Q_{2t_\perp} \right] b_{\vec{p}+\vec{t}}^\dagger b_{\vec{p}} . \end{aligned} \tag{80}$$

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