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The Riemann Zeta Function Applied on Glassy Systems and Neural Networks

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Abstract

Glassy systems and Neural networks share a simple model of study: the model of a set of N non interacting harmonic oscillators with energy. In this paper the author tries to describe these complex systems in order to find the energy involved. The mathematical method used is described in detail.

Key Words: Glassy systems, Zeta function, neural network,topology.

1. The Crisanti –Ritort model; the energy of complex glassy systems

Glass is a material which is in a metastable state where its free energy is higher than that of a crystal. This is obtained by stretching the liquid line from the space of the phases above the melting temperature, and the state that is glass is experimentally obtained through a rapid cooling of the liquid. Glass has properties which makes it similar both to liquid and solid materials, without being either of them. For example, what makes it resemble the liquid material is the random placement of the atoms, and what makes it resemble the solid materials is the reduced mobility of the atoms. Glass has various and remarkable properties. It presents anomalies of the behavior of its viscosity (its viscosity has a heteroclite behavior), anomalies that grows fast over a short temperature range The viscosity of glass may change for rapidly over a small variation of the melting temperature. This rapid variation is accurately described enough by the Vogel exponential law. Glass as a supercooled liquid has a relaxation function presenting two distinctive areas which correspond to different processes. Another interesting phenomenon characterizing glass is the phenomenon of aging. The viscosity grows with the time t according to a linear law, the growth being very slow. All these properties shows that the state of the glass is a particular state that must be considerate distinctive from the other states of the matter.

Many models have been suggested to simplify the study of such a complex system as glass. F. Ritort[1],then later with A. Crisanti[2] have proposed a model, defined by a set of N non-interacting harmonic oscillators with energy E, of the form.

$$E = \frac{k}{2} \quad \sum\nolimits_{i=1}^{N} x_i^2 \tag{1}$$

where k is a coupling constant (and $-\infty < x_i < \infty$). An effective interaction between oscillators is introduced through a parallel Monte Carlo dynamics characterized by small jumps in which

$$x_i \to x_i^{'} = x_i + \frac{r_i}{N}; \quad 1 \leq i \leq N, \tag{2}$$

where the variables r_i are extracted from a Gaussian distribution. At each Monte Carlo step all oscillators are updated following the previous rule. Changes in x_i should be small so as to give a smooth change in the energy for large N. This model must be discussed strictly within its domain of applicability. A. Crisanti and F. Ritort use it in order to study the relaxation after quenching to zero K. In this paper the Crisanti-Ritort model is used in order to express the energy of the complex glass system. For this we will use (1) considered after a finite number of jumps according to the formula (2). The number of oscillators is considered to be infinite. As a powerful method for doing a precise calculation the Riemann zeta function is introduced in the expression of the energy:

$$\zeta(s) = \sum\nolimits_{n=1}^{\infty} n^{-s}, s \in C, Res > 1 \tag{3}$$

Now expression (1) becomes:

$$E = \frac{k}{2}\zeta(-1) \tag{4}$$

The series (1) will be divergent and this formula implies an analytical continuation through the zeta function. That is why such regularization can be termed as a particular case of analytical continuation procedure. The Riemann zeta function $\zeta\,(s)$ is given by the series expression (3) in the region of the complex s-plane where Re $s > 1$. In the rest of the complex plane $\zeta\,(s)$ is not given by the series (1) (which is divergent). There is one and only one function that is meromorphic and is similar to our initial series within the domain of convergence. We extend its definition to the rest of the plan (except for the point $s = 1$ on the real axis). The generalized zeta function had already been used in [3] and [4] in a manner that we nowadays call regularization of one-loop groups [5],[6] and which essentially consist in attaching zeta functions not to path integrals, but to Feynman diagrams themselves. A modern formulation of this method has been developed in [7],[8]. In this paper the calculation of zeta function that is used in the expression of the energy is made starting from the relation

$$\zeta(s) = \frac{-\Gamma(1-s)}{2\pi i}\int_{\gamma}\frac{(-z)^{s-1}e^{-z}}{1-e^{-z}}dz \tag{5}$$

which is valid for the entire complex plane; the system γ is represented in the Figure 1.

In order to calculate the integral expression from the formula (5) we start from the Mac-Laurin development :

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n \tag{6}$$

With the help of (6) we can write:

$$B_n = \frac{n!}{2\pi i}\oint_{C_0}\frac{z}{e^z-1}\frac{dz}{z^{n+1}} \tag{7}$$

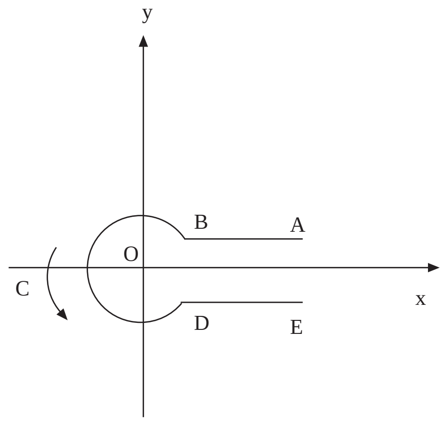

Figure 1

where B_n are the Bernoulli numbers and C_0 is a circle of radius $< 2\pi$ traversed in the clockwise direction about the region. The integrals (7) are solved for $n \geq 2$, deforming the contour C_0 as shown in Figure 2.

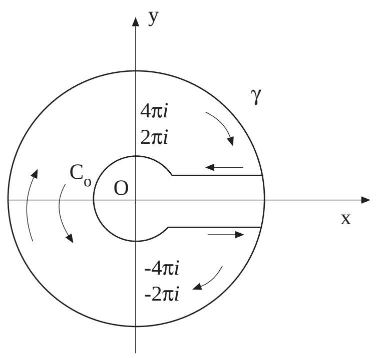

Figure 2

In the interior of the γ contour, supposing that $R \to \infty$, there are an infinity number of purely imaginary simple poles z_{-k} of the function that appears in the integral (7) and which have the form $z_k = \pm 2k\pi i$, $k \in N$. Due to the fact that the integrals on the x-axis are reduced, and the integral on the circle of radius R goes to zero, for $R \to \infty$ we have

$$\oint_{C_0} \frac{d_z}{(e^z - 1)z^n} = -2\pi i \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} Res\left[\frac{1}{(e^z - 1)z^n}, z_k\right] \tag{8}$$

where

$$Res\left[\frac{1}{(e^z - 1)z^n}, z_k\right] = \frac{1}{(2k\pi i)^n}, k \in Z - \{0\}. \tag{9}$$

If $n = 2m + 1$, $m \in N$, the residue in the points $z_k = k\pi i$ and $z_{-k} = -2k\pi i$ reduce themselves and from (7) and (8) we get $B_{2m+1} = 0$, $m \in N$; if $n = 2m$, then the residues in the points z_k and z_{-k} are equal, and from (7) and (8) we obtain:

$$B_{2m} = \frac{(2m)!}{2\pi i}(-2\pi i) \cdot 2\sum_{k=1}^{\infty} \frac{1}{k^{2m}(2\pi i)^{2m}} = (-1)^{m-1}\frac{2(2m)!}{(2\pi)^{2m}}\zeta(2m) \tag{10}$$

Now we can consider in (5) that $s = -1$ and (5) becomes

$$\zeta(-1) = +\frac{\Gamma(2)}{2\pi i} \int\limits_{\gamma} \frac{1}{e^z - 1} \cdot \frac{dz}{z^2} \tag{11}$$

From (7) we get

$$B_2 = \frac{1}{\pi i} \oint\limits_{C_0} \frac{dz}{(e^z - 1)z^2} \tag{12}$$

From (6) and (7) we obtain

$$\zeta(-1) = \frac{\Gamma(2)}{2} B_2 \tag{13}$$

From (4), the expression of the energy becomes:

$$E = \frac{k}{6} B_2 = \frac{k}{24} \tag{14}$$

That is, given the conditions of the formula (14), the expression of the energy depends only on the coupling constant which characterizes the studied glass system.

2. The Neural Networks Through Zeta Functions

The parametrical estimation of the neural networks is carried out considering a statistical model [9] $S = \{f(z;\theta) | \theta \in \Theta\}$ which is a family of probability density functions, where the space parameter θ belong to a domain in $\mathbf{R}^d$. We suppose that f(z, θ) > 0 for all z and θ, and that f(z, θ) are C^∞ on θ for any z. The *likelihood ratio* and the *Kullback-Leiber divergence* are defined by

$$L_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log \frac{f(z_i, \theta)}{f(z_i)} \tag{15}$$

and

$$D(\theta) = \int f(z) \log \frac{f(z)}{f(z_i;\theta)} dz. \tag{16}$$

Feed-forward neural networks like multilayer perceptrons can be described within this framework. Let $x \in \mathbf{R}$, $s(t) = \tanh(t)$ and $\theta = (a_j, b, c, d)^T$. The multilayer perceptron model with H hidden units is a family of functions defined by

$$\varphi(x, \theta) = \sum_{j=1}^{H} b(a_j x + c) + d. \tag{17}$$

This type of function is useful when studied in the form $H \to \infty$; $d = 0$ and

$$\varphi(x, \theta) = \sum_{j=1}^{\infty} b(a_j x + c). \tag{18}$$

Using the Truncated Epstein zeta function [10]

$$\zeta_E(-1;b,0;c) = \sum_{n=0}^{\infty} \left[b(n)^2 + c\right]^{-S} \tag{19}$$

and making the substitution $n^2 = a_j x$, we can write

$$\varphi(x,\theta) = \zeta_E(-1;b,0;c). \tag{20}$$

For this type of function, we can write, together with Elizalde [10], the asymptotic series :

$$\zeta_E(-1,b,0;c) \approx \sum_{m=0}^{\infty} \frac{(-b)^m \Gamma(m-1)}{m!\Gamma[1]c^{m-1}} \zeta_H(-2m,0) \tag{21}$$

where ζ_H is Hurwitz's zeta function. Elizalde also presents in [10] the modality of the analytical continuation of the function. The expression (20) offers, anyway, though the zeta function, an expression of the function family that describes the multilayer perceptron model.

3. The topology of the glassy system and neural network models

In this section, we present an application of the topological program for the study of the behavior of the dynamic systems introduced in [11],applied on the glassy systems and the neural networks discussed in the previous sections. Consider these systems as systems of free oscillators with the Hamiltonian

$$f(p,q) = p^2/2 - \omega^2 \cos q \tag{22}$$

where ω is a real parameter. The phase space is the cylinder $T*M = S^1 \times R$, and the Hamiltonian vector field

$$X_f = p\frac{\partial}{\partial q} + \omega^2 \sin q \frac{\partial}{\partial p}. \tag{23}$$

There are no other first integrals. There are two critical points to f : $a_1 = (0,0)$, a minimum, and $a_2 = (\pi, 0)$, a saddle point. Since $f(0,0) = -\omega^2$ and $f(\pi,0) = \omega^2$, we see that $K \subseteq R$ is the two-point set $K = \left\{-\omega^2\right\} \cup \left\{\omega^2\right\}$. From Morse theory we deduce that the maps $f|f^{-1}(-\omega^2,\omega^2)$ and $f|f^{-1}(\omega^2,\infty)$ are differentiable fibers. The level structure of f, such as the orbits of the flow of X_f, is summarized thus: for $-\infty < c < -\omega^2, I_c = 0$; for $\omega^2 < c < \infty, I_c = S^1 \cup S^1$, the disjoint union. There is a periodic current on each of the components. For $-\omega^2 < c < \omega^2, I_c = S^1$

$$I_{-\omega^2} = \{a_1\}; \quad I_{\omega^2} = S^1 \vee S^1,$$

the union being joined at the point a_2. This is a bouquet of circles, an eight shaped curve. The level set $I_{-\omega^2}$ separates the empty class of orbits from the small oscillations. The level set I_{ω^2} is also a separatrix; in its vicinity there are two types of motion: the small oscillations in I_c for $-\omega^2 < c < \omega^2$, and the full turns of the pendulum (with $c > \omega^2$). Obviously, the topological type of I_c changes in the neighborhood of K.

4. Summary

This paper is meant to emphasize the utility of using the zeta function in the study of complex systems such as glassy systems and neural networks. In both types of systems, the calculus of their internal energy can be done only by using certain models for the analysis. From the large variety of studied models, the paper focuses on the Crisanti-Ritort model, which is used as a starting point for the calculus of energy. The calculus problems can be solved by using the zeta function, but only together with the the analytical continuation of the function that has been developed in the past few years, especially given to the scientific activity of Elizalde and others.

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