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Quantization of Reparametrized Systems Using the WKB Method

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Abstract

The quantization of reparametrized systems is discussed using the WKB approximation. The Hamilton-Jacobi function, the equations of motion and the wave function—which the conditions constrain in the semiclassical limit—are determined. The success of this approach is then demonstrated for two applications. The first is an illustrative example of nonrelativistic dynamics. The second is the well-known motion of a relativistic particle in an external electromagnetic field.

Key Words: Singular Lagrangian, Parametrized systems, WKB Approximation

1. Introduction

In a previous paper [1], we proposed a general theory for the quantization of constrained systems using the WKB approximation. In the present work, we wish to apply the method to reparametrization systems by introducing an invariant parameter τ within the action integral, which will play the role of the time t. It is well known that any standard Hamiltonian system can be transformed to a constrained system with a vanishing Hamiltonian by going to an arbitrary reparametrization of time. In fact, time plays a central and peculiar role in quantum mechanics. In the standard nonrelativistic quantum mechanics, one can describe the motion of a system by using the canonical variables which are functions only of time. Time is the sole observable assumed to have a direct physical significance; but it is not a dynamical variable itself. It is an absolute parameter treated differently from the other coordinates, which turn out to be operators (observables) in quantum mechanics.

In the cases of nonrelativistic and relativistic point-particle mechanics, generally covariant systems may be obtained by promoting t to a dynamical variable. The idea behind this transformation is to treat symmetrically the time and the dynamical variables of the system. This is achieved by taking t as a function of an arbitrary parameter τ: $t=t(\tau)$,$q=q(\tau)$ (e.g., τ is the "proper time" in relativity theory) [2–6]. The arbitrariness of the label time τ is reflected in the invariance of the action under the time reparametrization. If S is the action integral, then

$$S = \int L(q, \frac{dq}{dt})dt = \int L^*(q, \frac{dq}{d\tau})d\tau.$$

Thus, we can express the action integral with respect to τ in the same form as with respect to t. This shows that the equations of motion, which follow from the action principle, must be invariant under the transformation from t to τ. The equations of motion do not refer to any absolute time. We have, therefore, a special form of Hamiltonian theory; but this form is not really so special because, starting with any Hamiltonian, it is always permissible to take the time variable as an extra coordinate and bring the theory into a form in which the Hamiltonian is equal to zero [2, 3]. The general rule for doing this is the following: we take t and set it equal to another dynamical coordinate, q_0. We then set up a new Lagrangian:

$$L^* = L^* \left(q_0, q_i, \frac{dq_0}{d\tau}, \frac{dq_i}{d\tau} \right); \qquad i=1,2, \ldots ,N.$$

L^* involves one more degree of freedom than the original L. L^* is not equal to L, but is related by:

$$\int L^* \left(q_0, q_i, \frac{dq_0}{d\tau}, \frac{dq_i}{d\tau} \right) d\tau = \int L \left(q_i, \frac{dq_i}{dt} \right) dt.$$

Thus, the action is the same whether it refers to L^* and τ or to L and t. This special case of the Hamiltonian formalism, where the Hamiltonian

$$H_0^* = p_0 \frac{dq_0}{d\tau} + p_i \frac{dq_i}{d\tau} - L^* \equiv 0,$$

is what is needed for a relativistic theory, because in such a theory we do not want to have one particular time playing a special role. Instead, we want to have the possibility of various times τ, which are all on the same footing. In reference [7], the reparametrization invariance treated as a gauge symmetry and a general structure of reparametrization and its connection with the zero Hamiltonian is established. Here and throughout the paper, Einstein's summation rule for repeated indices is used.

This paper is organized as follows. After this introductory section, a review of the preceding paper [1] is presented in Sec. 2. In Sec. 3, we discuss two physical applications: the first is an illustrative example of nonrelativistic dynamics; whereas the second is the motion of a well-known relativistic particle in an external electromagnetic field. Finally, the paper closes with a conclusion in Sec. 4.

2. Review of Previous Work

The starting point is the following. If we have a singular Lagrangian that has a Hessian matrix of rank $N - R$, $R < N$, then the set of Hamilton Jacobi partial differential equations (HJPDEs) is expressed as [8–12]

$$H'_\alpha \left(q_\beta, q_a, p_a = \frac{\partial S}{\partial q_a}, p_\beta = \frac{\partial S}{\partial q_\beta} \right) = 0; \tag{2.1}$$

$\alpha, = 0, N\text{-}R+1, \ldots, N$; $a=1, \ldots, N\text{-}R$,
where

$$H'_0 = p_0 + H_0;$$
$$H'_\mu = p_\mu + H_\mu.$$

Here, H'_μ represent the primary constraints, and S is the Hamilton-Jacobi function (action integral) to be determined.

In the preceding paper [1], we have proposed a general theory for solving this set of HJPDEs when they are separable in the form

$$S(q_a, q_\mu, t) = f(t) + W_a(E_a, q_a) + f_\mu(q_\mu) + A,$$

where E_a are the N-R constants of integration, and A is another constant.

However, for reparametrized systems, the constraint arises thanks to the fact that the time is somehow included in the configuration space by introducing the invariant parameter τ. Accordingly, Eq. (2.1) reduces to

$$H'_t = p_t + H_t = 0, \tag{2.2}$$

p_t being the generalized momentum associated with t and H_t of the originally noncovariant formulation.

In addition, the Hamilton-Jacobi function becomes

$$S(q_i, q_0) = f(q_0) + W_i(E_i, q_i) + A. \tag{2.3}$$

Once we have found S, the equations of motion can be obtained in the same manner as for regular systems, using the following canonical transformations [13, 14]:

$$\lambda_i = \frac{\partial S}{\partial E_i}; \tag{2.4a}$$

$$p_i = \frac{\partial S}{\partial q_i}, \tag{2.4b}$$

where λ_i are constants and can be determined from the initial conditions.

Equation (2.4a) can be solved to furnish q_i in terms of λ_i, E_i, and q_o:

$$q_i = q_i(\lambda_i, E_i, q_0);$$

and the momenta can be determined using Eq. (2.4b) as

$$p_i = p_i(\lambda_i, E_i, q_0).$$

Further, the connection between the classical and the quantum-mechanical equations of motion for constrained systems can readily be explored. It is well known that the H-J equation for unconstrained systems leads naturally to a semiclassical approximation; namely, WKB.

The Schrödinger equation in one dimension for a single particle in a potential $V(q)$ reads

$$i\hbar\frac{\partial\psi(q,t)}{\partial t} = \left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi(q,t).$$

Writing $\psi(q,t) = \exp\left(iS(q,t)/\hbar\right)$, and considering the expansion

$$S(q,t) = S_0 + \hbar S_1 + \hbar^2 S_2 + \ldots,$$

we get after some simplification

$$\psi(q_i,t) = \left[\prod_{i=1}^{N} \psi_{0i}(q_i)\right] \exp\left(\frac{i}{\hbar} S(q_i,t)\right). \tag{2.5}$$

This wave function represents our main result. In the semiclassical limit, it satisfies

$$\hat{H}'_t \psi = 0. \tag{2.6}$$

This condition is obtained when the dynamical coordinates and momenta are turned into their corresponding operators:

$$\begin{aligned} &q_i \to q_i; \\ &p_i \to \hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial q_i}; \\ &p_o \to \hat{p}_o = \frac{\hbar}{i}\frac{\partial}{\partial t}. \end{aligned}$$

3. Physical Applications

In this section, the quantization of totally constrained Hamiltonians, arising from the reparametrization of the system, will be investigated using the WKB approximation for two applications. The first is a simple illustrative model of the nonrelativistic motion of a particle in one-dimensional space. The second is the motion of a relativistic particle in an external electromagnetic field.

3.1. Nonrelativistic Dynamics as a Parametrized System

We first consider the following ordinary regular one-dimensional Lagrangian:

$$L = \frac{1}{2}\dot{q}^2 - V(q), \tag{3.1}$$

and introduce t as a function of τ, thereby yielding the two-dimensional singular Lagrangian

$$L^* = \frac{1}{2}\frac{\dot{q}^2}{\dot{t}} - V(q)\,\dot{t},$$

the dot now standing for $\frac{d}{d\tau}$.

From the definitions of momenta corresponding to this new Lagrangian, we have

$$p_q = \frac{\dot{q}}{\dot{t}}; \tag{3.2a}$$

$$p_t = -\frac{1}{2}\frac{\dot{q}^2}{\dot{t}^2} - V(q) = -\frac{1}{2}p_q^2 - V(q) = -H_t. \tag{3.2b}$$

Equation (3.2b) represents a primary constraint:

$$H'_t = p_t + \frac{1}{2}p_q^2 + V(q) = 0. \tag{3.3}$$

The canonical Hamiltonian H_0^* is identically zero:

$$H_0^* = p_t\dot{t} + p_q\dot{q} - L^* \equiv 0.$$

The corresponding HJPDE of the primary constraint Eq. (3.3) is

$$\frac{\partial S}{\partial q_0} + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + V(q) = 0, \tag{3.4}$$

since $q_0 = t$. Its solution is:

$$S = -Et + \int \sqrt{2(E-V)}\, dq + A, \tag{3.5}$$

where A is a constant.

Now, the equations of motion for q and p_q are

$$\lambda = \frac{\partial S}{\partial E} = -t + \int \frac{dq}{\sqrt{2(E-V)}},$$

which can be integrated to give

$$q = q(\lambda, E, t),$$

and

$$p_q = \frac{\partial S}{\partial q} = \sqrt{2(E-V)}.$$

Following the previous procedure for the quantization of constrained systems [1], we have the following wave function:

$$\Psi(q,t) = \frac{1}{\sqrt{p_q}} \exp\left(\frac{i}{\hbar}S\right). \tag{3.6}$$

In the classical limit $\hbar \to 0$, the primary constraint, Eq. (3.3), becomes a condition on this wave function:

$$H'_t\Psi = \left[\frac{\hbar}{i}\frac{\partial}{\partial t} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + V(q)\right]\Psi = 0. \tag{3.7}$$

To show this, we have

$$\begin{aligned}
\frac{\partial \Psi}{\partial t} &= -\frac{i}{\hbar}E\Psi;\\
\frac{\partial^2 \Psi}{\partial q^2} &= \left[\frac{1}{2}\frac{\partial^2 V}{\partial^2 q}\frac{1}{[2(E-V)]} + \frac{5}{4}\left(\frac{\partial V}{\partial q}\right)^2 \frac{1}{[2(E-V)]^2} - \frac{1}{\hbar^2}[2(E-V)]\right]\Psi.
\end{aligned}$$

Substituting this equation back in Eq. (3.7), then taking the classical limit $\hbar \to 0$, we get

$$H_t'\Psi = [-E + (E - V) + V]\Psi = 0.$$

Thus, the quantization procedure applied to the initial mechanical system, after promoting the time to become a dynamical variable, yields the correct equation for the wave function Ψ, (3.6), which is just the conventional time-dependent Schrödinger equation.

3.2. Motion of a Relativistic Particle in an External Electromagnetic Field

As a second application, let us consider a relativistic particle of charge e moving in an external electromagnetic field. The motion of such a particle is described by the following action [15, 16]:

$$S = \int L dt; \qquad L = -mc^2\sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c}\vec{u}.\vec{A} - e\Phi, \tag{3.8}$$

where $u = \frac{dq}{dt}$ is the velocity of the particle in a certain Lorentz frame, $\vec{A}$ is the vector potential, and Φ is the scalar potential.

In a covariant approach to the Lagrangian formulation, we introduce the particle's proper time τ into (3.8) through $dt = \gamma\, d\tau$. The action integral then becomes

$$S = \int \gamma\, L d\tau; \quad \gamma^{-1} = \sqrt{1 - \frac{u^2}{c^2}},$$

since the proper time is invariant. The condition that S also be invariant requires that $\gamma\, L$ be Lorentz invariant. In this case, the Lagrangian $L^* = \gamma\, L$ can be written in a covariant form as

$$L^* = -mc\sqrt{U_\alpha U^\alpha} - \frac{e}{c}U_\alpha A^\alpha, \quad \alpha = 0, 1, 2, 3, \tag{3.9}$$

where $U^\alpha = \left(U^0 = \gamma\, c, U^r = \gamma\, u^r\right)$ and $A^\alpha = \left(A^0 = \Phi, \vec{A}\right)$ are the four-vector velocity and four-vector potential, respectively; their covariant forms are $U_\alpha = (U_0 = \gamma\, c, U_r = \gamma\, u_r)$, $A_\alpha = \left(A^0 = \Phi, -\vec{A}\right)$, $r = 1, 2, 3$.

Now the generalized momenta read

$$p^\beta = \frac{\partial L^*}{\partial U^\beta} = -mcU_\beta[U_\alpha U^\alpha]^{-\frac{1}{2}} - \frac{e}{c}A_\beta; \quad \beta = 0, 1, 2, 3.$$

Because of the degeneracy of the Lagrangian, we cannot express all the components of velocity U^αvia p^α in terms of canonical variables; i.e., the components of momenta are not independent.

The time component of momenta is

$$p^0 = -mcU^0[U_\alpha U^\alpha]^{-\frac{1}{2}} - \frac{e}{c}A^0, \tag{3.10}$$

and the spatial components are

$$p^r = mcU^r[U_\alpha U^\alpha]^{\frac{-1}{2}} + \frac{e}{c}A^r; r = 1, 2, 3. \tag{3.11}$$

Making use of Eq. (3.11), one can solve for U_r in terms of p_r and U_0 as

$$U^r = \pm \frac{U_0 \left(p^r - \frac{e}{c}A^r\right)}{\left[\left|\vec{p} - \frac{e}{c}\vec{A}\right|^2 + m^2c^2\right]^{\frac{1}{2}}}.$$

If U^r is substituted into Eq. (3.10), we have

$$p_0 = -\left[\left|\vec{p} - \frac{e}{c}\vec{A}\right|^2 + m^2c^2\right]^{\frac{1}{2}} - \frac{e}{c}A_0 = -H_0,$$

which represents a primary constraint:

$$H' = p_0 + \sqrt{(\vec{p} - \frac{e}{c}\vec{A})^2 + m^2c^2} + \frac{e}{c}A_0 = 0. \tag{3.12}$$

This denotes the so-called "mass-shell condition", which represents the Klein-Gordon equation when quantized, as we shall see later.

The usual Hamiltonian H_0^* reads

$$H_0^* = p_\alpha U_\alpha - L^* = p_0 U_0 + p_r U_r - L^*.$$

In fact, direct calculations give

$$H_0^* = U_0 \left[\frac{m^2c^2 + \left|\vec{p} - \frac{e}{c}\vec{A}\right|^2}{\left[\left|\vec{p} - \frac{e}{c}\vec{A}\right|^2 + m^2c^2\right]^{\frac{1}{2}}} - \left[\left|\vec{p} - \frac{e}{c}\vec{A}\right|^2 + m^2c^2\right]^{\frac{1}{2}}\right] = 0.$$

This result, $H_0^* = 0$, reflects the fact that the system described by L^* does not have a unique time parameter. This system is time-reparametrization invariant.

We shall now examine the following two special cases.

Case 1: *Motion in a Uniform Electric Field*

Let us consider the motion of a particle initially at rest in a uniform electric field directed along the x-axis:

$$\vec{E} = E_0\hat{x},$$

where E_0 is a constant. In this case, the only nonzero component of A^α is

$$A^0 = \Phi = -\int E_0 dx = -E_0 x.$$

Thus, Eq. (3.12) becomes

$$H' = p_0 + \sqrt{p_x^2 + m^2c^2} - \frac{eE_0 x}{c} = 0, \tag{3.13}$$

and the corresponding HJPDE is

$$\frac{\partial S}{\partial x_0} + \sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + m^2c^2} - \frac{eE_0 x}{c} = 0.$$

Since $x_0 = ct$, the solution for this equation can be obtained as

$$S = f(t) + W(x, E).$$

With $f(t) =$ -Et, we find

$$\frac{\partial W}{\partial x} = \sqrt{\left(\frac{eE_0 x + E}{c}\right)^2 - m^2c^2};$$

or

$$W(x, E) = \int \sqrt{\left(\frac{eE_0 x + E}{c}\right)^2 - m^2c^2}\, dx.$$

Thus, the solution for S becomes

$$S = -Et + \int \sqrt{\left(\frac{eE_0 x + E}{c}\right)^2 - m^2c^2}\, dx.$$

The equations of motion for x and p_x can be determined as

$$\lambda_x = \frac{\partial S}{\partial E} = -t + \frac{1}{c}\int \frac{[(eE_0 x + E)/c]}{\sqrt{[(eE_0 x + E)/c]^2 - m^2c^2}} dx,$$

which can be integrated to give

$$x = \frac{c}{eE_0}\sqrt{[eE_0(\lambda_x + t)]^2 + m^2c^2} - \frac{E}{eE_0}.$$

The initial conditions imply that

$$x = \frac{c}{eE_0}\sqrt{(eE_0 t)^2 + m^2c^2} - \frac{mc^2}{eE_0}. \tag{3.14}$$

In addition, the solution for p_x is

$$p_x = \frac{\partial S}{\partial x} = \sqrt{\left(\frac{eE_0 x + E}{c}\right)^2 - m^2c^2}\,.$$

Substituting for x, we get

$$p_x = eE_0 t.$$

Now, the primary constraint (3.13) implies a condition on the wave function

$$\Psi = \frac{1}{\sqrt{p_x}} \exp\left[\frac{iS}{\hbar}\right]; \tag{3.15}$$

such that, in the semiclassical limit $\hbar \to 0$,

$$\hat{H}'\Psi = 0.$$

In fact, this is the Klein-Gordon equation in one-dimension.

Case 2: *Motion in a Uniform Magnetic Field*

Let us choose a constant magnetic field in the positive z-direction, $\vec{B} = B_0\hat{z}$. Since $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$, this choice of field gives the following components in cylindrical coordinates for the vector field:

$$A_\Phi = \frac{B_0\rho}{2}; A_0 = A_\rho = A_z = 0.$$

The primary constraint (3.12) can then be written as

$$p_0 + \sqrt{p_\rho^2 + p_z^2 + \left(p_\Phi - \frac{eB_0\rho}{2c}\right)^2 + m^2c^2} = 0. \tag{3.16}$$

Since the coordinates Φ and z are cyclic, the corresponding momenta p_z and p_Φ are constants of motion; call them α_z and α_Φ, respectively. This enables us to write the following HJPDE corresponding to (3.16):

$$\frac{\partial S}{\partial x_0} + \sqrt{\left(\frac{\partial S}{\partial \rho}\right)^2 + \alpha_z^2 + \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2 + m^2c^2} = 0.$$

Since $x_0 = cti$, the solution for this equation can be obtained as

$$S = f(t) + W_\rho(\rho, E) + \alpha_z z + \alpha_\Phi \Phi + A.$$

With $f(t) = -Et$, we find

$$\frac{\partial W_\rho}{\partial \rho} = \sqrt{\left(\frac{E'}{c}\right)^2 - \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2},$$

which has the solution

$$W_\rho = \int \sqrt{\left(\frac{E'}{c}\right)^2 - \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2}\, d\rho,$$

where $E' = \sqrt{E^2 - \alpha_z^2c^2 - m^2c^4}$.

Thus, the solution for S becomes

$$S = -Et + \alpha_z z + \alpha_\Phi \Phi + \int \sqrt{\left(\frac{E'}{c}\right)^2 - \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2} d\rho.$$

The equations of motion for ρ and p_ρ can now be determined as

$$\lambda_\rho = \frac{\partial S}{\partial E} = -t + \frac{E}{c^2} \int \frac{d\rho}{\sqrt{\left(\frac{E'}{c}\right)^2 - \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2}}.$$

This can be integrated directly to give

$$\lambda_\rho + t = -\frac{2E}{eB_0 c} \sin^{-1}\left[\frac{c}{E'}\left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)\right];$$

and ρ can be extracted from this equation as

$$\rho = \frac{2c}{eB_0}\left[\alpha_\Phi + \frac{E'}{c}\sin(\omega t + \lambda)\right], \tag{3.17}$$

where

$$\omega = \frac{eB_0 c}{2E}; \quad \lambda = \omega \lambda_\rho;$$

and

$$p_\rho = \frac{\partial S}{\partial \rho} = \sqrt{\left(\frac{E'}{c}\right)^2 - \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2}. \tag{3.18}$$

If we substitute for ρ, we get

$$p_\rho = \frac{E'}{c}\cos(\omega t + \lambda). \tag{3.19}$$

The solution for z can also be determined as

$$\lambda_z = \frac{\partial S}{\partial \alpha_z} = z - \alpha_z \int \frac{d\rho}{\sqrt{\left(\frac{E'}{c}\right)^2 - \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2}},$$

which, upon integration, gives

$$z = \lambda_z + \frac{\alpha_z c^2}{E}(\lambda_\rho + t). \tag{3.20}$$

Similarly, the solution for Φ is

$$\lambda_\phi = \frac{\partial S}{\partial \alpha_\phi} = \phi - \int \frac{\left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right) d\rho}{\sqrt{\left(\frac{E'}{c}\right)^2 - \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2}}.$$

After integration, we have

$$\phi = \lambda_\phi + \frac{2c}{eB_0}\sqrt{\left(\frac{E'}{c}\right)^2 - \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2};$$

or, using Eq. (3.19),

$$\phi = \frac{2E'}{eB_0}\cos(\omega t + \lambda) + \lambda_\phi. \tag{3.21}$$

These results are in agreement with those discussed in [11,12].

Now, the primary constraint (3.12) implies a condition on the wave function

$$\Psi(\rho, E, t) = \frac{1}{\sqrt{p_\rho}}\exp\left(\frac{iS}{\hbar}\right) = \left[\left(\frac{E'}{c}\right)^2 - \left(\alpha_\Phi - \frac{eB_0\rho}{2c}\right)^2\right]^{\frac{-1}{4}}\exp\left(\frac{iS}{\hbar}\right); \tag{3.22}$$

such that, in the semiclassical limit $\hbar \to 0$,

$$\hat{H}'\Psi = 0.$$

In fact, this is the well-known Klein-Gordon equation, which can be written as

$$\left[(i\hbar\partial_\mu - eA_\mu)^2 + m^2\right]\Psi = 0. \tag{3.23}$$

4. Conclusion

In this work we have extended the method of the quantization of constrained systems using the WKB approximation to be applicable for reparametrized systems, for which the promotion of time as a function of the proper-time leads to a constrained system with vanishing Hamiltonian. Thus, we have an extended phase space that includes a new coordinate, the time, whose conjugate momentum represents the total energy of the system.

Further, we have shown how the HJPDE can be set up for these reparametrized systems which enables us to determine the Hamilton-Jacobi function S. Then the equations of motion can be derived from this function in the same manner as for regular systems.

Moreover, the action integral can be employed to determine the wave function for these parametrized systems. We have shown that the constraints become conditions on the wave function to be satisfied in the semiclassical limit.

The success of our approach has been demonstrated for two applications. The first is an illustrative example in one-dimensional dynamics that describes the concept of parametrized systems. The second application is the well-known motion of a relativistic particle in an external electromagnetic field described by the usual reparametrization invariant action. In the second application we have shown how the proper-time formulation can be quantized to give the Klein-Gordon equation.

The challenge now is to think of further relativistic applications, especially the motion of a spinning particle in an external electromagnetic field.

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