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# A Study on Breit Interaction Through Heavy Mesons Spectra

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**Abstract**

In the present work the adequacy of the complete Breit interaction terms in describing the strong interaction can be tested through studying $c\bar{c}$ and $b\bar{b}$ spectra. Breit equation is solved using the complete Breit interaction form. The Breit equation can be transformed into three categories of differential equations according to the spectroscopic signature of the different states. The resonance masses of $c\bar{c}$ and $b\bar{b}$ states are calculated to order $\alpha^4$. One can see a good agreement between our results and the corresponding experimental data in all considered states.

**Key Words:** Quarkonium spectroscopy, Breit equation, Bound states.

## 1. Introduction

The investigation of the quarkonia properties in the framework of the quark model is an important problem in elementary particles physics. Comparison of the theoretical predictions with the corresponding experimental data of these systems has given reasonable information about the interacting potential between the particle system constituents. This information is of a great practical interest as it is not possible to obtain the potential of the interquark interaction through the whole range from the bases of quantum chromodynamics (QCD). As is also well known, the large value of the strong coupling constant makes a perturbation theory inapplicable. Therefore it becomes necessary to include non-perturbative effects connected with the complicated structure of the vacuum. This leads to a theoretical uncertainty in the quark potential at large, intermediate, and small distances [1]. Potential models [2–8], both non-relativistic as well as relativistic versions, have been successfully used to describe the different properties of these systems.

In this work the funnel potential beside Breit interaction terms are used to compute the energy levels of $c\bar{c}$ and $b\bar{b}$ systems.

## 2. Formalism

Breit equation has been extensively used in the past [9–11]. It describes the interaction between two fermion masses $m_1$ and $m_2$. This equation can be written as

$$\left[E-\gamma_0^{(1)}(\vec{\gamma}^{(1)}\cdot\vec{p}+m_1)-\gamma_0^{(2)}(-\vec{\gamma}^{(2)}\cdot\vec{p}+m_2)-V_{int}(\vec{r})\right]\Psi(\vec{r})=0, \tag{1}$$

where $E$ is the total energy in the center of mass system, $\gamma_0^{(i)}(\vec{\gamma}^{(i)}\cdot\vec{p}+m_i)$ is the Dirac operator of $i^{th}$ particle, the superscript $i(i=1,2)$ refers to the particles, and $V_{int}(\vec{r})$ is the interacting potential and is given by

$$V_{int}(r)=V_s(r)+V_v(r)+V_r(r), \tag{2}$$

where $V_s(r), V_v(r)$ and $V_r(r)$ refer to the scaler, vector and Breit interactions, respectively. These different interactions are defined as

$$V_s(r)=-(\gamma_0^{(1)}\otimes\gamma_0^{(2)})S(r) \tag{3}$$

$$V_v(r)=[(\gamma_0^{(1)}\gamma_u^{(1)})\otimes(i\gamma_0^{(2)}\gamma_u^{(2)})]_{u=0}V(r) \tag{4}$$

$$V_r(r)=-V(r)[\frac{\vec{\gamma}_0^{(1)}.\vec{\gamma}_0^{(2)}}{2}+\frac{(\vec{\gamma}^{(1)}.\vec{r})(\vec{\gamma}^{(2)}.\vec{r})}{2r^2}], \tag{5}$$

where $V(r)$and $S(r)$ have the form

$$V(r)=-\frac{4}{3}\frac{\alpha_s}{r} \quad \text{and} \quad S(r)=kr.$$

The form of equation (1) acquires dependence on the representation of the $\gamma-$ matrices, so we shall consider the Dirac-Pauli representation, for which the $\gamma-$ matrices are given by

$$\gamma_0=\beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \qquad \gamma_5=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad \vec{\gamma}=\begin{pmatrix}0&\vec{\sigma}\\-\vec{\sigma}&0\end{pmatrix}.$$

The wave vector $\Psi(\vec{r})$ is a sixteen-component wave function and it can be represented as a $4\times4$ matrix

$$\Psi(\vec{r})=\Psi_{\gamma_0^{(1)}\gamma_0^{(2)}}=\begin{pmatrix}\psi_{++}&\psi_{+-}\\\psi_{-+}&\psi_{--}\end{pmatrix}, \tag{6}$$

where the indices $+ve$ and $-ve$ refer to the eigenvalues $(+1,\ -1)$ of the Dirac matrices $\gamma_0^{(i)}$ for the so-called double Dirac representation [12–14]. It should be noticed that the left index refers to the first particle and the right index refers to the second particle. Inserting equations (2–5) into (1), one gets

$$E\begin{pmatrix}\psi_{++}&\psi_{+-}\\\psi_{-+}&\psi_{--}\end{pmatrix}-\sigma^{(1)}\cdot p\begin{pmatrix}\psi_{-+}&\psi_{--}\\\psi_{++}&\psi_{+-}\end{pmatrix}-m_1\begin{pmatrix}\psi_{++}&\psi_{+-}\\-\psi_{-+}&-\psi_{--}\end{pmatrix}+\sigma^{(2)}\cdot p\begin{pmatrix}\psi_{+-}&\psi_{++}\\\psi_{--}&\psi_{-+}\end{pmatrix}$$

$$-m_2\begin{pmatrix}\psi_{++}&-\psi_{+-}\\\psi_{-+}&-\psi_{--}\end{pmatrix}-V(r)\begin{pmatrix}\psi_{++}&\psi_{+-}\\\psi_{-+}&\psi_{--}\end{pmatrix}+V(r)\tfrac{H(r)}{2}\begin{pmatrix}\psi_{--}&\psi_{-+}\\\psi_{+-}&\psi_{++}\end{pmatrix}+S(r)-\begin{pmatrix}\psi_{++}&-\psi_{+-}\\\psi_{-+}&\psi_{--}\end{pmatrix}=0 \tag{7}$$

where,

$$H(r) = \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} + (\vec{\sigma}^{(1)} \cdot \vec{r})(\vec{\sigma}^{(2)} \cdot \vec{r}). \tag{8}$$

In order to simplify equation (7), one can introduce the following components [12]:

$$\begin{aligned}
&\left.\begin{matrix}\phi \\ \phi^0\end{matrix}\right\} = P_0 \tfrac{i}{\sqrt{2}}(\Psi_{++} \mp \Psi_{--}) \\
&\left.\begin{matrix}\chi \\ \chi^0\end{matrix}\right\} = P_0 \tfrac{i}{\sqrt{2}}(\Psi_{+-} \mp \Psi_{-+}) \\
&\left.\begin{matrix}\vec{\phi} \\ \vec{\phi}^0\end{matrix}\right\} = \vec{\phi}^0 = \tfrac{1}{2}(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})P_1 \tfrac{1}{\sqrt{2}}(\Psi_{+-} \pm \Psi_{-+}) \\
&\left.\begin{matrix}\vec{\chi} \\ \vec{\chi}^0\end{matrix}\right\} \tfrac{1}{2}(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})P_1 \tfrac{1}{\sqrt{2}}(\Psi_{++} \pm \Psi_{--}),
\end{aligned} \tag{9}$$

where $P_0 = \frac{1}{4}(1 - \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})$ and $P_1 = \frac{1}{4}(3 - \vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})$ are the projection operators of states with total spin $S = 0$ and $S = 1$, respectively; while $\vec{\sigma}^{(i)}$ is the Pauli matrix of the $i^{th}$-particle. The components $\phi, \phi^0$, $\chi,\ \chi^0$ and $\vec{\phi}, \vec{\phi}^0$, $\overleftarrow{\chi}, \vec{\chi}^0$ correspond to the scalar and vector states, respectively.

Using, the relations [15, 16]

$$\begin{aligned}
&P_0(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)}) = (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})P_1, \quad P_0(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}) = (\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})P_0 = 0 \\
&P_0(\sigma_i^{(1)}\sigma_k^{(2)} + \sigma_k^{(1)}\sigma_i^{(2)}) = -2\delta_{ik}P_0, \quad P_0(\sigma_l^{(1)} - \sigma_i^{(2)})(\sigma_k^{(1)} - \sigma_k^{(2)}) = 4P_0\delta_{ik} \\
&P_0(\sigma_i^{(1)}\sigma_k^{(2)} - \sigma_k^{(1)}\sigma_i^{(2)}) = iP_0 \in_{ikl} (\sigma_l^{(1)} - \sigma_l^{(2)}), \\
&P_0(\sigma_l^{(1)} - \sigma_i^{(2)})(\sigma_k^{(1)} + \sigma_k^{(2)}) = 2iP_0 \in_{ikl} (\sigma_l^{(1)} - \sigma_l^{(2)}) \\
&P_0(\vec{\sigma}^{(1)}.\vec{\sigma}^{(2)}) = -3P_0, \quad P_0(\vec{\sigma}^{(1)}.\hat{r})(\vec{\sigma}^{(2)}.\hat{r}) = -P_0 \\
&(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})P_1(\vec{\sigma}^{(1)}.\vec{\sigma}^{(2)}) = (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})P \\
&(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})P_1[(\vec{\sigma}^{(1)}.\hat{r})(\vec{\sigma}^{(2)}.\hat{r})] = (\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})P_1 - 2\hat{r}[\hat{r}.(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})P_1]
\end{aligned}$$

equation (7) are rewritten in the following form:

$$
\begin{aligned}
&\tfrac{1}{2}[E+S-3V]\phi_0 - \tfrac{m_1+m_2}{2}\phi - i\vec{P}.\vec{\phi} = 0\\
&\tfrac{1}{2}[E+S+V]\phi - \tfrac{m_1+m_2}{2}\phi^0 = 0\\
&\tfrac{1}{2}[E-S-V]\chi^0 - \tfrac{m_1-m_2}{2}\chi - i\vec{P}.\vec{\chi} = 0\\
&\tfrac{1}{2}[E-S+V]\chi - \tfrac{m_1-m_2}{2}\chi^0 = 0\\
&\tfrac{1}{2}[E+S]\vec{\chi} - \tfrac{V}{2}\hat{r}(\hat{r}.\vec{\chi}) - \tfrac{m_1+m_2}{2}\vec{\chi}^0 + i\vec{P}.\chi^0 = 0\\
&\tfrac{1}{2}[E+S-2V]\vec{\chi}^0 + \tfrac{V}{2}\hat{r}(\hat{r}.\vec{\chi}^0) - \tfrac{m_1+m_2}{2}\vec{\chi} + i\vec{P}\times\vec{\phi}^0 = 0\\
&\tfrac{1}{2}[E-S]\vec{\phi} - \tfrac{V}{2}\hat{r}(\hat{r}.\vec{\phi}) - \tfrac{m_1-m_2}{2}\vec{\phi}^0 + i\vec{P}.\phi^0 = 0\\
&\tfrac{1}{2}[E-S-2V]\vec{\phi}^0 + \tfrac{V}{2}\hat{r}(\hat{r}.\vec{\phi}^0) - \tfrac{m_1-m_2}{2}\vec{\phi} + i\vec{P}\times\vec{\chi}^0 = 0
\end{aligned}
\tag{10}
$$

To separate the angular dependence from the radial one, the following relations should be used [12–13]:

$$
\hat{r}\times\vec{L} = i\frac{\partial}{\partial\hat{r}_\perp}, \quad \vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\hat{r}_\perp}, \quad \hat{r}\times\frac{\partial}{\partial\hat{r}_\perp} = i\vec{L}, \quad \frac{\partial}{\partial\hat{r}_\perp}\cdot\hat{r} = 2,
$$

$$
\frac{\partial}{\partial\hat{r}_\perp}\times\frac{\partial}{\partial\hat{r}_\perp} = \hat{r}\times\frac{\partial}{\partial\hat{r}_\perp}, \quad \hat{r}\times\hat{(}\hat{r}\times\frac{\partial}{\partial\hat{r}_\perp}) = -\frac{\partial}{\partial\hat{r}_\perp}, \quad \hat{r}\cdot\frac{\partial}{\partial\hat{r}_\perp} = 0 \tag{11}
$$

At the same time, we shall introduce the expansions of the wave functions given in equation (9) as follows:
(i) For scalar functions ($S=0$),

$$
B(\vec{r}) = \sum_{j,m}\beta(r)Y_{jm}(\hat{r}_\perp), \tag{12}
$$

where $B(\vec{r})$ stand for any of the four $S=0$ states, $\beta(r)$ is the radial part and $Y_{jm}(\hat{r}_\perp)$ refers to the angular spherical harmonics dependence.

(ii) For vector states ($S=1$),

$$
\vec{A}(\vec{r}) = \hat{r}A_e(\vec{r}) - \frac{\partial}{\partial\hat{r}_\perp}\frac{A_l(\vec{r})}{J(J+1)} - \hat{r}\times\frac{\partial}{\partial\hat{r}_\perp}\frac{A_m(\vec{r})}{J(J+1)} \tag{13}
$$

where, $A_i(\vec{r}) = \sum\limits_{j,m} A_i(r)Y_{jm}(\hat{r}_\perp)$, $(i=e,l,m)$,
which represent the electric, longitudinal and magnetic parts, respectively.

In addition, let us define the spin operator $S_k$, which acts on one component of any of the four vector components as follows:

$$
S_k\vec{A}(\vec{r})|_i = \frac{1}{2}P_0(\vec{\sigma}^{(1)} - \vec{\sigma}^{(2)})_i\frac{1}{\sqrt{2}}(\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)})_k\frac{1}{2}(\cdots)
$$

where the $(\cdots)$ refers to $(\Psi_{+-}+\Psi_{-+})$ for $\vec{\phi}$, $(\Psi_{+-}-\Psi_{-+})$ for $\vec{\phi}^{0}$, and so on, which means that $S_k\vec{A}(\vec{r})|_i = i\in_{ikl} A_l$. Also, we can define the total angular momentum $\vec{J}=\vec{L}+\vec{S}$ as acting on only one component of the vector state

$$J_k\vec{A}_e(\vec{r})|_i = S_k\vec{A}_e(\vec{r})|_i + L_k\vec{A}_e(\vec{r})|_i = -i\in_{kla}\hat{r}_l\hat{r}_iA_e(r)\frac{\partial Y_{jm}}{\partial \hat{r}_{\perp a}} = \hat{r}_iL_kA_e(r)Y_{jm}$$

so that

$$J^2\vec{A}_e(\vec{r}) = \hat{r}L^2A_e(\vec{r}) = J(J+1)\vec{A}_e(\vec{r}),$$

which implies that $J$ is the total angular momentum.

As mentioned before, one can eliminate the angular dependence by inserting equations (11–13) into equations (10), with which one gets a set of sixteen radial equations for the functions $\phi$, $\phi^0$, $\phi_e$, $\phi_e^0$, $\phi_m$, $\phi_m^0$, $\phi_l$, $\phi_l^0$, $\chi$, $\chi^0$, $\chi_e$, $\chi_e^0$, $\chi_m$, $\chi_m^0$, $\chi_l$ and $\chi_l^0$. Also, for any vector state (suppose $\chi$) there are three states characterized by the spectroscopic signature $^3(J+1)_J$, $^3(J-1)_J$ and $^3J_J$. These states are described by the wave function $\chi_{L=J-1}$, $\chi_{L=J+1}$ and $\chi_{L=J}$, respectively, which satisfy the relations [13–14]

$$\begin{aligned}
\chi_e &= \sqrt{\tfrac{J}{2J+1}}\chi_{L=J-1} + \sqrt{\tfrac{J+1}{2J+1}}\chi_{L=J+1}\\
\chi_l &= J(J+1)\left(\sqrt{\tfrac{J+1}{2J+1}}\chi_{L=J-1} + \sqrt{\tfrac{J}{2J+1}}\chi_{L=J+1}\right)\\
\chi_m &= J(J+1)\chi_{L=J}
\end{aligned} \tag{14}$$

Now it is easy to see that $\phi$, $\phi^0$, $\phi_e$, $\phi_e^0$, $\phi_l$, $\phi_l^0$, $\chi_m$, and $\chi_m^0$ have parity $P=\eta(-)^J$, while the components $\phi_m$, $\phi_m^0$, $\chi$, $\chi^0$, $\chi_e$, $\chi_e^0$, $\chi_l$ and $\chi_l^0$ have parity $P=-\eta(-)^J$, where $\eta$ is the intrinsic parity $+1$ for two-fermion or two-antifermion, and $-1$ for fermion-antifermion systems). In addition, the first group can be classified into two subgroups: the first one ($\phi$, $\phi^0$, $\phi_e$ and $\phi_l$) describes states of spectroscopic signature $^1J_J$ and the second one ($\phi_e^0$, $\phi_l^0$, $\chi_m$ and $\chi_m^0$) describes states of spectroscopic signature $^3J_J$. The second group ($\phi_m$, $\phi_m^0$, $\chi$, $\chi^0$, $\chi_e$, $\chi_e^0$, $\chi_l$ and $\chi_l^0$) describes states of spectroscopic signature $^3(J\pm1)_J$.

According to the above classification, the radial equations can be rewritten as

(i) Equations that describe states $^1J_J$ with $P=\eta(-)^J$:

$$\begin{aligned}
&\frac{1}{2}(E+S-3V)\phi^0 - m\phi - \left(\frac{d}{dr}+\frac{2}{r}\right)\phi_e - \frac{1}{r}\phi_l = 0\\
&\frac{1}{2}(E+S+V)\phi - m\phi^0 = 0\\
&\frac{1}{2}(E+S-V)\chi_e - m\chi_e^0 + \frac{d\chi^0}{dr} = 0\\
&\frac{1}{2}(E-S)\phi_l - \frac{J(J+1)}{r}\phi^0 = 0
\end{aligned} \tag{15}$$

(ii) Equations that describe states $^3J_J$ with $P=\eta(-)^J$:

$$
\begin{aligned}
&\frac{1}{2}(E+S)\chi_m - m\chi_m^0 = 0\\
&\frac{1}{2}(E+S-2V)\chi_m^0 - m\chi_m + \left(\frac{d}{dr}+\frac{1}{r}\right)\phi_l^0 + \frac{J(J+1)}{r}\phi_e^0 = 0\\
&\frac{1}{2}(E-S-V)\phi_e^0 + \frac{1}{r}\chi_m = 0\\
&\frac{1}{2}(E-S-2V)\phi_l^0 - \left(\frac{d}{dr}+\frac{1}{r}\right)\chi_m^0 = 0
\end{aligned}
\tag{16}
$$

(iii) Equations that describe states ${}^3(J\pm1)_J$ with $P=-\eta(-)^J$:

$$
\begin{aligned}
&\tfrac{1}{2}(E-S-3V)\chi^0 - \left(\tfrac{d}{dr}+\tfrac{2}{r}\right)\chi_e - \tfrac{1}{r}\chi_l = 0\\
&\frac{1}{2}(E+S-V)\chi_e - m\chi_e^0 + \frac{d\chi^0}{dr} = 0\\
&\frac{1}{2}(E+S)\chi_l - m\chi_l^0 - \frac{J(J+1)}{r}\chi^0 = 0\\
&\frac{1}{2}(E+S-V)\chi_e^0 - m\chi_e + \frac{1}{r}\phi_m^0 = 0\\
&\frac{1}{2}(E+S-2V)\chi_l^0 - m\chi_l - \left(\frac{d}{dr}+\frac{1}{r}\right)\phi_m^0 = 0\\
&\frac{1}{2}(E-S-2V)\phi_m^0 + \left(\frac{d}{dr}+\frac{1}{r}\right)\chi_l^0 + \frac{J(J+1)}{r}\chi_e^0 = 0
\end{aligned}
\tag{17}
$$

It should be noticed that the components $\chi$ and $\phi$ vanish in the event $m=m_1=m_2$. On the other hand, equations (15–17) can be reduced to the following differential equations:

States with $J=L$ and $S=0$:

$$
\begin{aligned}
&\frac{d^2\phi^0}{dr^2} + \left[\frac{2}{r} - \frac{k-\frac{4}{3}\frac{\alpha_s}{r^2}}{E+kr+\frac{4}{3}\frac{\alpha_s}{r}}\right]\frac{d\phi^0}{dr} + \left[-\frac{J(J+1)}{r^2}\frac{E+kr+\frac{4}{3}\frac{\alpha_s}{r}}{(E+kr)}\right.\\
&\left.+\frac{1}{4}\left\{\left(E+\frac{4}{3}\frac{\alpha_s}{r}\right)^2 - k^2r^2\right\} - m^2\frac{E+kr+\frac{4}{3}\frac{\alpha_s}{r}}{E-kr-\frac{4}{3}\frac{\alpha_s}{r}} + \frac{2}{3}\frac{\alpha_s}{r}(E+kr+\frac{4}{3}\frac{\alpha_s}{r})\right]\phi^0 = 0
\end{aligned}
\tag{18}
$$

States with $J=L$, $S=1$:

$$
\begin{aligned}
&\frac{d^2\chi_m^0}{dr^2} + \left[\frac{2}{r} - \frac{k-\frac{8}{3}\frac{\alpha_s}{r^2}}{E+kr+\frac{8}{3}\frac{\alpha_s}{r}}\right]\frac{d\chi_m^0}{dr} + \left[-\frac{J(J+1)}{r^2}\left\{1+\frac{\frac{4}{3}\frac{\alpha_s}{r}}{E+kr+\frac{4}{3}\frac{\alpha_s}{r}}\right\}\right.\\
&\left.+\frac{1}{4}\left\{(E+\frac{8}{3}\frac{\alpha_s}{r})^2 - k^2r^2\right\} - m^2\frac{E+kr+\frac{8}{3}\frac{\alpha_s}{r}}{E-kr} - \frac{1}{r}\frac{k-\frac{8}{3}\frac{\alpha_s}{r^2}}{E+kr+\frac{8}{3}\frac{\alpha_s}{r}}\right]\chi_m^0 = 0
\end{aligned}
\tag{19}
$$

States with $J=L\pm1$, $S=1$

$$\frac{d^2\chi_l^0}{dr^2}+\left[\frac{2}{r}-\frac{k-\frac{8}{3}\frac{\alpha_s}{r^2}}{E+kr+\frac{8}{3}\frac{\alpha_s}{r}}\right]\frac{d\chi_l^0}{dr}+\left[\frac{E+kr+\frac{8}{3}\frac{\alpha_s}{r}}{E-kr}\left(\frac{1}{4}(E-kr+\frac{8}{3}\frac{\alpha_s}{r})(E-kr-m^2)\right)\right.$$
$$\left.-\frac{J(J+1)}{r^2}\frac{(E+\frac{8}{3}\frac{\alpha_s}{r})^2-k^2r^2}{(E-kr)(E+kr+4\frac{\alpha_s}{r})}-\frac{k-\frac{8}{3}\frac{\alpha_s}{r^2}}{(E+kr+\frac{8}{3}\frac{\alpha_s}{r})}\right]\chi_l^0+\left[\frac{J(J+1)}{r^2}\left(1-\frac{(E-kr+\frac{4}{3}\frac{\alpha_s}{r})(E+kr+\frac{8}{3}\frac{\alpha_s}{r})}{(E-kr)(E+kr+4\frac{\alpha_s}{r})}\right)\right.$$
$$\frac{d\chi_e^0}{dr}+\left[\frac{J(J+1)}{r}\frac{E+kr+\frac{8}{3}\frac{\alpha_s}{r}}{(E-kr)(E+kr+4\frac{\alpha_s}{r})}\right.$$
$$\left.\left(k+\frac{4}{3}\frac{\alpha_s}{r^2}-\frac{2}{r}(E-kr+\frac{4}{3}\frac{\alpha_s}{r})\right)-\frac{J(J+1)}{r}\frac{E-\frac{8}{3}\frac{\alpha_s}{r^2}}{E+kr+\frac{8}{3}\frac{\alpha_s}{r}}\right]\chi_e^0=0 \text{ and} \tag{20}$$

$$\frac{d^2\chi_e^0}{dr^2}+\left[\frac{2}{r}-\frac{k-4\frac{\alpha_s}{r^2}}{E+kr+4\frac{\alpha_s}{r}}-\frac{2(k+\frac{4}{3}\frac{\alpha_s}{r^2})}{E-kr+\frac{4}{3}\frac{\alpha_s}{r}}\right]\frac{d\chi_e^0}{dr}+\left[\frac{k+\frac{4}{3}\frac{\alpha_s}{r^2}}{E-kr+\frac{4}{3}\frac{\alpha_s}{r}}\{\frac{k+4\frac{\alpha_s}{r^2}}{E+kr+4\frac{\alpha_s}{r}}-\frac{2}{r}\}\right.$$
$$-\frac{2}{r^2}-\frac{J(J+1)}{r^2}\frac{E+kr+4\frac{\alpha_s}{r}}{E+kr+\frac{8}{3}\frac{\alpha_s}{r}}-\frac{2}{r}\frac{k+4\frac{\alpha_s}{r^2}}{E-kr+4\frac{\alpha_s}{r}}-\frac{1}{E-kr+\frac{4}{3}\frac{\alpha_s}{r}}$$
$$\left.\left\{-\frac{8}{3}\frac{\alpha_s}{r^3}+m^2(E+kr+4\frac{\alpha_s}{r})\right\}+\frac{1}{4}(E+kr+4\frac{\alpha_s}{r})(E-kr+\frac{4}{3}\frac{\alpha_s}{r})\right]\chi_e^0$$
$$+\left[\frac{E-kr+\frac{8}{3}\frac{\alpha_s}{r}}{r(E-kr+\frac{4}{3}\frac{\alpha_s}{r})}-\frac{E+kr+4\frac{\alpha_s}{r}}{r(E+kr+\frac{8}{3}\frac{\alpha_s}{r})}\right]\frac{d\chi_l^0}{dr} \tag{21}$$
$$+\left[-\frac{1}{r(E-kr+\frac{4}{3}\frac{\alpha_s}{r})}\left(k+\frac{8}{3}\frac{\alpha_s}{r^2}+\frac{(k-4\frac{\alpha_s}{r^2})(E-kr+\frac{8}{3}\frac{\alpha_s}{r})}{E+kr+\frac{8}{3}\frac{\alpha_s}{r}}\right)\right.$$
$$\left.-\frac{E-kr+\frac{8}{3}\frac{\alpha_s}{r}}{r^2(E-kr+\frac{4}{3}\frac{\alpha_s}{r})}-\frac{E+kr+4\frac{\alpha_s}{r}}{r^2(E+kr+\frac{8}{3}\frac{\alpha_s}{r})}\right]\chi_l^0=0\,.$$

## 3. Results and Discussion

In the present work the adequacy of the complete Breit interaction in describing the strong interaction is tested through studying $c\bar{c}$ and $b\bar{b}$ spectra. The Breit equation is solved using the complete Breit interaction form, whereby the equation is transformed into three categories of differential equations (18–21) and classified by spectroscopic signature. From these equations one can notice that the funnel interaction contain two singularities: one at $r=0$ and the other at $r=\infty$. These two points are outside the interested range. Beside of these two singularities, there is in equation (18) an additional singular point, which arises from the non-perturbative treatment of Breit interaction, at $r \prec 1.0$ fm. For this reason the energy levels given by the solution of that equation are excluded in this study. On the other hand, in equations (19)–(21) there are singular points at $r \succ 5$ fm, which is out of the range of the interacting systems.

Using the Rune-Kutta method, equations (19)–(21) have been solved numerically. This is a bound state problem for which the initial value of the wave function and its derivative can be obtained as an expansion from the asymptotic behavior of the differential equations at $r \to 0$ to start the calculation. At the same time, the behavior of the wave function at $r \to \infty$ looks like $\exp(-\frac{1}{4}kr^2)$. Thus the wave function can be estimated between these boundaries.

Tisibidis [16] has applied Breit equation on a quarkonium system. He solved Breit equation taking into account the funnel potential and considered the Breit interaction as a perturbed term.

In the present work the funnel and Breit interactions are used to solve the Breit equation exactly without any perturbation treatment. It is worthwhile to mention that equations (19)–(20) contain only four fitting parameters namely, the quark masses $(m_c, m_b)$, the strong coupling constant $\alpha_s$ and the string constant $k$. The best values of the fitting parameters are given in Table 1.

Using these values the error width of the energies are calculated. A comparison between our results with both that given in reference [16] and the corresponding experimental data is given in Table 2.

**Table 1.** Parameter values used in the present calculations.

| Parameter | Value |
|---|---|
| $k$ | $0.129 \pm 0.001$ Gev$^2$ |
| $\alpha_s$ | $0.4775 \pm 0.0005$ |
| $m_c$ | $1.362 \pm 0.18$ GeV |
| $m_b$ | $4.72 \pm 0.19$ GeV |

**Table 2.** Calculated masses of different $c\bar{c}$ and $b\bar{b}$ states in comparison with Ref. [16] and the corresponding experimental data in [19].

| State | | $J^{PC}$ | Present | Ref. [16] | Expt. [19] |
|---|---|---|---|---|---|
| $c\bar{c}$ | $J/\psi(1S)$ | $1^-$ | $3098.9 \pm 2.1$ | $3096.87 \pm 0.04$ | $3096.87 \pm 0.04$ |
| | $\psi(2S)$ | $1^-$ | $3683 \pm 3$ | $3685.6 \pm 0.09$ | $3685.6 \pm 0.09$ |
| | $\psi(3S)$ | $1^-$ | $3768 \pm 1$ | — | $3769.9 \pm 2.5$ |
| | $\psi(4S)$ | $1^-$ | $4044.5 \pm 4.5$ | $4106 \pm 10$ | $4040 \pm 10$ |
| | $\psi(5S)$ | $1^-$ | $4181 \pm 3$ | — | $4159 \pm 20$ |
| | $\psi(6S)$ | $1^-$ | $4419.5 \pm 4.5$ | $4454 \pm 8$ | $4415 \pm 6$ |
| | $\chi_{c0}(1P)$ | $0^{++}$ | $3415.5 \pm 0.5$ | $3422 \pm 1$ | $3415 \pm 0.8$ |
| | $\chi_{c1}(1P)$ | $1^{++}$ | $3510.25 \pm 0.25$ | $3510.51 \pm 0.12$ | $3510.51 \pm 0.12$ |
| | $\chi_{c2}(1P)$ | $2^{++}$ | $3561 \pm 5$ | $3547.9 \pm 0.2$ | $3556.18 \pm 0.13$ |
| $b\bar{b}$ | $\gamma(1S)$ | $1^-$ | $9461.7 \pm 1.4$ | $9460.3 \pm 0.26$ | $9460.3 \pm 0.26$ |
| | $\gamma(2S)$ | $1^-$ | $10030 \pm 7$ | $10023.26 \pm 0.31$ | $10023.26 \pm 0.31$ |
| | $\gamma(3S)$ | $1^-$ | $10354.6 \pm 0.6$ | $10355.2 \pm 0.5$ | $10355.2 \pm 0.5$ |
| | $\gamma(4S)$ | $1^-$ | $10580 \pm 0.5$ | $10580.0 \pm 3.5$ | $10580.0 \pm 3.5$ |
| | $\gamma(5S)$ | $1^-$ | $10862 \pm 3$ | $10882 \pm 8$ | $10865 \pm 8$ |
| | $\gamma(6S)$ | $1^-$ | $11016.5 \pm 3$ | $11037 \pm 9$ | $11019 \pm 8$ |
| | $\chi_{b0}(1P)$ | $0^{++}$ | $9857 \pm 3$ | $9858.9 \pm 0.4$ | $9859.9 \pm 1$ |
| | $\chi_{b1}(1P)$ | $1^{++}$ | $9890.5 \pm 1.75$ | $9892.7 \pm 0.6$ | $9892.7 \pm 0.6$ |
| | $\chi_{b2}(1P)$ | $2^{++}$ | $9908.8 \pm 3.8$ | $9914.7 \pm 0.9$ | $9912.6 \pm 0.5$ |
| | $\chi_{b0}(2P)$ | $0^{++}$ | $10231.5 \pm 0.5$ | $10234.3 \pm 0.2$ | $10232.1 \pm 0.6$ |
| | $\chi_{b1}(2P)$ | $1^{++}$ | $10253.6 \pm 1.6$ | $10255.2 \pm 0.5$ | $10255.2 \pm 0.5$ |
| | $\chi_{b2}(2P)$ | $2^{++}$ | $10264.75 \pm 3.75$ | $10266.9 \pm 0.7$ | $10268.5 \pm 0.4$ |

The obtained mass spectrum using the Breit equation, in case of $c\bar{c}$, is satisfactory for $\psi(3770)$, $\psi(4040)$, $\psi(4160)$ and $\psi(4415)$ assigned as$\psi(3S)$, $\psi(4S)$, $\psi(5S)$ and $\psi(6S)$, respectively. It may be interesting to note that, these assignments lead to an anomalous mass relation: that $m(3S) - m(2S)$ is smaller than $m(4S) - m(3S)$ and $m(5S) - m(4S)$ is smaller than $m(6S) - m(5S)$. These anomalous mass relations may imply that in the energy region just above thresholds of many opened channels (e.g., in 3.7–4.4 GeV for $c\bar{c}$). The resonance masses can be significantly distorted. These difficulties can be due to other possibilities such as $c\bar{c}q\bar{q}$ or $c\bar{c}g$ states [17, 18].

From the present work one can conclude that, the resonance masses of $c\bar{c}$ and $b\bar{b}$ states can be reproduced successfully by using the funnel potential and the complete Breit interaction without any perturbative treatment.

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