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Projective Geometry and $\mathcal{PT}$-Symmetric Dirac Hamiltonian

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Abstract

The $(3+1)$-dimensional (generalized) Dirac equation is shown to have the same form as the equation expressing the condition that a given point lies on a given line in 3-dimensional projective space. The resulting Hamiltonian with a γ_5 mass term is not Hermitian, but is invariant under the combined transformation of parity reflection $\mathcal{P}$ and time reversal $\mathcal{T}$. When the $\mathcal{PT}$ symmetry is unbroken, the energy spectrum of the free spin-$\frac{1}{2}$ theory is real, with an appropriately shifted mass.

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I. INTRODUCTION AND SUMMARY

Conventional quantum mechanics requires the Hamiltonian H of any physical system be Hermitian (transpose + complex conjugation) so that the energy spectrum is real. But as shown in the seminal paper by Bender and Boettcher [1], there is an alternative formulation of quantum mechanics in which the requirement of Hermiticity is replaced by the condition of space-time $\mathcal{PT}$ reflection symmetry. (For a recent review, see Ref. [2].) If H has an unbroken $\mathcal{PT}$ symmetry, then the energy spectrum is real. Examples of $\mathcal{PT}$-symmetric non-Hermitian quantum-mechanical Hamiltonians include the class of Hamiltonians with *complex* potentials: $H = p^2 + x^2(ix)^\epsilon$ with $\epsilon > 0$. Incredibly the energy levels of these Hamiltonians turn out to be real and positive. [1] Now Hermiticity is an algebraic requirement whereas the condition of $\mathcal{PT}$ symmetry appears to be more geometric in nature. Thus one may wonder whether a purely geometric consideration can naturally lead to a Hamiltonian which is $\mathcal{PT}$-symmetric rather than its Hermitian counterpart. In this note we provide one such example.

In the next section, we "derive" the $(3+1)$-dimensional Dirac equation from a consideration of the condition that a given point lies on a given line in 3-dimensional projective space. By associating the (homogeneous) coordinates of the point with the Dirac spinor components $\psi(\mathbf{x}, t)$, and the coordinates of the line with the four-momentum and two real mass parameters m_1 and m_2 of the Dirac particle, we are led to an equation taking on the form of a generalized Dirac equation with Hamiltonian density

$$\mathcal{H}(\mathbf{x}, t) = \bar{\psi}(\mathbf{x}, t)(-i\not{\nabla} + m_1 + m_2\gamma_5)\psi(\mathbf{x}, t) \quad (m_2 \text{ real}). \tag{1}$$

As noted in Ref. [3], the Hamiltonian $H = \int d\mathbf{x}\,\mathcal{H}(\mathbf{x}, t)$ associated with the above $\mathcal{H}$ is not Hermitian but is invariant under combined $\mathcal{P}$ and $\mathcal{T}$ reflection. For $\mu^2 \equiv m_1^2 - m_2^2 \geq 0$, it is equivalent to a Hermitian Hamiltonian for the conventional free fermion field theory with mass μ. Studies of spin-$\frac{1}{2}$ theories in the framework of projective geometry have been undertaken before. See, e.g., Ref. [4]. [1] But the idea that there may be a natural *connection*

[1] These papers are rather mathematical and technical. The authors of the first two papers discuss the Dirac equation in terms of the Plucker-Klein correspondence between lines of a three-dimensional projective space and points of a quadric in a five-dimensional projective space. The last paper shows that the Dirac equation bears a certain relation to Kummer's surface, viz., the structure of the Dirac ring of matrices is

between the projective geometrical approach (perhaps also other geometrical approaches) and $\mathcal{PT}$-symmetric Hamiltonians as pointed out in this note appears to be novel.

II. PROJECTIVE GEOMETRY AND $\mathcal{PT}$-SYMMETRIC DIRAC EQUATION

It is convenient to use homogeneous coordinates to express the geometry in a projective space. [5] A point $\mathbf{x} \equiv (x, y, z)$ in three-dimensional Euclidean space can be expressed by the ratios of four coordinates (x_1, x_2, x_3, x_4) which are called the homogenous coordinates of that point. One possible definition of (x_1, x_2, x_3, x_4), in terms of $\mathbf{x}$ is $x_1 = \frac{x}{d}, x_2 = \frac{y}{d}, x_3 = \frac{z}{d}, x_4 = \frac{1}{d}$, with d being the distance of the point from the origin. Obviously, for any constant c, (cx_1, cx_2, cx_3, cx_4) and (x_1, x_2, x_3, x_4) represent the same point $\mathbf{x}$.

Consider the line through two points (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4). For (x_1, x_2, x_3, x_4) to lie on that line, the following determinant has to vanish, for any (r_1, r_2, r_3, r_4),

$$\begin{vmatrix} a_1 & b_1 & x_1 & r_1 \\ a_2 & b_2 & x_2 & r_2 \\ a_3 & b_3 & x_3 & r_3 \\ a_4 & b_4 & x_4 & r_4 \end{vmatrix} = 0. \tag{2}$$

This gives, for any r_1,

$$\begin{vmatrix} a_2 & b_2 & x_2 \\ a_3 & b_3 & x_3 \\ a_4 & b_4 & x_4 \end{vmatrix} = 0. \tag{3}$$

With the aid of the Plucker coordinates of the line defined by

$$p_{ij} = -p_{ji} \equiv \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}, \tag{4}$$

Eq. (3) can be written as

$$p_{34}x_2 - p_{24}x_3 + p_{23}x_4 = 0. \tag{5}$$

related to that of Kummer's 16_6 configuration. All these authors, explicitly or implicitly, put one of the two masses, viz., m_2 in (1), to be zero by hand. In this note, we "derive" the generalized Dirac equation from the projective geometrical approach in a relatively simple way and point out that there is no need to put $m_2 = 0$ and perhaps it is even natural to keep *both* masses m_1 and m_2.

Similarly, for any r_2, r_3, r_4, the following equations respectively must hold

$$
\begin{aligned}
p_{41}x_3 - p_{31}x_4 + p_{34}x_1 &= 0, \\
p_{12}x_4 - p_{42}x_1 + p_{41}x_2 &= 0, \\
p_{23}x_1 - p_{13}x_2 + p_{12}x_3 &= 0.
\end{aligned} \tag{6}
$$

Note that the Plucker line coordinates are not independent; the identical relation that connects them can be found by expanding the determinant

$$
\begin{vmatrix} a_1 & b_1 & a_1 & b_1 \\ a_2 & b_2 & a_2 & b_2 \\ a_3 & b_3 & a_3 & b_3 \\ a_4 & b_4 & a_4 & b_4 \end{vmatrix} = 0, \tag{7}
$$

from which

$$
p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0. \tag{8}
$$

Next, we relabel the homogeneous coordinates (x_1, x_2, x_3, x_4) as the four Dirac spinor components ψ. Let us first use the Dirac representation for the 4×4 Dirac matrices

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{9}
$$

where 0 is a 2×2 zero matrix, 1 is a 2×2 unit matrix, and σ are the three 2×2 Pauli matrices. Let us further write the six Plucker coordinates (under the so-called Klein transformation) in terms of p_μ with μ running over $0, 1, 2, 3$ (to be interpreted as the four-momentum of the Dirac particle) and m_1 and m_2 (to be interpreted as two real mass parameters) as follows:

$$
\begin{aligned}
p_{34} &= +p_0 + m_1, \quad & p_{12} &= -p_0 + m_1, \\
p_{13} &= +p_1 - ip_2, \quad & p_{24} &= -p_1 - ip_2, \\
p_{41} &= +p_3 + m_2, \quad & p_{23} &= -p_3 + m_2.
\end{aligned} \tag{10}
$$

Then we can rewrite Eqs. (5) and (6) as

$$
(\gamma^0 p_0 + \gamma \cdot \mathbf{p} + m_1 + m_2\gamma_5)\psi = 0, \tag{11}
$$

the generalized Dirac equation in energy-momentum space! In coordinate space, we get

$$
\left(i\partial\!\!\!/ - m_1 - m_2\gamma_5\right)\psi(\mathbf{x}, t) = 0. \tag{12}
$$

The above choice (10) of p_{ij} in terms of p_μ, m_1 and m_2 is dictated by the representation of the Dirac matrices we have adopted. A different representation would result in a different choice. To wit, if we use the Weyl or chiral representation for the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{13}$$

we have to choose the Plucker coordinates according to

$$\begin{aligned} p_{34} = +m_1 - m_2, \quad p_{12} = +m_1 + m_2, \\ p_{13} = +p_1 - ip_2, \quad p_{24} = -p_1 - ip_2, \\ p_{41} = +p_0 + p_3, \quad p_{23} = +p_0 - p_3, \end{aligned} \tag{14}$$

to yield (11) or (12).

Associated with the generalized Dirac equation (12) is the Hamiltonian density for the free Dirac particle given in (1). Following Bender et al. [3], one can check that the Hamiltonian H is not Hermitian because the m_2 term changes sign under Hermitian conjugation. However H is invariant under combined $\mathcal{P}$ and $\mathcal{T}$ reflection given by

$$\begin{aligned} \mathcal{P}\psi(\mathbf{x},t)\mathcal{P} &= \gamma_0\psi(-\mathbf{x},t), \\ \mathcal{P}\bar{\psi}(\mathbf{x},t)\mathcal{P} &= \bar{\psi}(-\mathbf{x},t)\gamma_0, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \mathcal{T}\psi(\mathbf{x},t)\mathcal{T} &= \mathbf{C}^{-1}\gamma_5\psi(\mathbf{x},-t), \\ \mathcal{T}\bar{\psi}(\mathbf{x},t)\mathcal{T} &= \bar{\psi}(\mathbf{x},-t)\gamma_5\mathbf{C}, \end{aligned} \tag{16}$$

where $\mathbf{C}$ is the charge-conjugation matrix, defined by $\mathbf{C}^{-1}\gamma_\mu\mathbf{C} = -\gamma_\mu^{\mathrm{T}}$. Therefore, the projective geometrical approach yields (at least in this particular example) a $\mathcal{PT}$-symmetric Hamiltonian rather than a Hermitian Hamiltonian. [6]

By iterating (12), one obtains

$$\left(\partial^2 + \mu^2\right)\psi(\mathbf{x},t) = 0. \tag{17}$$

Thus, the physical mass that propagates under this equation is real for $\mu^2 \geq 0$, i.e.,

$$m_1^2 \geq m_2^2, \tag{18}$$

which defines the parametric region of unbroken $\mathcal{PT}$ symmetry. If (18) is not satisfied, then the $\mathcal{PT}$ is broken. [7] And one recovers the Hermitian case only if $m_2 = 0$.

Of course, it would be nice if the geometrical picture alluded to in this paper could give us some additional insight and/or predictions. For example, one may ask whether the values of the special cases $m_2 = 0$, which corresponds to the standard Dirac equation, and $m_1 = m_2$, i.e., $\mu = 0$, which marks the onset of broken $\mathcal{PT}$ symmetry, have any particular geometrical significance. [8] Eq. (10) (Eq. (14)) shows that $m_2 = 0$ is given by the condition $p_{14} = p_{23}$ ($p_{12} = p_{34}$) and that $\mu = 0$ corresponds to $p_{12} + p_{34} = p_{41} + p_{23}$ ($p_{34} = 0$) for the Plucker coordinates for the case of the Dirac representation (the Weyl representation) of the Dirac matrices. Unfortunately since these conditions are representation-dependent, any potential geometrical significance that can be attached to these two special cases will probably be hard to identify. On the other hand, as shown above, the projective geometrical method of "deriving" the Dirac equation is very general. It includes both the standard Dirac equation and the generalized Dirac equation which yields a non-Hermitian yet $\mathcal{PT}$-symmetric Hamiltonian. One can trace this feature to the simple fact that there are six Plucker coordinates which, in general, can *naturally* accommodate two types of masses (in addition to the four energy-momentum) for the spin-$\frac{1}{2}$ particles.

Finally we note that (17) in the form of $(-p^\mu p_\mu + m_1^2 - m_2^2)\psi = 0$ is simply a reflection of the relation (8) among the Plucker coordinates when they are written in terms of p^μ , m_1 and m_2 given by either (10) or (14).

For completeness, we should mention that Bender and collaborators [3] have constructed a Hermitian Hamiltonian h that corresponds to the non-Hermitian Hamiltonian H of (1) for $\mu^2 = m_1^2 - m_2^2 \geq 0$. The two Hamiltonians are related by the similarity transformation

$$h = e^{-Q/2} H e^{Q/2}, \tag{19}$$

where

$$Q = -\tanh^{-1}\varepsilon \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}, t)\gamma_5\psi(\mathbf{x}, t), \tag{20}$$

with $\varepsilon = m_2/m_1$. The resulting h is given by

$$h = \int d\mathbf{x}\, \bar{\psi}(\mathbf{x}, t)(-i\nabla\!\!\!/ + \mu)\psi(\mathbf{x}, t), \tag{21}$$

in agreement with (17).

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[6] If we use im_2 instead of m_2 (for real m_2) in Eq. (10) or Eq. (14), then instead of Eq. (12), we get $\left(i\partial\!\!\!/ - m_1 - im_2\gamma_5\right)\psi(\mathbf{x}, t) = 0$. The resulting H would be Hermitian but not $\mathcal{PT}$-symmetric. (Recall that under both Hermitian conjugation and $\mathcal{T}$ reflection, i changes sign.) And in that case, the shifted mass (see Eq. (17)) is given by $\mu = (m_1^2 + m_2^2)^{\frac{1}{2}}$.

[7] This would correspond to the case of a conventional tachyonic spin-$\frac{1}{2}$ particle. But we note that additional physics is required to discover that a Dirac equation for an imaginary value of mass actually *cannot* describe a fermionic tachyon. See H. van Dam, Y. J. Ng and L. C. Biedenharn, Phys. Lett. **158B**, 227 (1985).

[8] This issue is raised by an anonymous referee.