Some Open Problems in Combinatorial Physics

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Contents

1. Problem A: Multiplicities in diag.

1.1. Setting

Let $\mathcal{H}(F,G)$ be the Hadamard exponential product as defined below by

$$F(z)=\sum_{n\geq 0} a_n\frac{z^n}{n!},\ G(z)=\sum_{n\geq 0} b_n\frac{z^n}{n!},\ \mathcal{H}(F,G):=\sum_{n\geq 0} a_nb_n\frac{z^n}{n!}\ . \tag{1}$$

In the case of free exponentials, that is if we write the functions as

$$F(z)=\exp\left(\sum_{n=1}^{\infty}L_n\frac{z^n}{n!}\right),\qquad G(z)=\exp\left(\sum_{n=1}^{\infty}V_n\frac{z^n}{n!}\right)\ , \tag{2}$$

and using the expansion with Bell polynomials in the sets of variables $\L=\{L_n\}$, $\mathbb{V}=\{V_m\}$ (see [6, 10] for details), we obtain

$$\mathcal{H}(F,G)=\sum_{n\geq 0}\frac{z^n}{n!}\sum_{P_1,P_2\in UP_n}\L^{Type(P_1)}\mathbb{V}^{Type(P_2)} \tag{3}$$

where UP_n is the set of unordered partitions of $[1\cdots n]$.
ßAn unordered partition P of a set X is a subset of $P\subset\mathfrak{P}(X)-\{\emptyset\}$‡ (that is an unordered collection of blocks, i. e. non-empty subsets of X) such that

- the union $\bigcup_{Y\in P}Y=X$ (P is a covering)
- P consists of disjoint subsets, i. e.
  $Y_1,Y_2\in P$ *and* $Y_1\cap Y_2\neq\emptyset\Longrightarrow Y_1=Y_2$.

The type of $P\in UP_n$ (denoted above by $Type(P)$) is the multi-index $(\alpha_i)_{i\in\mathbb{N}^+}$ such that α_k is the number of k-blocks, that is the number of members of P with cardinality k.
ßLet P_1,P_2 be two unordered partitions of the same set. To each labelling of the blocks

$$P_r=\{B_i^{(r)}\}_{1\leq i\leq n_r}\ ;\ r=1,2 \tag{4}$$

one can associate the intersection matrix

$$M=\left(\mathrm{card}(B_i^{(1)}\cap B_j^{(2)})\right)_{1\leq i\leq n_1\ ;\ 1\leq j\leq n_2}\ . \tag{5}$$

As (P_1,P_2) are, in essence, unlabelled, the arrow so constructed

$$(P_1,P_2)\mapsto class(M)=d \tag{6}$$

aims at classes of packed matrices [7] under permutations of rows and columns.
These classes have been shown [2, 3] to be in one to one correspondence with Feynman-Bender diagrams [1] which are bicoloured graphs with p ($=\mathrm{card}(P_1)$) black spots, q ($=\mathrm{card}(P_2)$) white spots, no isolated vertex and integer multiplicities. We denote the set of such diagrams by **diag** [8, 9].
Then, the correspondence goes as showed below.

‡ The set of subsets of X is denoted by $\mathfrak{P}(X)$ (this notation [4] is that of the former German school).

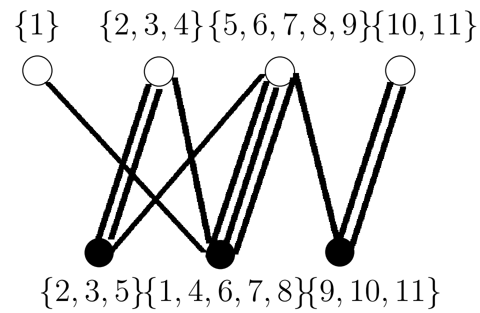

Fig 1. — *Diagram from* P_1, P_2 *(set partitions of* $[1 \cdots 11]$*).*
$P_1 = \{\{2,3,5\},\{1,4,6,7,8\},\{9,10,11\}\}$ *and* $P_2 = \{\{1\},\{2,3,4\},\{5,6,7,8,9\},\{10,11\}\}$ *(respectively black spots for* P_1 *and white spots for* P_2*).*
The incidence matrix corresponding to the diagram (as drawn) or these partitions is $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. *But, due to the fact that the defining partitions are unordered, one can permute the spots (black and white, between themselves) and, so, the lines and columns of this matrix can be permuted. The diagram could be represented by the matrix* $\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ *as well.*

Noting $mult(d)$ the cardinality of each fibre of (6), formula (3) reads

$$\mathcal{H}(F,G) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{\substack{d \in diag \\ |d|=n}} mult(d) \mathbb{L}^{\alpha(d)} \mathbb{V}^{\beta(d)} \tag{7}$$

where $\alpha(d)$ (resp. $\beta(d)$) is the "white spots type" (resp. the "black spots type") i.e. the multi-index $(\alpha_i)_{i \in \mathbb{N}^+}$ (resp. $(\beta_i)_{i \in \mathbb{N}^+}$) such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i (i lines connected to the spot) and $mult(d)$ is the number of pairs of unordered partitions of $[1 \cdots |d|]$ (here $|d| = |\alpha(d)| = |\beta(d)|$ is the number of lines of d) with associated diagram d.

1.2. Problem A

Give a formula (as smart as possible) for $mult(d)$ as a function of d (in the language of [7], as a function of the class of a packed matrix under the permutation of rows and columns).
ß**Hint**. — For practical computations, one of the two partitions may be kept fixed, say P_1 and the result of the enumeration multiplied by $\frac{n!}{|stab(P_1)|}$.

2. Problem B: Combinatorics of Riordan-Sheffer one-parameter groups.

We start with the (vector) space $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ of complex bi-infinite matrices.
Let $\mathcal{RF}(\mathbb{N},\mathbb{C}) = (\mathbb{C}^{(\mathbb{N})})^{\mathbb{N}}$ the space of row-finite matrices (i. e. matrices for which every row is finitely supported). To every matrix $T \in \mathcal{RF}(\mathbb{N},\mathbb{C})$, one can associate the sequence transformation

$$(a_k)_{k\in\mathbb{N}} \mapsto (b_n)_{n\in\mathbb{N}} \tag{8}$$

given by

$$b_n = \sum_{k\in\mathbb{N}} T[n,k]a_k \tag{9}$$

this sum is finitely supported as $T \in \mathcal{RF}(\mathbb{N},\mathbb{C})$. One can prove that the set $\mathcal{RF}(\mathbb{N},\mathbb{C})$ is exactly the algebra of continuous endomorphisms of $\mathbb{C}^{\mathbb{N}}$ endowed with the topology of pointwise convergence.
This transformation can be transported on EGFs by

$$f = \sum_{k\in\mathbb{N}} a_k \frac{z^k}{k!} \mapsto \hat{f} = \sum_{n\in\mathbb{N}} b_n \frac{z^n}{n!} \tag{10}$$

and, in case $\hat{f}$ is given by

$$\hat{f}(z) = \Phi_{g,\phi}[f](z) = g(z)f(\phi(z)). \tag{11}$$

with

$$g(z) = 1 + \text{higher terms } \textit{and } \phi(z) = z + \text{higher terms}. \tag{12}$$

we say that the matrix is a matrix of substitutions with prefunction.
In classical combinatorics (for OGF and EGF), the matrices $M_{g,\phi}(n,k)$ are known as *Riordan matrices* (see [11, 12] for example). One can prove, using a Zariski-like argument, the following proposition [10, 5].

Proposition 2.1 *[10] Let M be the matrix of a substitution with prefunction; then so is M^t for all $t \in \mathbf{C}$.*

2.1. Problem B

a) Provide a combinatorial proof of the preceding proposition for $t \in \mathbb{Q}$ (without using the "pro-algebraic" structure of the group of substitutions with prefunctions, directly or indirectly).
b) Give a combinatorial interpretation of $M^{1/2}$ for some Sheffer matrices.

3. Problem C: A corpus for combinatorial vector fields.

With the preceding notations one can show that, if M is a matrix of substitution with prefunction, the limit

$$\lim_{q\to+\infty} q(M^{\frac{1}{q}} - I) \tag{13}$$

exists (call it L) and the associated transformation of sequences (see above) is the sum of a vector field and a scalar field. One can see that

$$M \in \mathbb{Q}^{\mathbb{N}\times\mathbb{N}} \Longrightarrow L \in \mathbb{Q}^{\mathbb{N}\times\mathbb{N}} \ . \tag{14}$$

in addition, if M is a matrix of substitution (i. e. the prefunction is $\equiv 1$) then the scalar field is zero and so the associated differential operator is a pure vector field (with coefficients in $\mathbb{Q}$ if M is in $\mathbb{Q}^{\mathbb{N}\times\mathbb{N}}$).
On the other hand, if $\mathcal{C}$ is a class of labelled graphs for which the exponential formula applies, the matrix M such that

$$M[n,k] = \textit{Number of graphs labelled by } [1..n] \textit{ and with } k \textit{ connected components} \tag{15}$$

is a matrix of substitution [10]. For example with the graphs of equivalence relations on finite sets, the substitution is $z \mapsto e^z - 1$; for graphs of idempotent endofunctions, the substitution is $z \mapsto ze^z$.

3.1. Problem C

a) What is the combinatorial interpretation of the coefficients of the vector field for the two preceding examples ?
b) Can we give any insight of the form of this vector field for general classes of graphs ?
ß**Hint**. — $M^z = e^{z log(M)}$ where $log(M)$ is the matrix of a differential operator of the form $q(z)\frac{d}{dz} + v(z)$.

4. Problem D Probabilistic study of approximate substitutions

Our motivation, in this section, consists in approximating the matrices of infinite substitutions by finite matrices of (approximate) substitutions. We are then interested in the probabilistic study of these matrices. To this end, we randomly generate unipotent (unitriangular) matrices and observe the number of occurrences of matrices of substitutions.

We start by giving some examples of our experiment which are summarized in the table below:

Size	Number of drawings	Range of variables	Probability
$[3 \times 3]$	300	$[1 \cdots 10]$	1
		$[1 \cdots 100]$	1
		$[1 \cdots 10000]$	1
$[4 \times 4]$	275	$[1 \cdots 10]$	0.0473
		$[1 \cdots 100]$	0.0001
		$[1 \cdots 10000]$	0^+
$[10 \times 10]$	1500	$[1 \cdots 10]$	0.0327
		$[1 \cdots 100]$	0^+
		$[1 \cdots 10000]$	0^+

According to the results obtained, we observe that the (approximate) substitution matrices are not very frequent. However, in meeting certain conditions such as size, the number of drawings and the range of the variables, we can obtain positive probabilities that these matrices appear.
Let us note that the smaller the size of the matrix the more probable one obtains a matrix of substitution in a reasonable number of drawings.
We also notice that, if we vary the range of variables, and this in an increasing way and by keeping unchanged the number of drawings and size, the probability tends to zero.
We also notice that the unipotent matrices of size 3 are all matrices of approximate substitutions. This is easy to see because the exponential generating series of the 3^{rd} column will always have the form $c_k = \dfrac{x^2}{2!}$.
Thus, we can say that the test actually starts from the matrices of size higher or equal to 4.

Result 4.1 *Let r represent the cardinality of the range of variables and $n \times n$ be the size of the matrix.*
According to the results obtained; we can say that the probability p_n of appearance of the matrices of substitutions depends on r and n and we have the following upper bound:

$$p_n \leq \frac{r^{2n-3}}{r^{\frac{n(n-1)}{2}}} \tag{16}$$

which shows that

$$p_n \longrightarrow 0 \quad as \quad n \longrightarrow \infty \tag{17}$$

4.1. Problem D

One can conjecture that the effect of the range selection vanishes when n tends to infinity. More precisely:

$$p_n \sim \frac{r^{2n-3}}{r^{\frac{n(n-1)}{2}}} \tag{18}$$

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