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Radiation Transfer for Exponential c(x)

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Abstract

The emergent flux for radiation transfer in inhomogeneous half-space for exponential $c\left(x\right)=\frac{1+ke^{-mx}}{1+be^{-mx}}$ is obtained by using Modified-Eddington method, where m, b, k are constants.

As a result of this work, angular flux is obtained in terms of Jacobi polynomials.

Key Words: The emergent flux, Radiation transfer, Modified-Eddington

1. Introduction

The radiation problems for the homogeneous medium have been solved using various methods [1, 2, 3]. These kind of problems are then extended to problems of the form

$$\mu\partial_x\varphi\left(x,\mu\right)+\varphi\left(x,\mu\right)=\frac{c\left(x\right)}{2}\int_{-1}^{1}\varphi\left(x,\mu'\right)d\mu';0\leq x\leq\infty \tag{1}$$

$$\varphi\left(0,\mu\right)=g\left(\mu\right),0\leq\mu\leq 1 \tag{2}$$

$$\lim_{x\to\infty}\varphi\left(x,\mu\right)<\infty, \tag{3}$$

where c(x) is a continuous function of position [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In most of these works c(x) is considered to have the forms,

$$c(x)=c_0e^{-x/s} \tag{4}$$

or

$$c(x)=\int_0^\infty\alpha\,(t)e^{-x/t}dt, \tag{5}$$

where

$$\int_{0}^{\infty} |t\,(\alpha\ (t))|^{2}\ dt < \infty, \tag{6}$$

For these choices of c(x), the singular eigenfunction method of solution is used [4, 5].

Pomraining and Larsen [6] obtained the solutions of the transport equation using Case's method. G.C. Pomraning has calculated [11] the radiation diffuse problem and hence the emergent flux for the homogeneous medium. Wakil, Saad and Hendi [16] used the Modified-Eddington approximation for the exponential medium.

In our work we use the Modified-Eddington method to determine

$$c\,(x) = \frac{1 + ke^{-mx}}{1 + be^{-mx}} \tag{7}$$

with which the emergent flux is calculated.

2. The Modified-Eddington Method

The angular flux in the Modified-Eddington method is defined as

$$\varphi\,(x,\mu) = M\,(x)\,E\,(x,\mu) + H\,(x)\,O\,(x,\mu)\,, \tag{8}$$

where the zeroth and first moments of the flux are given by

$$M\,(x) = \frac{1}{2}\int_{-1}^{1} \varphi\,(x,\mu)d\mu \tag{9}$$

and

$$H\,(x) = \frac{1}{2}\int_{-1}^{1} \varphi\,(x,\mu)\mu d\mu \tag{10}$$

E(x,μ) and O(x,μ) in the definition of the angular flux are even and odd functions of μ, respectively, and normalized as

$$\frac{1}{2}\int_{-1}^{1} E\,(x,\mu)d\mu = 1; \tag{11}$$

$$\frac{1}{2}\int_{-1}^{1} O\,(x,\mu)d\mu = 1. \tag{12}$$

Multiplying Eq.(1) by $\mu^n, n = 0, 1$, respectively and integrating over μ in the range (-1,1) we get, respectively for each n,

$$\frac{d}{dx}H\left(x\right)+\left[1-c\left(x\right)\right]M\left(x\right)=0 \tag{13}$$

$$\frac{d}{dx}\left[D\left(x\right)M\left(x\right)\right]+H\left(x\right)=0, \tag{14}$$

where

$$D\left(x\right)=\frac{1}{2}\int_{-1}^{1}E\left(x,\mu\right)\mu^{2}d\mu. \tag{15}$$

In order to eliminate spatial derivatives Eqs. (1, 8, 14, 15) are used and

$$E\left(x,\mu\right)=\frac{c\left(x\right)}{1-\nu^{2}\mu^{2}} \tag{16}$$

$$O\left(x,\mu\right)=\mu\frac{E\left(x,\mu\right)}{D\left(x\right)} \tag{17}$$

are obtained, where

$$\nu^{2}\left(x\right)=\frac{1-c\left(x\right)}{D\left(x\right)} \tag{18}$$

and

$$2\nu\left(x\right)c\left(x\right)=\ln\left[\left(1+\nu\left(x\right)\right)/\left(1-\nu\left(x\right)\right)\right]. \tag{19}$$

Equations (13, 14) give

$$\frac{d^{2}}{dx^{2}}\left[D\left(x\right)M\left(x\right)\right]-\left(1-c\left(x\right)\right)M\left(x\right)=0. \tag{20}$$

For$m << 1, D(x)$ can be assumed as a constant.

3. The Calculations

We can solve Eq.(20) for c(x) given in Eq.(17) using the Modified-Eddington method:

$$D\frac{d^{2}}{dx^{2}}M\left(x\right)-\left(1-\frac{1+ke^{-mx}}{1+be^{-mx}}\right)M\left(x\right)=0. \tag{21}$$

When the variable x is changed to s as

$$s=-e^{-mx}, \tag{22a}$$

the following equation is found:

$$\frac{d^2M}{ds^2} + \frac{1}{s}\frac{dM(s)}{ds} + \frac{1}{s^2(1-bs)}\left\{\frac{(b-k)s}{Dm^2}\right\}M(s) = 0. \tag{22b}$$

Lets choose

$$M(s) = (1-bs)\,y(s). \tag{23}$$

Placing it into Eq.(22b), we get:

$$\frac{d^2y}{ds^2} + \frac{(1-3bs)}{s(1-bs)}\frac{dy}{ds} + \frac{1}{s(1-bs)}\left[-b + \frac{b-k}{Dm^2}\right]y(s) = 0 \tag{24}$$

Using t=bs ;

$$\frac{d^2y}{dt^2} + \frac{(1-3t)}{t(1-t)}\frac{dy}{dt} + \frac{1}{t(1-t)}\left[-1 + \frac{b-k}{Dm^2b}\right]y = 0 \tag{25}$$

and with $t = \frac{1}{2}(1-z)$ we obtain

$$\left(1-z^2\right)\frac{d^2y}{dz^2} + (1-3z)\frac{dy}{dz} + \left[-1 + \frac{b-k}{Dm^2b}\right]y = 0. \tag{26}$$

This differential equation is in the form similar to the Jacobi differential equation which is given as

$$\left(1-z^2\right)\frac{d^2y}{dz^2} + \{\alpha - \beta + (\alpha+\beta-2)z\}\frac{dy}{dz} + (n+1)(n+\alpha+\beta)\,y = 0, \tag{27}$$

whereα, β are constants [18, 19].

$(1-z)^{\alpha}(1+z)^{\beta}P_n^{(\alpha,\beta)}(z)$ satisfies this differential equation. Comparing the two equations (26, 27) we have

$$\alpha = 0 \tag{28a}$$

$$\beta = -1 \tag{28b}$$

$$n = \mp\sqrt{\frac{b-k}{Dm^2b}}. \tag{28c}$$

The solution to Eq.(21) can now be written in the form

$$M(x) = P^{(0,-1)}_{\mp\sqrt{\frac{b-k}{Dm^2b}}}\left(1+2be^{-mx}\right). \tag{29}$$

From Eq.(14), we get

$$H(x) = -D\frac{dM(x)}{dx} = 2bmDP^{(0,-1)'}_{\pm\sqrt{\frac{b-k}{Dm^2b}}}\left(1+2be^{-mx}\right). \tag{30}$$

Using Eq.(8) and the experessions in Eq.(16, 17, 29, 30) the angular flux can be obtained as

$$\begin{aligned}\varphi(x,\mu) \;=\; & \left(\frac{1+ke^{-mx}}{1+be^{-mx}}\right)\left(\frac{1}{1-\nu^2\mu^2}\right)P^{(0,-1)}_{\mp\sqrt{\frac{b-k}{Dm^2b}}}\left(1+2be^{-mx}\right)\\ & +\left(\frac{1+ke^{-mx}}{1+be^{-mx}}\right)\left(\frac{2bm\mu}{1-\nu^2\mu^2}\right)P^{(0,-1)'}_{\mp\sqrt{\frac{b-k}{Dm^2b}}}\left(1+2be^{-mx}\right), \end{aligned} \tag{31}$$

where b, D, m, k are constants.

Using the solution given in Eq.(31), one can calculate the constans C and D

$$C = 1-2\int_0^1 \varphi(0,-\mu)\mu d\mu, \tag{32}$$

$$D = \frac{1}{3}+2\int_0^1 \varphi(0,-\mu)\mu^2 d\mu. \tag{33}$$

as

$$C = 1+\frac{1}{\nu^2}\left(\frac{1+k}{1+b}\right)\left\{\ln\left(1-\nu^2\right)P^{(0,-1)}_{\mp\sqrt{\frac{b-k}{Dm^2b}}}(1+2b)+\frac{4bm}{\nu}\left(-\nu+\arctan h\nu\right)P^{(0,-1)'}_{\mp\sqrt{\frac{b-k}{Dm^2b}}}(1+2b)\right\} \tag{34}$$

and

$$D = \frac{1}{3}+\frac{2}{\nu^3}\left(\frac{1+k}{1+b}\right)\left\{(-\nu+\arctan h\nu)P^{(0,-1)}_{\mp\sqrt{\frac{b-k}{Dm^2b}}}(1+2b)+\frac{\nu^2+\ln\left(1-\nu^2\right)}{\nu}P^{(0,-1)'}_{\mp\sqrt{\frac{b-k}{Dm^2b}}}(1+2b)\right\}. \tag{35}$$

4. Conclusion

The emergent flux for radiation transfer in inhomogeneous half-space is calculated using $c(x) = (1+ke^{-mx})/(1+be^{-mx})$. This calculation has been previously handled by Pomraning using the linear transport equation. In our work the Modified. Eddington method is used and analytical expressions in terms of Jacobi polynomials are obtained.

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