

Schrödinger Quantization and Excited States of Zitterbewegung

Selahattin GÖNEN

Dicle University, Department of Physics, Diyarbakır-TURKEY

Ali HAVARE

Mersin University, Department of Physics, Mersin-TURKEY

Received 12.03.2001

Abstract

General wavefunction is expanded as power series of a variable which describes internal dynamics of particle, to show that it contains excited states of Schrödinger's Zitterbewegung.

1. Introduction

The electron is a fundamental particle in the nature and its understanding is based on the construction of electrodynamics. First studies concerning electrons were made by Lorentz in 1905's. Dirac contributed these studies extending Lorentz equation to contain radiative processes, and is now known as the Lorentz-Dirac equation [1, 2]. In these studies, the electron was considered to be a classical, point-like spinless particle. This classical picture changed with the development of quantum mechanics giving new degrees of freedom to the electron, the fundamental among which being spin [3]. In his equation, Pauli considered the electron as a non-relativistic particle and quantum spin described with his exclusion principle [4]. In the two quantum mechanical descriptions proposed by Schrödinger, one assumes the electron to be a spinless point-like particle with non-relativistic energies and described by Schrödinger equation, while the other allows relativistic energies and described by the Klein-Gordon equation. However the best description of the electron is given by the Dirac equation [5]. All of these equations contains a spin concept which can not be understood by classical mechanics and it is not obvious how the correspondence principle would operate.

In recent years, there has been an increasing effort to understand both the classical meaning of spin and the behaviour of a classical system according to the rules of quantum mechanics [6, 7, 8, 9, 10]. In these works, in addition to usual the coordinate and momentum degrees of freedom, the electron's internal degrees of freedom is also included. Though a classical system both orbital momentum and spin angular momentum may have any value, in quantum mechanics spin angular momentum may assume only integer multiples of $\hbar/2$ [11]. Similar situations apply in vector-field equations of \hbar -spin and Rarita-Schwinger of $3\hbar/2$ -spin [12].

In this work we obtained a wave function by quantizing the classical model and then expand this function in terms of the eigenvalues of spin operator. Finally it is seen that this quantized wave function, which has spin eigenvalues $0, 1/2, 1, 3/2, 2, \dots, n/2$ is a generalized wave function.

2. Equation of Motion for the Classical System

The action of a classical spinning particle is given by Barut and Zhanghi [7] as

$$I = \int d\tau \left[i\bar{z} \dot{z} + P_\mu \dot{x}^\mu - \pi_\mu \bar{z} \gamma^\mu z \right]. \quad (1)$$

Here, x^μ are the space-time coordinates, $z, i\bar{z} \in C^4$ are internal dynamics variables, and π_μ are general momentum. In this case, the complete phase space has four sets of variables: $(x^\mu, P_\mu, z, i\bar{z})$. We use a proper time formalism as $z = z(\tau)$ and $x^\mu = x^\mu(\tau)$, where τ being the proper time of center of mass. Although the mass term does not enter into the Lagrangian it exist due to integral motion. The classical theory already contains the notion of anti-particles in the form of positive and negative internal frequencies for a given momentum P.

The equations of motion for the classical system are

$$\dot{z} = -i\pi_\mu \gamma^\mu z \quad (2)$$

$$\dot{\bar{z}} = i\bar{z} \gamma^\mu \pi_\mu \quad (3)$$

$$\dot{x}^\mu = v^\mu \bar{z} \gamma^\mu z \quad (4)$$

$$P^\mu = eA_{\alpha,\mu} v^\alpha - \pi_\alpha \bar{z} \gamma^\alpha z \quad (5)$$

and the Hamiltonian of the classical system, which is equal to the proper mass, takes the form

$$H = iz\bar{z} + P_\mu \dot{x}^\mu - L = \pi_\mu \bar{z} \gamma^\mu z. \quad (6)$$

The variation of the Hamiltonian is

$$\delta H = \frac{\partial H}{\partial(i\bar{z})} \delta(i\bar{z}) + \frac{\partial H}{\partial P_\mu} \delta P_\mu + \frac{\partial H}{\partial z} \delta z + \frac{\partial H}{\partial x^\mu} \delta x^\mu + \frac{\partial H}{\partial \tau} \delta \tau. \quad (7)$$

We obtain Hamilton equations of motion from the Hamiltonian in Eq.(6) as

$$\frac{\partial H}{\partial P_\mu} = \frac{dx^\mu}{d\tau}, \quad (8)$$

$$\frac{\partial H}{\partial(i\bar{z})} = \frac{dz}{d\tau}, \quad (9)$$

$$\frac{\partial H}{\partial z} = -\frac{d(i\bar{z})}{d\tau}, \quad (10)$$

$$\frac{\partial H}{\partial x^\mu} = -\frac{dP^\mu}{d\tau}. \quad (11)$$

We can rewrite Eq.(7) by using this equations of motion as

$$\delta H = \frac{dz}{d\tau} \delta(i\bar{z}) + \frac{dx^\mu}{d\tau} \delta P_\mu - \frac{d(i\bar{z})}{d\tau} \delta z - \frac{dP^\mu}{d\tau} \delta x^\mu + \frac{\partial H}{\partial \tau} \delta \tau. \quad (12)$$

Using this expression we can write variation of action in Eq.(1) as

$$\delta I = (P_\mu dx^\mu + i\bar{z} dz - H d\tau). \quad (13)$$

3. Quantization

The phase function, which is defined by quantum mechanics, depends on classical action as

$$S = \frac{1}{\hbar} I. \quad (14)$$

The Hamilton-Jacobi, or phase function S, describes a family of possible classical trajectories $(z(\tau), x^\mu(\tau))$ with momentum $P_A = (i\bar{z}, P_\mu)$. In quantum theory, these possible classical trajectories are the rays which belong to a wave with a definite momentum P_A and proper mass $H = m$ described by the wave function as ($\hbar = 1$)

$$\Psi_P(z, x; \tau) = A e^{iS(z, x; \tau)}, \quad (15)$$

where

$$S(z, x^\mu; \tau) = \begin{cases} \int (i\bar{z}dz + P_\mu dx^\mu - Hd\tau); & P_A \text{ depends on } z, x^\mu \\ i\bar{z}z + P_\mu x^\mu - H\tau & ; \bar{z}, P_\mu \text{ are constants} \end{cases} \quad (16)$$

By the substitution of S (in the special case of constants $+i\bar{z}, P_\mu$) into Eq.(15), it takes the form

$$\Psi_{P_A}(z, x; \tau) = Ae^{i(+i\bar{z}z + P_\mu x^\mu - H\tau)}. \quad (17)$$

The wave function that represents the particle will be a superposition of eigenfunctions. This general situation must be solution of Schrödinger wave equation. For a given wave function $\Psi(\bar{z}, x; \tau)$ covariant Schrödinger equation is

$$i\frac{\partial\Psi}{\partial\tau} = H\Psi, \quad (18)$$

where \bar{z} is the canonical conjugate of z . It is possible to obtain excited states of Zitterbewegung by expanding a general wave function $\Psi(\bar{z}, x; \tau)$ power series of \bar{z} :

$$\begin{aligned} \Psi(\bar{z}, x; \tau) = & \psi(x; \tau) + \frac{1}{1!}\bar{z}^\alpha\psi_\alpha(x; \tau) + \frac{1}{2!}\bar{z}^\alpha\bar{z}^\beta\psi_{\alpha\beta}(x; \tau) + \\ & \frac{1}{3!}\bar{z}^\alpha\bar{z}^\beta\bar{z}^\nu\psi_{\alpha\beta\nu}(x; \tau) + \dots \end{aligned} \quad (19)$$

If we substitute this expansion into Eq.(18) and separate the terms in ordered form we get

$$i\frac{\partial}{\partial\tau}[\psi(x; \tau)] = \bar{z}\gamma^\mu z\pi_\mu\psi_\alpha(x; \tau), \quad (20)$$

$$i\frac{\partial}{\partial\tau}[\bar{z}^\alpha\psi_\alpha(x; \tau)] = \bar{z}^\eta\gamma^\mu_{(\eta\sigma)}z^\sigma\pi_\mu\bar{z}^\alpha\psi_\alpha(x; \tau), \quad (21)$$

$$i\frac{\partial}{\partial\tau}[\bar{z}^\alpha\bar{z}^\beta\psi_{\alpha\beta}(x; \tau)] = \bar{z}^\eta\gamma^\mu_{(\eta\sigma)}z^\sigma\pi_\mu\bar{z}^\alpha\bar{z}^\beta\psi_{\alpha\beta}(x; \tau), \quad (22)$$

$$i\frac{\partial}{\partial\tau}[\bar{z}^\alpha\bar{z}^\beta\bar{z}^\nu\psi_{\alpha\beta\nu}(x; \tau)] = \bar{z}^\eta\gamma^\mu_{(\eta\sigma)}z^\sigma\pi_\mu\bar{z}^\alpha\bar{z}^\beta\bar{z}^\nu\psi_{\alpha\beta\nu}(x; \tau), \quad (23)$$

⋮

$$i\frac{\partial}{\partial\tau}[\bar{z}^{\alpha_1}\bar{z}^{\alpha_2}\dots\bar{z}^{\alpha_n}\psi_{\alpha_1,\alpha_2,\dots,\alpha_n}(x; \tau)] = \bar{z}^\eta\gamma^\mu_{(\eta\sigma)}z^\sigma\pi_\mu\bar{z}^{\alpha_1}\bar{z}^{\alpha_2}\dots\bar{z}^{\alpha_n}\psi_{\alpha_1,\alpha_2,\dots,\alpha_n}(x; \tau). \quad (24)$$

Starting from Eq.(20) we can obtain explicit form of these equations. It is known that from Euler-Lagrange equations of classical system

$$\dot{x}^\mu = v^\mu = \bar{z}\gamma^\mu z. \quad (25)$$

Using this and introducing wave function $\psi(x; \tau)$ as

$$\psi(x; \tau) = U(x)e^{-im\tau} \quad (26)$$

Eq.(20) takes the form

$$[v^\mu\pi_\mu - m]U(x) = 0. \quad (27)$$

In the spinless case it is seen that

$$v^\mu v_\mu = 1 \quad (28)$$

and

$$\pi^\mu\pi_\mu = m^2. \quad (29)$$

Comparing these two conditions we obtain

$$v^\mu = \frac{1}{m}\pi^\mu. \quad (30)$$

Substituting Eq.(30) in Eq.(27) we get the following equation, and is known as Klein-Gordon equation:

$$[\pi^\mu\pi_\mu - m^2]U(x) = 0. \quad (31)$$

In Eq.(21), if choose the dynamics variable \bar{z}^α as time independent and use Eq.(26) we obtain

$$[\gamma^\mu \pi_\mu - m^2]U(x) = 0. \quad (32)$$

This is known as the Dirac equation describing spin-1/2 particles.

Similarly for Eq.(22) we can simplify calculations using

$$\bar{z}^\alpha \bar{z}^\beta \psi_{\alpha\beta}(x; \tau) = \frac{1}{2} \bar{z}^\alpha \bar{z}^\beta (\psi_{\alpha\beta} + \psi_{\beta\alpha}), \quad (33)$$

where $\bar{z}^\alpha \bar{z}^\beta = \bar{z}^\beta \bar{z}^\alpha$. Then Eq.(22) takes the form

$$i \frac{\partial}{\partial \tau} [\psi_{\alpha\beta} + \psi_{\beta\alpha}] = [\gamma_{(\alpha\beta)}^\mu \otimes \mathbf{I}_{(\alpha\sigma)} + \mathbf{I}_{(\alpha\sigma)} \otimes \gamma_{(\alpha\beta)}^\mu] \pi^\mu (\psi_{\alpha\beta} + \psi_{\beta\alpha}). \quad (34)$$

Here, if we introduce

$$\Psi_{\alpha\beta} = (\psi_{\alpha\beta} + \psi_{\beta\alpha}) \quad (35)$$

and

$$\beta^\mu = [\gamma_{(\alpha\beta)}^\mu \otimes \mathbf{I}_{(\alpha\sigma)} + \mathbf{I}_{(\alpha\sigma)} \otimes \gamma_{(\alpha\beta)}^\mu], \quad (36)$$

and use Eq.(26), Eq.(34) gives

$$[\beta^\mu \pi_\mu - m]U_{\alpha\beta}(x) = 0. \quad (37)$$

This is the Kemmer equation and describes spin-1 particles. The β^μ are 16×16 Hermitian matrices and are known as Kemmer matrices.

If we use in Eq.(23) the following expression

$$\bar{z}^\alpha \bar{z}^\beta \bar{z}^\nu \psi_{\alpha\beta\nu}(x; \tau) = \frac{1}{6} \bar{z}^\alpha \bar{z}^\beta \bar{z}^\nu (\psi_{\alpha\beta\nu} + \psi_{\alpha\nu\beta} + \psi_{\beta\alpha\nu} + \psi_{\beta\nu\alpha} + \psi_{\nu\alpha\beta} + \psi_{\nu\beta\alpha}) \quad (38)$$

we get

$$i \frac{\partial}{\partial \tau} \Psi_{\alpha\beta\nu} = [\gamma^\mu \otimes \mathbf{I}_{\alpha\nu} \otimes \mathbf{I}_{\nu\alpha} + \mathbf{I}_{\beta\nu} \otimes \gamma^\mu \otimes \mathbf{I}_{\nu\beta} + \mathbf{I}_{\sigma\nu} \otimes \mathbf{I}_{\nu\sigma} \otimes \gamma^\mu] \pi_\mu \Psi_{\alpha\beta\nu}. \quad (39)$$

If we introduce

$$\alpha^\mu = [\gamma^\mu \otimes \mathbf{I}_{\alpha\nu} \otimes \mathbf{I}_{\nu\alpha} + \mathbf{I}_{\beta\nu} \otimes \gamma^\mu \otimes \mathbf{I}_{\nu\beta} + \mathbf{I}_{\sigma\nu} \otimes \mathbf{I}_{\nu\sigma} \otimes \gamma^\mu], \quad (40)$$

$$\Psi_{\alpha\beta\nu} = \frac{1}{6} (\psi_{\alpha\beta\nu} + \psi_{\alpha\nu\beta} + \psi_{\beta\alpha\nu} + \psi_{\beta\nu\alpha} + \psi_{\nu\alpha\beta} + \psi_{\nu\beta\alpha}), \quad (41)$$

and use Eq.(26) we obtain the following equation which is known as Rarita-Schwinger equation describing spin-3/2 particles:

$$[\alpha^\mu \pi_\mu - m]U_{\alpha\beta\nu}(x) = 0. \quad (42)$$

Now, general term Eq.(24) gives

$$[\alpha^\lambda \pi_\lambda - m]U_{\alpha_1, \alpha_2, \dots, \alpha_n}(x) = 0, \quad (43)$$

where

$$\alpha^\lambda = [\gamma^\lambda \otimes \mathbf{I} \otimes \dots \otimes \mathbf{I} + \mathbf{I} \otimes \gamma^\lambda \otimes \dots \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{I} \otimes \dots \otimes \gamma^\lambda]. \quad (44)$$

These $n \times n$ matrices satisfies

$$\sum_{(P)} \alpha_{\mu_1} \dots \alpha_{\mu_L} (\alpha_{\mu_L+1} \alpha_{\mu_L+2} - \delta_{\mu_L+1, \mu_L+2}) = 0, \quad (45)$$

where the sum $\sum_{(P)}$ is performed over all possible permutations of $\mu_1, \mu_2, \dots, \mu_{L+1}, \mu_{L+2}$ indices. Equation (43) describes the dynamics of spin- $n/2$ particles.

4. Conclusions

In this work, to understand the Schrödinger equation, after quantizing the classical system with spin, a generalized wave function was expanded into a power series in terms of internal coordinates of the system. Then it was shown that this expansion includes zero and positive integer values $\hbar/2$, hence containing all fermion states (odd integer of $\hbar/2$) and all boson states (even integer of $\hbar/2$).

References

- [1] H.A. Lorentz, “The Theory of Electrons”, Dover, N. Y. (1952).
- [2] P.A.M. Dirac, Proc. Roy. Soc. (London) **A167**, 148(1958).
- [3] H.A. Kramers, “Quantum Mechanics”, Dover, N.Y.(1964).
- [4] W. Pauli, Z. Phys., **31**, 765(1925).
- [5] T-Y Wu and W-Y Pauchy Hwang, “Relativistic Quantum Mechanics and Quantum Fields”, World Scientific(1991).
- [6] A.T. Ogelski and J. Sobczyk, J. Math. Phys., **22**(1981).
- [7] A.O. Barut and N. Zanghi, Phys. Rev. Lett., **52**, 2009(1984).
- [8] A.O. Barut, “What is Electron? Relativistic Electron Theory and Radiative Process”, Quantum Optics, Experimental Gravitation and Measurement Theory (ed. P.Mystre and M.O. Scully), Plenum Press, N. Y., pp.155, (1983).
- [9] A.O. Barut and I.H. Duru, Phys.Rev. Lett., **53**,2355(1984).
- [10] A.O. Barut and N. Ünal, Phys. Rev.Lett. **A40**, 5404(1989).
- [11] A.O. Barut, Phys. Rev. Lett. **B**,436(1990).
- [12] Y. Takahashi, “An Introduction to Field Quantization”, Pergamon Press, London(1969).