

# The Contributions of Anomalous Magnetic Moment to Relativistic Landau Orbits

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## Abstract

In this work, the contributions of anomalous magnetic moment to relativistic Landau orbits are calculated by the method of self-field quantum electrodynamics. In addition to contributions from positive energy states new terms from negative energy states that have not been taken into account before are now also included. Contributions from these terms are found to be in the same order as positive terms.

## 1. Introduction

In order to explain the spectra of atoms in magnetic fields, Uhlenbeck and Goudsmit [1] postulated that the electron has an intrinsic (spin) angular momentum  $\hbar/2$  and a magnetic dipole moment  $e\hbar/2mc$ , the Bohr magneton. Later, Dirac showed that both properties of the electron are the consequences of relativistically invariant quantum mechanics [2].

The magnitude of the electron magnetic dipole moment and spin angular momentum is  $e/mc$ , that is, Lande g-factor for electron spin is 2 as predicted by Dirac theory which neglects coupling of the electron to the radiation field [3]. As in the case of the Lamb shift, radiative corrections give small departure from this prediction. Just prior to the first accurate measurements by Kusch *et al.* [4], Schwinger calculated the “anomaly”  $(g-2)/2$  as  $\alpha/2\pi \approx 0.00116$  [5]. The experimentalists reported a value  $0.00119 \pm 0.00005$ . Like the Lamb shift, the anomalous moment of the electron also provides one of the most sensitive tests of QED. Accurate measured value of anomalous magnetic moment is 0.00115652188(4) and its theoretical value by QED, up to the fourth order in the fine-structure constant  $\alpha$ , is 0.001159652192(74) [6,7].

The self-field quantum electrodynamics is introduced to complete standard quantum electrodynamics [8]. As it is known, standard quantum electrodynamics first quantizes free fields and then takes the interactions between the particles as perturbative effects. However, in the self-field quantum electrodynamics both free fields and interactions are taken as a whole and quantized. No perturbation is taken into account in this method. In this respect, the self-field quantum electrodynamics use a similar method with classical electrodynamics. A. O. Barut *et al.* first used the self-field quantum electrodynamics to calculate anomalous magnetic moment for non-relativistic Pauli equation [9, 10, 11, 12]. Here, the same method is applied to calculate the contributions of anomalous magnetic moment to relativistic Landau orbits.

## 2. Action of the System

For a relativistic electron in an external arbitrary field, the action is given as

$$W = \int dx \left[ \bar{\Psi}(x) (\gamma^\mu i \partial_\mu - M) \Psi(x) + J^\mu A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]. \quad (2.1)$$

Here, the first term is the kinetic energy of electron, the second term is the contribution from the interaction of electron with external electromagnetic field, and the last term is the self-energy of electron. In the first term  $\Psi(x)$  describes the electron matter field which is defined at time-position point  $(x_0, \vec{x})$ , and  $\gamma_\mu$  are Dirac gamma matrices. The mass and current of the electron are defined as  $M$  and  $J^\mu$  ( $J^\mu = -e\bar{\Psi}\gamma^\mu\Psi$ ), respectively.  $A_\mu$  is the total electromagnetic field defined as  $A_\mu = A_\mu^{Self} + A_\mu^{Ext}$ , where  $A_\mu^{Ext}$  is not a dynamical function. In the last term  $F_{\mu\nu}$  is a self-field tensor and is given as  $F_{\mu\nu} = A_{\nu,\mu}^{Self} - A_{\mu,\nu}^{Self}$ .

If we expand the time dependence of wavefunction of electron,  $\Psi(x_0, \vec{x})$ , by a Fourier series ( $\hbar = 1$ )

$$\Psi(x) = \Psi(x_0, \vec{x}) = \sum_n \Psi_n(\vec{x}) e^{-iE_n x_0} \quad (2.2)$$

(where the summation goes over all discrete and continuous states) and substitute  $\vec{\pi} = i\vec{\nabla} - e\vec{A}$ ,  $\vec{A}^{Ext} \neq 0$ ;  $\bar{\Psi}(x)\gamma^o = \Psi^\dagger(x)$ ;  $\bar{\Psi}(x)\vec{\gamma} = \Psi^\dagger(x)\vec{\alpha}$ ;  $\gamma^o = \beta$ ;  $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$  (where  $\vec{\sigma}$  are Pauli matrices); and

$$A_\mu^{Self}(x) = -\frac{1}{(2\pi)^4} \int dy \int d^4k \frac{e^{-ik(x-y)}}{k^2} J_\mu(y) \quad (2.3)$$

in Eq.(2.1) it is found that non-linear self energy part of the total action is

$$W_1 = \frac{e^2}{2} \sum_{nmrs} 2\pi\delta(E_n - E_m + E_r - E_s) \int d\vec{x}d\vec{y} [\Psi_n^\dagger(\vec{x})\Psi_m(\vec{x})\Psi_r^\dagger(\vec{y})\Psi_s(\vec{y}) - \Psi_n^\dagger(\vec{x})\vec{\alpha}\Psi_m(\vec{x})\Psi_r^\dagger(\vec{y})\vec{\alpha}\Psi_s(\vec{y})] \int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{[(E_n - E_m)^2 - |\vec{k}|^2]} \quad (2.4)$$

and the linear part is

$$W_o = \sum_n \int d\vec{x} 2\pi\Psi_n^\dagger(\vec{x})(E_n - \vec{\alpha}\cdot\vec{\pi} - \beta M)\Psi_n(\vec{x}) \quad (2.5)$$

### 3. The Calculation of Anomalous Magnetic Moment Contributions

In order to provide a transition from Dirac equation to Pauli equation for each labeled state we use in Eq. (2.4) unitary similarity transformation matrix  $S$ ; so, instead of Dirac wavefunction, our wavefunction for electron takes the form

$$\Psi(\vec{x}) = S \begin{pmatrix} \varphi(\vec{x}) \\ \chi(\vec{x}) \end{pmatrix} = K \begin{pmatrix} E + M & -\vec{\sigma}\cdot\vec{\pi} \\ \vec{\sigma}\cdot\vec{\pi} & E + M \end{pmatrix} \begin{pmatrix} \varphi(\vec{x}) \\ \chi(\vec{x}) \end{pmatrix} \quad (3.1)$$

Thus we get

$$W_1 = \frac{e^2}{2} \sum_{nmrs} 2\pi\delta(E_n - E_m + E_r - E_s) \int d\vec{x}d\vec{y} \left\{ \left[ S \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \right]_n^\dagger \left[ S \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \right]_m \left[ S \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \right]_r^\dagger \left[ S \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \right]_s - \left[ S \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \right]_n^\dagger \vec{\alpha} \left[ S \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \right]_m \left[ S \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \right]_r^\dagger \vec{\alpha} \left[ S \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \right]_s \right\} \int \frac{d\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{[(E_n - E_m)^2 - |\vec{k}|^2]}, \quad (3.2)$$

where  $K = 1/\sqrt{2E(E+M)}$  is the normalization constant.

When we expand Eq.(3.7), it is seen that some terms are proportional to the power of  $(\vec{\sigma}\cdot\vec{\pi})$ . Using  $\chi = \frac{\vec{\sigma}\cdot\vec{\pi}}{E+M}\varphi$  we express all terms in the expansion of Eq.(3.2) in terms of  $\varphi$ . Most of the terms in the final expression are seen to be proportional to non-linear powers of  $(\vec{\sigma}\cdot\vec{\pi})$ . However we are interested only in terms that are proportional to external magnetic field and we neglect all the other terms. Taking the properties of Dirac-delta function into account we write the remaining terms as follows:

### 3.1. Case I: $n = m$ , $r = s$ (Vacuum polarization)

$$\begin{aligned}
 W_1 = & -\frac{e^2}{(2\pi)^3} \sum_{nm} 4\pi \int d\vec{x}d\vec{y} K_n^2 K_m^2 \{ (E_n + M)^2 \varphi_n^\dagger(\vec{x}) \varphi_n(\vec{x}) \varphi_m^\dagger(\vec{y}) (\vec{\sigma} \cdot \vec{\pi})^2 \varphi_m(\vec{y}) \\
 & + (E_m + M)^2 \varphi_n^\dagger(\vec{x}) (\vec{\sigma} \cdot \vec{\pi})^2 \varphi_n(\vec{x}) \varphi_m^\dagger(\vec{y}) \varphi_m(\vec{y}) - 4(E_n + M)(E_m + M) \\
 & \times \varphi_n^\dagger(\vec{x}) \vec{\pi} \varphi_n(\vec{x}) \varphi_m^\dagger(\vec{y}) \vec{\pi} \varphi_m(\vec{y}) \} \int \frac{d\vec{k}}{|\vec{k}|^2} e^{i\vec{k} \cdot (\vec{x} - \vec{y})}
 \end{aligned} \tag{3.3}$$

### 3.2. Case II: $n = s$ , $m = r$ (Lamb shift and other contributions)

$$\begin{aligned}
 W_1 = & -\frac{e^2}{(2\pi)^3} \sum_{nm} 4\pi \int d\vec{x}d\vec{y} K_n^2 K_m^2 \{ (E_n + M)(E_m + M) [\varphi_n^\dagger(\vec{x}) \varphi_m(\vec{x}) \varphi_m^\dagger(\vec{y}) \\
 & \times (\vec{\sigma} \cdot \vec{\pi})^2 \varphi_n(\vec{y}) + \varphi_n^\dagger(\vec{x}) (\vec{\sigma} \cdot \vec{\pi})^2 \varphi_m(\vec{x}) \varphi_m^\dagger(\vec{y}) \varphi_n(\vec{y})] - 2(E_n + M)(E_m + M) \\
 & \times [\varphi_n^\dagger(\vec{x}) \vec{\pi} \varphi_m(\vec{x}) \varphi_n^\dagger(\vec{y}) \vec{\pi} \varphi_m(\vec{y}) - \varphi_n^\dagger(\vec{x}) (\vec{\sigma} \times \vec{\pi}) \varphi_m(\vec{x}) \varphi_m^\dagger(\vec{y}) (\vec{\sigma} \times \vec{\pi}) \varphi_n(\vec{y})] - \\
 & 2(E_n + M)^2 [\varphi_n^\dagger(\vec{x}) \vec{\pi} \varphi_m(\vec{x}) \varphi_n^\dagger(\vec{y}) \vec{\pi} \varphi_m(\vec{y}) - \varphi_n^\dagger(\vec{x}) (\vec{\sigma} \times \vec{\pi}) \varphi_m(\vec{x}) \varphi_m^\dagger(\vec{y}) \\
 & \times (\vec{\sigma} \times \vec{\pi}) \varphi_n(\vec{y})] \} \int \frac{d\vec{k}}{(E_n - E_m)^2 - |\vec{k}|^2} e^{i\vec{k} \cdot (\vec{x} - \vec{y})}
 \end{aligned} \tag{3.4}$$

In Eq.(3.3) and Eq.(3.4) we first used the dipole approximation (i.e.,  $\lambda \ll r_o$  or  $e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \approx 1$ , where  $r_o$  and  $\lambda$  are atomic radia and radiated wavelength, respectively); we replaced  $E = \varepsilon + M$  for energy to pass into non-relativistic region; and, finally, we used equation  $(\vec{\sigma} \cdot \vec{\pi})^2 = \pi^2 - e\vec{\sigma} \cdot \vec{B}$  to get the non-linear self-field part of action as

$$W_1 = \frac{4\alpha\Lambda}{M^2} \sum_n [\langle n | e\vec{\sigma} \cdot \vec{B}^{Ext} | n \rangle + \langle n | \vec{\pi}^2 | n \rangle] \tag{3.5}$$

and the linear kinetic part

$$W_o = \sum_n 2\pi [\langle n | [\varepsilon - \frac{\vec{\pi}^2}{2M} + \frac{e}{2M} \vec{\sigma} \cdot \vec{B}^{Ext}] | n \rangle]. \tag{3.6}$$

Summing Eq.(3.5) and Eq.(3.6) we get

$$W = \sum_n 2\pi \langle n | [\varepsilon + \Delta E - \frac{\vec{\pi}^2}{2M} \left(1 - \frac{8\alpha\Lambda}{2\pi M}\right) + \frac{e}{2M} \vec{\sigma} \cdot \vec{B}^{Ext} \left(1 + \frac{8\alpha\Lambda}{2\pi M}\right)] | n \rangle, \tag{3.7}$$

where  $\alpha = e^2/4\pi$ . Defining a mass renormalization as

$$M_R^{-1} = \frac{1}{M} \left(1 - \frac{8\alpha\Lambda}{2\pi M}\right) \tag{3.8}$$

and rearranging Eq.(3.7) we find

$$W = \sum_n 2\pi \{ \langle n | [\varepsilon + \Delta E - \frac{\vec{\pi}^2}{2M_R} + \frac{e}{2M_R} \vec{\sigma} \cdot \vec{B}^{Ext} (1 + \frac{16\alpha\Lambda}{2\pi M} + O(\alpha^2))] | n \rangle. \tag{3.9}$$

The fourth term in the above equation has the factor

$$\left(1 + \frac{16\alpha\Lambda}{2\pi M_R} + O(\alpha^2)\right). \tag{3.10}$$

where the second term is the gyromagnetic ratio. Since we are only interested in the terms that are linear in  $\alpha$ , we neglect all the others.

## 4. Conclusion

The main contribution of this work to the previous investigations is the addition of new terms due to negative energy states which are at the same order positive terms. If the cutoff value of the momentum integral is chosen as  $\Lambda = M_R/16$  our final expression for anomalous magnetic moment is in agreement with standard quantum electrodynamics ( $a_e = \alpha/2\pi$ ) [12, 13].

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