On a Geometrical Description of Quantum Mechanics

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We show that Quantum Mechanics can be interpreted as a modification of the Euclidean nature of 3-d space into a particular Weyl affine space which we call Q-wis. This is proved using the Bohm-de Broglie causal formulation of Quantum Mechanics. In the Q-wis geometry, the length of extended objects changes from point to point. In our proposed geometrical formulation, deformation of the standard rulers used to measure physical distances are in the core of quantum effects.

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I. INTRODUCTION

The early years of quantum mechanics were marked by intense debates and controversies related to the meaning of the new behavior of matter. While one group was convinced that was unavoidable to abandon the classical picture, the other group tried incessantly to save its main roots and conceptual pillars. To be able to reproduce the atypical quantum effects, the latter group was forced to introduce new ingredients such as de Broglie's pilot wave [1] or Mandelung's hydrodynamical picture [2].

However, the lack of physical explanations for these ad hoc modifications weakened these pictures. At the same time, the former group leaded by Schrödinger, Bohr and Heisenberg was increasingly gaining new adepts until its climax in the 1927 Solvay's conference when this picture was finally accepted as the orthodox interpretation of quantum mechanics - the Copenhagen interpretation [3].

Notwithstanding, a marginal group of physicists continued to develop other approaches [4] to describe quantum mechanics that are more adequate to connect to a classical picture[1]. One of the most prominent amongst these alternative interpretations is the causal interpretation of quantum mechanics also known as Bohm-de Broglie interpretation [5].

The development of quantum cosmological scenario brought to light some difficulties intrinsic to the Copenhagen interpretation. More specifically, the measurement process in a quantum closed universe seems inevitably inconsistent [6]. Fortunately, there are some alternative interpretations that are consistently applied simultaneously to cosmology and to the micro-world. As two examples we mention the many-worlds interpretation [7] and the consistent histories formulation [8].

In the present work we will focus only on the Bohm-de Broglie interpretation since it is amongst the well defined interpretation that can be applied to any kind of system, including the universe as a whole, and up to date it is completely equivalent to the Copenhagen interpretation when applied to the micro-world.

We will show that it is possible to interpret all quantum phenomena as a modification of the geometrical properties of the physical space. Hence, we will deal with a generalization of Euclidean geometry that was first introduced by Weyl [9].

Weyl proposed a different modification of the Euclidean geometry than the one developed by Riemann. In fact, a Riemannian geometry can be understood as a special case of a Weyl geometry. In appendix A we describe in more details the geometrical properties of a Weyl geometry, but it is worth to mention its main difference that is related to the notion of a standard ruler.

In a Weyl geometry the length of an extended object changes from point to point. This means that a ruler of length l will change by an amount

$$\delta l = l\, f_a\, \mathrm{d}x^a \quad .$$

This effect may become an obstacle to the notion of a local ruler and thus to local measurement of distance. However, there is a special class of Weyl geometries known as Weyl Integrable Space (Q-wis) that is free of such difficulty.

This is provided by the condition that the vector f_i is a gradient of a function, i.e. $f_a = f_{,a}$. Q-wis is distinguished precisely by the fact that the length of the ruler transported along a closed curve does not change. Hence, if the change of the ruler's length is $\mathrm{d}l$, for a closed path in Q-wis we have

$$\oint \mathrm{d}l = 0 \qquad ,$$

which guarantees the uniqueness of any local measurement. The allowance of an intrinsic modification of the standard rulers is the main geometrical hypothesis of the present work. Similarly to London [10] and Santamato [11], we shall argue how this geometrical modification can be in the origin of quantum effects. For the sake of clarity we will deal with the simplest system possible,

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[1] Since we are not concerned with relativistic phenomena, the term classical physics should be understood as pre-relativistic physics unless otherwise specified.

namely an isolated point-like particle possibly subjected to an external potential.

The outline of the article is as follows. In the next section we briefly review the main points of quantum mechanics and in section III we describe how to connect the Q-wis space to quantum theory. We show that quantum mechanics can be derived from a geometrical variational principle. In section IV we present our final remarks. Appendix A is reserved to describe the main properties of the Q-wis geometry.

II. QUANTUM MECHANICS

Quantum mechanics is a modification of the classical laws of physics to incorporate the uncontrolled disturbance caused by the macroscopic apparatus necessary to realize any kind of measurement. This statement, known as Bohr's complementary principle, contains the main idea of the Copenhagen interpretation of quantum mechanics. The quantization program continues with the correspondence principle promoting the classical variables into operators and the Poisson brackets into commutation relations.

In this non-relativistic scenario, the Schrödinger equation establishes the dynamics for the wave function describing the system. Note that as in newtonian mechanics time is only a external parameter and the 3-d space is assumed to be endowed with the Euclidean geometry.

Using the polar form for the wave function, $\Psi = A\, e^{iS/\hbar}$, the Schrödinger equation can be decomposed in two equations for the real functions $A\,(x)$ and $S\,(x)$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S.\nabla S + V - \frac{\hbar^2}{2m}\frac{\nabla^2 A}{A} = 0 \quad , \tag{1}$$

$$\frac{\partial A^2}{\partial t} + \nabla\left(A^2\frac{\nabla S}{m}\right) = 0 \quad . \tag{2}$$

Solving these two equations is completely analogous to solving the Schrödinger equation. The probabilistic interpretation of quantum mechanics associate $\|\Psi\|^2 = A^2$ with the probability distribution function on configuration space. Hence, eq. (2) has exactly the form of a continuity equation with $A^2\nabla S/m$ playing the role of current density.

A. Bohm-de Broglie interpretation

The causal interpretation which is an ontological hidden variable formulation of quantum mechanics, propose that the wave function does not contain all the information about the system.

An isolated system describing a free particle (or a particle subjected to a potential V) is defined simultaneously by a wave function and a point-like particle. In this case, the wave function still satisfies the Schrödinger equation but it should also works as a guiding wave modifying the particle's trajectory.

Note that eq.(1) is a Hamilton-Jacobi like equation with an extra term that it is often called quantum potential

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 A}{A} \quad , \tag{3}$$

while, as already mentioned, eq.(2) is a continuity-like equation. The Bohm-de Broglie interpretation take these analogies seriously and postulate an extra equation associating the velocity of the point-like particle with the gradient of the phase of the wave function. Hence,

$$\dot{x} = \frac{1}{m}\nabla S \quad . \tag{4}$$

Integrating eq. (4) yields the quantum bohmian trajectories. The unknown or hidden variables are the initial positions necessary to fix the constant of integration of the above equation.

The quantum potential is the sole responsible for all novelties of quantum effects such as non-locality or tunneling processes. As a matter of fact, the Bohm-de Broglie interpretation has the theoretical advantage of having a well formulated classical limit. Classical behavior is obtained as soon as the quantum potential, which has dimensions of energy, becomes negligible compared to other energy scales of the system.

In what follows, we will show that it is possible to reinterpret quantum mechanics as a manifestation of non-Euclidean structure of the 3-dimensional space. Hence, we propose a geometrical interpretation to describe quantum effects.

III. NON-EUCLIDEAN GEOMETRY

Since ancient times, Euclidean geometry was considered as the most adequate mathematical formulation to describe the physical space. However, its validity can only be established a posteriori as long as its construction yields useful notions to connect physical quantities such as the Euclidean distance between two given points.

Special relativity modified the notion of 3-dimensional Euclidean space to incorporate time in a four-dimensional continuum (Minkowski spacetime). Later on, General Relativity generalized the absolute Minkowski spacetime to describe gravitational phenomena. General Relativity considers the spacetime manifold as a dynamical field that can be deformed and stretched but in such a way that it always preserves its Riemannian structure. It is worth noting that both the Euclidean and Minkowskian spaces are nothing more than special cases of Riemaniann spaces.

Nonetheless, Riemannian manifold are not the most general type of geometrical spaces. In the same way as above, Riemannian geometries can be understood as a special subclass of a more general structure known as

Weyl space. As to the matter of which geometry is actually realized in Nature, it has to be determined by physical experiments.

Instead of imposing a priori that quantum mechanics has to be constructed over an Euclidean background as it is traditionally done, we shall argue that quantum effects can be interpreted as a manifestation of a non-Euclidean structure derived from a variational principle. The validity of the specific geometrical structure proposed can be checked a posteriori comparing it to the usual non-relativistic quantum mechanics.

Thus, consider a point-like particle with velocity $v = \nabla S/m$ and subjected to a potential V. Following Einstein's idea to derive the geometrical structure of space from a variational principle by considering the connection as an independent variable, we start with

$$I = \int \mathrm{d}t\mathrm{d}^3x\sqrt{g}\,\Omega^2\left(\lambda^2\mathcal{R} - \frac{\partial S}{\partial t} - \mathcal{H}_m\right) \quad , \qquad (5)$$

and consider the connection of the 3-d space Γ^i_{jk}, the Hamilton's principal function S and the scalar function Ω as our independent variables.

Each one of the terms in equation (5) is understood as follows: we are considering the line element in cartesian coordinates given by

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

with

$$g = \det g_{ij} \quad .$$

The Ricci curvature tensor is defined in term of the connection through

$$\mathcal{R}_{ij} = \Gamma^m_{mi\,,j} - \Gamma^m_{ij\,,m} + \Gamma^l_{mi}\Gamma^m_{jl} - \Gamma^l_{ij}\Gamma^m_{lm}$$

and its trace defines the curvature scalar $\mathcal{R} \equiv g^{ij}\mathcal{R}_{ij}$ which has dimensions of inverse length squared, $[\mathcal{R}] = L^{-2}$. The constant λ^2 has dimension of energy times length squared, $[\lambda^2] = E.L^2$, and the $\frac{\partial S}{\partial t}$ term is related to the particle's energy. In the case of our point-like particle the matter hamiltonian is simply

$$\mathcal{H}_m = \frac{1}{2m}\nabla S.\nabla S + V \quad .$$

¿From equation (5), variation of the action I with respect to the independent variables results (see appendix A for details)

$$\delta\Gamma^i_{jk} : \qquad g_{ij;k} = -4\left(\ln\Omega\right)_{,k}\, g_{ij} \quad , \qquad (6)$$

where ";" denotes covariant derivative and a common "," simple spatial derivative. Equation (6) characterize the affine properties of the physical space. Hence, the variational principle naturally defines a Weyl Integrable Space. Variation with respect to Ω gives

$$\delta\Omega : \qquad \lambda^2\mathcal{R} = \frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S.\nabla S + V \quad . \qquad (7)$$

The right-hand side of this equation has dimension of energy while the curvature scalar has dimension of $[\mathcal{R}] = L^{-2}$. Furthermore, apart from the particle's energy, the only extra parameter of the system is the particle's mass m. Thus, there is only one way to combine the unknown constant λ^2, which has dimension of $[\lambda^2] = E.L^2$, with the particle's mass such as to form a physical quantity. Multiplying them, we find a quantity that has dimension of angular momentum squared $[m.\lambda^2] = \hbar^2$.

In terms of the scalar function Ω, the curvature scalar is given by (see appendix A)

$$\mathcal{R} = 8\frac{\nabla^2\Omega}{\Omega} \quad . \qquad (8)$$

Hence, setting $\lambda^2 = \hbar^2/16m$, equation (7) becomes

$$\delta\Omega : \qquad \frac{\partial S}{\partial t} + \frac{1}{2m}\nabla S.\nabla S + V - \frac{\hbar^2}{2m}\frac{\nabla^2\Omega}{\Omega} = 0 \quad , \qquad (9)$$

Finally, varying the Hamilton's principal function S we find

$$\delta S : \qquad \frac{\partial\Omega^2}{\partial t} + \nabla\left(\Omega^2\frac{\nabla S}{m}\right) = 0 \quad . \qquad (10)$$

Equations (9) and (10) are identical to equations (1) and (2) if we identify $\Omega = A$. Thus, the "action" of a point-like particle non-minimally coupled to geometry given by

$$I = \int \mathrm{d}t\mathrm{d}^3x\sqrt{g}\,\Omega^2\left[\frac{\hbar^2}{16\,m}\mathcal{R} - \left(\frac{\partial S}{\partial t} + \mathcal{H}_m\right)\right] \quad , \qquad (11)$$

exactly reproduce the Schrödinger equation and thus the quantum behavior.

The straightest way to compare this geometrical approach to the common quantum theories is to relate it to the Bohm-de Broglie interpretation[2]. Note that this formulation has the advantage of giving a physical explanation of the appearance of the quantum potential, eq. (3). In Q-wis, this term is simply the curvature scalar of the Weyl integrable space. The inverse square root of the curvature scalar defines a typical length L_w (Weyl length) that can be used to evaluate the strength of quantum effects

$$L_w \equiv \frac{1}{\sqrt{\mathcal{R}_w}} \quad .$$

As we have already mentioned, the classical limit of Bohm-de Broglie interpretation is achieved when the quantum potential is negligible compared to other energy scales of the system. In the scope of this geometrical approach, the classical behavior is recovered when

[2] Up to date, all interpretation of quantum mechanics are on equal footing. Thus, establishing the connection with the causal interpretation automatically links this geometrical interpretation with all others.

the length defined by the Weyl curvature scalar is small compared to the typical length scale of the system. Once the Weyl curvature becomes non-negligible the system goes into a quantum regime.

A. Geometrical uncertainty principle

As long as we accept that quantum mechanics is a manifestation of a non-Euclidean geometry, we are faced with the need of reinterpreting geometrically all theoretical issues related to quantum effects. As a first step, we derive the uncertainty principle as a break down of the classical notion of a standard ruler.

It is well known amongst relativistic physicists that there is no absolute notion of spatial distance in curved spacetime. However, this is no longer true when there is an absolute newtonian time and only the spatial manifold is allowed to be curved. In this case, it is possible to define distance as the smallest length between two given points calculated along geodesics in 3-d space. This is a consistent definition since the 3-d space has a true metric in the mathematical sense that its eigenvalues are all positives. However, this definition does not encompass the classical definition of a standard ruler.

Hence, we are unable to perform a classical measurement to distances smaller than the Weyl curvature length. In other words, the size of a measurement has to be bigger than the Weyl length

$$\Delta L \geq L_w = \frac{1}{\sqrt{\mathcal{R}_w}} \quad . \tag{12}$$

The quantum regime is extreme when the Weyl curvature term dominates. Thus, from equations (8) and (9) we have

$$\mathcal{R}_w = 2\left(\frac{2\Delta p}{\hbar}\right)^2 - \frac{16m}{\hbar^2}\left(\frac{\partial S}{\partial t} - V\right) \leq 2\left(\frac{2\Delta p}{\hbar}\right)^2 \tag{13}$$

and finally combining equations (12) and (13) we obtain

$$\Delta L . \Delta p \geq \frac{\hbar}{2\sqrt{2}} \quad .$$

We should emphasize that now the Heisenberg's uncertainty relation has a pure geometrical meaning. Our argument closely resembles Bohr's complementary principle inasmuch as the impossibility of applying the classical definitions of measurements. However, we strongly diverge with respect to the fundamental origin of the physical limitation.

Bohr's complementary principle is based on the uncontrolled interference of a classical apparatus of measurement. On the other hand, we argue that the notion of a classical standard ruler breaks down because its meaning is intrinsically dependent on the validity of Euclidian geometry. Once it becomes necessary to include the Weyl curvature, we are no longer able to perform a classical measurement of distance.

There is another way to interpret the uncertainty principle. For a given particle of mass m and energy E there is only one combination with the free parameter of the theory ($\hbar$) that furnishes a quantity with dimensions of length. We take this value as a definition of the classical size of the particle, namely

$$l_{part} \equiv \sqrt{\frac{\hbar^2}{E\, m}} \quad . \tag{14}$$

Note that this definition coincides with the Compton's wavelength of the particle which is related to the limits of validity of non-relativistic quantum mechanics.

Considering a free stationary particle, from equation (7) we have

$$E = \frac{\hbar^2}{16m}\mathcal{R}_W \;\Rightarrow\; l_{part} = \frac{4}{\sqrt{\mathcal{R}_W}} \quad ,$$

and from equation (13)

$$l_{part} . \Delta p \geq \sqrt{2}\, \hbar \quad . \tag{15}$$

¿From this point of view, the uncertainty principle indicates that it is impossible to perform a measurement smaller than the classical size of the particle defined by equation (14). In other words, it is impossible to perform a classical measurement inside the particle.

IV. CONCLUSIONS

It is well known that as soon as we consider high velocities or high energies one has to abandon the Euclidean geometry as a good description of the physical space. These brought two completely different modifications where the physical space loses its absolute and universal character. In fact, these are the core of classical relativistic physical theories, namely Special and General Relativity.

In a similar way, one should be allowed to consider that the difficulties that appears while going from classical to quantum mechanics comes from an inappropriate extrapolation of the Euclidean geometry to the microworld. Hence, the unquestioned hypothesis of the validity of the 3-d Euclidean geometry to all length scales might be intrinsically related to quantum effects.

In the present work, we have shown that there is a close connection between the Bohm-de Broglie interpretation of quantum mechanics and the Q-wis Weyl integrable space. In fact, we point out that the Bohmian quantum potential can be identified with the curvature scalar of the Q-wis. Moreover, we present a variational principle that reproduces the Bohmian dynamical equations considered up to date as equivalent to Schrödinger's quantum mechanics.

The Palatini-like procedure, in which the connection acts as an independent variable while varying the action, naturally endows the space with the appropriate Q-wis structure. Thus, the Q-wis geometry enters into the theory less arbitrarily than the implicit ad hoc Euclidean hypothesis of quantum mechanics.

The identification of the Q-wis curvature scalar as the ultimate origin of quantum effects leads to a geometrical version of the uncertainty principle. This geometrical description considers the uncertainty principle as a break down of the classical notion of standard rulers. Thus, it arises an identification of quantum effects to the length variation of the standard rulers.

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APPENDIX A: Q-WIS GEOMETRY

In this session we shall briefly review the mathematical properties of such 3-d Weyl Integrable Space (Q-wis).

Contrary to the Riemannian geometry which is completely specified by a metric tensor, the Weyl space defines an affine geometry. This means that the covariant derivative which is defined in terms of a connection Γ^m_{ik} depends not only on the metric coefficients but also on a vector field $f_a(x)$.

For instance, given a vector X_a its covariant derivative is

$$X_{a\,;b} = X_{a\,,\,b} - \Gamma^m_{ab}\,X_m \quad . \tag{A1}$$

The non-metricity of the Weyl geometry implies that rulers, which are standards of length measurement, changes while we transport it by a small displacement $\mathrm{d}x^i$. This means that a ruler of length l will change by an amount

$$\delta\, l = l\, f_a\, \mathrm{d}x^a \quad . \tag{A2}$$

As a consequence, the covariant derivative of the metric tensor does not vanishes as in a Riemannian geometry but instead it is given by

$$g_{ab\,;\,k} = f_k\, g_{ab} \quad . \tag{A3}$$

Using cartesian coordinates, it follows that the expression for the connection in terms of the vector f_k takes the form

$$\Gamma^k_{ab} = -\,\frac{1}{2}\,\left(\delta^k_a\, f_b + \delta^k_b\, f_a - g_{ab}\, f^k\right) \quad . \tag{A4}$$

The particular case of Weyl Integrable Space is provided by the condition that the vector f_i is a gradient of a function, i.e. $f_a = f_{,\,a}$. This property ensures that the length does not changes its value along a closed path

$$\oint \mathrm{d}l = 0 \quad . \tag{A5}$$

As a matter of convenience, we define

$$f = -4\, ln\, \Omega \quad . \tag{A6}$$

Then the Ricci tensor

$$\mathcal{R}_{ij} = \Gamma^m_{mi,j} - \Gamma^m_{ij,m} + \Gamma^l_{mi}\Gamma^m_{jl} - \Gamma^l_{ij}\Gamma^m_{lm}$$

constructed with the above connection equation (A4) is given by

$$\mathcal{R}_{ij} = 2\frac{\Omega_{,ij}}{\Omega} - 6\frac{\Omega_{,i}\Omega_{,j}}{\Omega^2} + 2g_{ij}\left[\frac{\nabla^2\Omega}{\Omega} + \frac{\vec{\nabla}\Omega.\vec{\nabla}\Omega}{\Omega^2}\right]$$

and the scalar of curvature $\mathcal{R} \equiv g^{ij}\mathcal{R}_{ij}$ becomes

$$\mathcal{R} = 8\,\frac{\nabla^2\Omega}{\Omega} \quad . \tag{A7}$$

In the present paper we have used a variational principle to arrive at the Q-wis structure. The proof is as follows. Consider the action

$$I = \int \mathrm{d}t\, \mathrm{d}^3x\sqrt{g}\;\, \Omega^2\, \mathcal{R} \tag{A8}$$

then, variation of the connection yields

$$\begin{aligned} \delta I \;&=\; \int \mathrm{d}t\, \mathrm{d}^3x\sqrt{g}\;\, \Omega^2\, g^{ab}\, \delta R_{ab} \\ &=\; \int \mathrm{d}t\, \mathrm{d}^3x\; Z^{ab}_m\, \delta\Gamma^m_{ab} \end{aligned} \tag{A9}$$

with

$$Z^{ab}_m \equiv (\sqrt{g}\, g^{ab}\, \Omega^2)_{;\,m} - \frac{1}{2}\,(\sqrt{g}\, g^{ak}\, \Omega^2)_{;\,k}\, \delta^b_m - \frac{1}{2}\,(\sqrt{g}\, g^{bk}\, \Omega^2)_{;\,k}\, \delta^a_m \tag{A10}$$

Taking the trace of this expression yields

$$\left(\sqrt{g}\, g^{ak}\, \Omega^2\right)_{;\,k} = 0 \quad . \tag{A11}$$

Substituting (A11) in (A10) we finally obtain the condition for a Weyl integrable geometry

$$g_{ab\,;\,k} = -\,4\,\frac{\Omega_{,\,k}}{\Omega}\, g_{ab} \quad , \tag{A12}$$

or using that $g^{ik}g_{kj} = \delta^i_j$ we find the contra-variant expression

$$g^{ab}{}_{;\,k} = 4\,\frac{\Omega_{,\,k}}{\Omega}\, g^{ab} \quad . \tag{A13}$$

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