

Entanglement generation by collisions of quantum solitons in the Born's approximation

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We present analytic expressions describing generation of the entanglement in collisions of initially uncorrelated quantum solitons. The results, obtained by means of the Born's approximation (for fast solitons), are valid for both integrable and non-integrable quasi-one-dimensional systems supporting soliton states.

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I. INTRODUCTION

Entanglement is a fundamental property of bipartite quantum systems [1]. Apart from being a major resource for quantum-information techniques, entanglement exhibits itself perhaps in the most spectacular form in the breakdown of Bell's inequalities and Einstein-Podolsky-Rosen "paradox". In experiments, strongly entangled states, and in particular those exhibiting "nonlocality", are typically created with microscopic particles produced by the same source, or interacting prior to the detection, such as pairs of photons [2, 3] or ions [4]. In their original paper [5], Einstein, Podolsky and Rosen considered two particles whose ordinary degrees of freedom, center-of-mass positions and relative momenta, were correlated. This situation is realized with the entanglement of Gaussian states of light [6] or atomic ensembles [7].

Particularly interesting is the possibility to generate entanglement between *macroscopic* (or mesoscopic) objects that may be transmitted over long distances. In this connection, *solitons*, i.e., stable solitary waves that propagate without distortion [9], may be considered as possible robust quantum-information carriers. In particular, the solitons as collective excitations in nonlinear media (unlike ions or other material objects) may be created with a desirable shape and effective mass, and admit a much greater degree of control by means of various "management" techniques [11].

The objective of the present work is to analyze the possibility of the creation of entanglement between solitons by means of collisions between them. To the best of our knowledge, this problem was not considered before (except for a preliminary version of the present work [12]). However, a very recent work [13] has presented a rigorous proof of the generation of entanglement between constituent solitons in oscillating two- and three-soliton bound states, in the integrable model based on the NLS (nonlinear Schrödinger) equation with the cubic nonlinearity. It is relevant to note that multi-soliton solutions of the integrable NLS equations are actually unstable against slow separation, hence the entanglement, generated by the interactions between the solitons while

they remained bound, may be kept after the separation.

Natural candidates for the study of the collision-induced entanglement are matter-wave solitons, that can be built of ultracold bosonic atoms, which form the Bose-Einstein condensate (BEC) [8], and photonic solitons in nonlinear optical waveguides [9, 10]. The studies of BECs has led to the creation of dark [14, 15] and bright [16, 17, 19] solitons in trapped Bose-Einstein condensates (with repulsive and attractive interactions, respectively). Recently, the observation of stable bright-dark soliton pairs has been reported too [21]. In addition, similarities to nonlinear optics [9] have triggered the interest in theoretical studies of discrete (lattice) BEC solitons [22, 23], and have also led to the seminal observation of gap solitons, i.e., robust localized matter-wave packets supported by the interplay of repulsive interactions and an effective negative mass of collective excitations in the condensate, induced by the periodic optical-lattice potential [20]. The analysis reported below suggests that the matter-wave solitons in BEC with attractive inter-atomic interactions have the best potential for the generation of the entanglement through collisions; in particular, the velocity of the moving solitons, which is the crucial parameter, which determines the collision-induced entanglement (see below), can be easily controlled for BEC solitons.

While most of the previous studies of matter-wave solitons were concentrated on their classical and mean-field aspects, more recently considerable interest has been devoted to the role of thermal noise [23, 25] and quantum fluctuations. The latter may cause filling up of the dark-soliton's core through the quantum depletion process, as was predicted using the Bogoliubov-de Gennes (BdG) equations [26]. Making use of the discrete NLS equation and the time-evolving block-decimation algorithm [27] in the framework of the Bose-Hubbard model, it was confirmed that quantum effects lead to the filling in of dark BEC solitons, and it has been demonstrated that collisions between them become inelastic [28]. The BdG approach and its generalizations were also employed to study the solitons' stability [29, 30, 31] and excitations caused by opening of the trap [32]. In the latter work, exact Lieb-Liniger solutions were also used, to calculate

internal correlation functions of positions of the particles. A noisy version of standing bright solitons was studied by means of the exact diagonalization and quantum Monte-Carlo method [33].

Being typically associated with integrable 1D classical models, stable solitons are known too in non-integrable systems, including multidimensional ones. One of remarkable properties of solitons is their stability to perturbations. The stability can be extended to quantum settings, in which the mean-field description admits quantum solitons in the semi-classical form. As mentioned above, the robustness of mean-field solitons against quantum fluctuations (including finite-temperature effects) was studied, using the time-dependent Hartree-Fock-Bogoliubov equations, for both ordinary matter-wave solitons, supported by attractive interactions between atoms [30], and for gap solitons [31]. Although under extreme conditions the quantum fluctuations may split a mean-field soliton [30], it has been concluded that, in a broad range of parameters relevant to the experiments, the matter-wave solitary waves predicted by the mean-field description (i.e., found as stable localized solutions to the Gross-Pitaevskii equation, GPE [34]) are completely robust objects – in fact, in perfect agreement with the experimental observations of these solitons [17]-[20]. As concerns the relation between the mean-field and quantum descriptions of localized objects in BEC, it is relevant to mention that an alternative derivation of the GPE from a consistent many-body quantum theory, based on the variational approach in the multi-configurational space, was recently presented in Refs. [35].

In this work, we analyze the collision-induced generation of the entanglement in pairs of fast solitons for a class of equations of the NLS type by means of the Born approximation, which is valid when the kinetic energy of the moving objects (solitons) is much larger than the potential of the interaction between them, hence the interaction may be treated as a perturbation [36] (therefore, this is the case opposite to that studied in Ref. [13], where the entanglement was considered between bound solitons created with zero relative velocity). We aim to present simple analytical expressions for the collision-generated entanglement of two quantum solitons, which are valid for generic quasi-1D systems – integrable or not – that admit soliton solutions. Another approach is possible for nearly quiescent solitons, when the entanglement-generating perturbation is the weak interaction between them, assuming that the distance between the solitons is large enough. Recently, this approach was developed in Ref. [37] for kinks in the sine-Gordon equation. In that model, the kinks were maintained in the quiescent state by boundary conditions, as the model was defined in a domain of a finite size.

The rest of the paper is organized as follows. In Section II, we introduce the model, starting with the known system of classical equations of motion for a pair of well-separated solitons [10], and then proceed to the respec-

tive quantum system. We confine the analysis to the basic case of the pair of symmetric solitons with equal amplitudes. Mismatch between the amplitudes makes the interaction effectively incoherent [38], thus suppressing the entanglement generation (we estimate the size of the mismatch up to which it may be neglected). The initial quantum state, corresponding to far separated solitons, is taken as a product of two independent wave packets. In Section III, the calculation of the correction to this factorized state, generated by the collision between the fast solitons, is performed in an analytical form, by means of the above-mentioned Born approximation. The collision-induced correction to the wave function features *explicit entanglement* in terms of two relevant degrees of freedom, *viz.*, the distance between the solitons and their relative phase, r and χ . The results are reported for the initial condition of two types: a more sophisticated one, with the Gaussian localization of χ around a definite value, and also for a simple phase-independent initial distribution. Both types of the initial quantum states may be realized in the experiment, under different specific conditions. The paper is concluded by Section IV, where, in particular, we discuss the robustness of the predicted entanglement against external noise, and possible extensions of the work.

II. THE MODEL

A. The classical soliton pair

For the condensate with attractive interactions between atoms, the scaled form of the GPE in the free 1D space, which describes the BEC in the mean-field approximation, is tantamount to the integrable NLS equation [34], i.e.,

$$iu_t + (1/2)u_{xx} + |u|^2 u = 0. \quad (1)$$

The commonly known soliton solution to Eq. (1) is

$$u_{\text{sol}} = \eta \operatorname{sech}[\eta((x - \xi(t)))] \exp[i\phi(t) + icx], \quad (2)$$

where η is the soliton's amplitude, $d\xi/dt = c$ its velocity, and

$$-d\phi/dt = (c^2 - \eta^2)/2 \quad (3)$$

the intrinsic frequency.

More realistic forms of the GPE in 1D include various terms which break the integrability of the NLS equation. In particular, the full equation must include the axial trapping potential, but, as concerns interactions between solitons, the axial potential is not a crucially important factor, according to the available experimental results [17] and theoretical analysis [39, 41]. Another physically relevant feature that may affect soliton-soliton collisions is a quintic nonlinear term. It may account for three-body collisions in the condensate, provided that

they are lossless [42], but a more general (in fact, universal) source of the quintic term is the deviation of the condensate loaded into a cigar-shaped trap from the one-dimensionality. This term can be derived by means of a perturbative analysis [43], or by the expansion of a more general equation, that takes into regard the underlying three-dimensionality via the nonpolynomial nonlinearity [44]. The universal quintic term always corresponds to *self-attraction* (irrespective of the sign of the binary interactions between atoms), with the coefficient in front of it proportional to the transverse-confinement frequency and square of the collisional scattering length. By means of straightforward rescalings, the equation with the combination of attractive cubic and quintic terms can be cast in the form that contains no free parameters,

$$iu_t + (1/2)u_{xx} + |u|^2 u + |u|^4 u = 0. \quad (4)$$

Although the cubic-quintic (CQ) NLS equation (4) is not integrable, its exact soliton solutions are well known [45], the entire family being stable [46]:

$$\tilde{u}_{\text{sol}} = \frac{\sqrt{2}\eta \exp[i\phi(t) + icx]}{\sqrt{\sqrt{1 + (8/3)\eta^2 \cosh[2\eta(x - \xi(t))]} - 1}}, \quad (5)$$

where $\phi(t)$ and $\xi(t)$ have the same meaning as in Eq. (2). The CQ model based on equation (4) finds other physical realizations in nonlinear optics [47].

The analysis which treats the interaction between well separated quasi-classical (mean-field) solitons (2) with common amplitude η as a perturbation leads to the conclusion that they may be considered as a pair of quasi-particles with an effective Hamiltonian [10],

$$H_{\text{tot}} = (1/2) \left(\dot{\rho}^2 - \dot{\theta}^2 + \dot{r}^2 - \dot{\chi}^2 \right) - e^{-|r|} \cos \chi, \quad (6)$$

where the overdot stands for the differentiation with respect to rescaled time $\tau \equiv 2\sqrt{2}\eta^2 t$, the normalized distance between the solitons and center-of-mass coordinate are defined as

$$r \equiv \eta(\xi_1 - \xi_2), \quad \rho \equiv \eta(\xi_1 + \xi_2), \quad (7)$$

while the relative and overall phases of the soliton pair are

$$\chi \equiv \phi_1 - \phi_2, \quad \theta \equiv \phi_1 + \phi_2 - \eta^2 t. \quad (8)$$

A noteworthy fact, which is obvious in expression (6), is that the effective mass corresponding to the phase degrees of freedom in the two-soliton set is *negative* [48].

Actually, Hamiltonian (6) is universal, applying to any model which supports exponentially localized solitons with an intrinsic phase degree of freedom [40]. In particular, this Hamiltonian governs the interaction between solitons (5) of the NLS equation with the CQ nonlinearity, Eq. (4), up to a proper rescaling of coefficients [41].

B. Quantization

Proceeding to the quantum version of the model considered above, we treat, as usual [49], each soliton as a quantum particle with two degrees of freedom, the position and phase (in the experimentally relevant situation, effects of quantum fluctuations around the quasi-classical shape of the solitons may be negligible for matter-wave solitons, as discussed above). Thus, the quantum counterpart of classical Hamiltonian 6 gives rise to the following linear Schrödinger equation for the wave function of the soliton pair, $\Psi^{(\text{tot})}$, which depends on the total set of four degrees of freedom describing the pair:

$$i\Psi_T^{(\text{tot})} = -(1/2) \left[\Psi_{rr}^{(\text{tot})} - \Psi_{\chi\chi}^{(\text{tot})} + \Psi_{\rho\rho}^{(\text{tot})} - \Psi_{\theta\theta}^{(\text{tot})} \right] - \varepsilon e^{-|r|} (\cos \chi) \Psi^{(\text{tot})}, \quad (9)$$

where $T \equiv \hbar\tau$ and $\varepsilon \equiv 1/\hbar^2$, with \hbar the renormalized Planck's constant [measured in scaled units in which Hamiltonian (6) was written]. The reduced form of Eq. (9) for the wave function which depends only on the relative variables, r and χ , is

$$i\Psi_T = - \left[\frac{1}{2} (\Psi_{rr} - \Psi_{\chi\chi}) + \varepsilon e^{-|r|} \cos \chi \right] \Psi. \quad (10)$$

Our objective is to analyze collisions between rapidly moving quantum solitons, with large relative momentum K_0 . The initial state is taken as a naturally expected non-entangled factorized one, centered around definite initial values of the dynamical variables, $\pm\xi_0$ and $\pm\chi_0$:

$$\begin{aligned} \Psi_0^{(\text{tot})}(\xi_1, \xi_2, \phi_1, \phi_2) &= \\ &\equiv \exp \left[-\frac{\eta_0^2 (\xi_1 - \xi_0)^2}{2\xi^2} - \frac{(\phi_1 - \chi_0)^2}{2\Phi^2} \right] e^{iK_0\eta(\xi_1 - \xi_0)} \\ &\times \exp \left[-\frac{\eta_0^2 (\xi_2 + \xi_0)^2}{2\xi^2} - \frac{(\phi_2 + \chi_0)^2}{2\Phi^2} \right] e^{-iK_0\eta(\xi_2 + \xi_0)}. \end{aligned} \quad (11)$$

This state assumes equal widths of the wave packets for the two solitons, Ξ and Φ , with the pair's center of mass set at $x = 0$, and initial separation $2\xi_0 \equiv \xi_1(T = 0) - \xi_2(T = 0)$. The mean value of the initial overall phase is also fixed to be zero, while the initial phase difference between the solitons is $2\chi_0 \equiv \phi_1(T = 0) - \phi_2(T = 0)$. Finally, initial state (11) can be written in terms of the variables defined in Eqs. (7) and (8),

$$\begin{aligned} \Psi_0^{(\text{tot})}(\xi_1, \xi_2, \phi_1, \phi_2) &= e^{iK_0(r - r_0)} \\ &\times \exp \left[-\frac{(r - r_0)^2 + \rho^2}{4\xi^2} - \frac{(\chi - \chi_0)^2 + \theta^2}{4\Phi^2} \right], \end{aligned} \quad (12)$$

where and $r_0 \equiv 2\eta\xi_0$.

To conclude the formulation of the model, it is relevant to consider the possibility of the excitation of an

intrinsic mode (IM) in the colliding solitons (if the IM exists). A known principle is that solitons of integrable equations do not support IMs, but a nonintegrable model may feature an IM. In particular, exactly one IM exists in excited states of solitons (5) of the CQ NLS equation, with the self-focusing sign of both nonlinear terms [46]. A possibility of using solitons' IMs as carriers of quantum information was proposed in Ref. [37]. However, in the case of the collision between fast solitons, which is considered below, the excitation of the IM, as well as generation of nonsoliton modes ("radiation"), may be neglected, in the lowest approximation, simply because the intensity of these effects is inversely proportional to the square of the collision velocity [10].

III. GENERATION OF ENTANGLEMENT BY COLLISIONS BETWEEN SOLITONS

A. The Born's approximation

Scattering solutions to reduced equation (10) are generated by the incident wave, $\Psi(r \rightarrow \infty) = \exp[iKr + i\kappa\chi - i(K^2 - \kappa^2)T/2]$, where κ is an integer. The full solution is sought for as

$$\Psi(T, r, \chi) = V(r, \chi) \exp[-i(K^2 - \kappa^2)T/2], \quad (13)$$

with the stationary part of the wave functions obeying the following equation,

$$V_{rr} - V_{\chi\chi} + [2\varepsilon e^{-|r|} \cos \chi + (K^2 - \kappa^2)] V = 0. \quad (14)$$

As said above, our basic assumption is that we consider the collision between fast solitons, i.e., K^2 is a large parameter in comparison with ε (κ^2 may be large too). In other words, the kinetic energy of the relative motion is much larger than the potential of the soliton-soliton interaction. In fact, the two-particle Hamiltonian (6) can be used for the description of collisions between solitons only in this case; otherwise, one cannot neglect deformation of the solitons in the course of the collision, as well as the excitation of the IM, if it exists in the solitons, and the generation of the radiation modes.

We apply the Born's approximation [36], looking for a solution to Eq. (14) as

$$V(r, \chi) = [1 + \varphi(r, \chi)] \exp(iKr + i\kappa\chi), \quad (15)$$

where perturbation φ , which is assumed to be a slowly varying function of r in comparison with $\exp(iKr)$, obeys the simplified equation: $-iK\varphi_r + (1/2)\varphi_{\chi\chi} + i\kappa\varphi_\chi = \varepsilon e^{-|r|} \cos \chi$. Solutions to this equation satisfying the necessary boundary condition, $\varphi(r = -\infty) = 0$, can be readily found in an analytical form. The outcome of the collision is determined by the asymptotic form of the solution, which is found to be

$$\begin{aligned} & \varphi(r \rightarrow +\infty, \chi) = (i\varepsilon/K) \\ & \times \left[e^{i\chi} e^{i(\kappa-1/2)r/K} + e^{-i\chi} e^{-i(\kappa+1/2)r/K} \right]. \end{aligned} \quad (16)$$

B. Analysis of the collision-induced entanglement

In the case of the fast collision, the overall variables, ρ and θ , may be treated as "frozen" ones in the wave packet generated by initial state (12). Then, it is natural to decompose the initial state over the set of plane waves with respect to variables r and χ :

$$\begin{aligned} \Psi_0^{(\text{tot})}(\xi_1, \xi_2, \phi_1, \phi_2) &= \exp\left(-\frac{\rho^2}{4\Xi^2} - \frac{\theta^2}{4\Phi^2}\right) \\ &\times (\Xi\Phi/\pi) \int_{-\infty}^{+\infty} dK \int_{-\infty}^{+\infty} d\kappa e^{iK(r-r_0) + i\kappa(\chi-\chi_0)} \\ &\times \exp\left[-\Xi^2(K-K_0)^2 - \Phi^2\kappa^2\right]. \end{aligned} \quad (17)$$

Note that the Gaussian distribution of angular wavenumber κ , which must be integer, is valid for $|\kappa| \gg 1$, i.e., $\Phi \ll 2\pi$ [recall that Φ is the width of the initial distribution of the phase variables introduced in Eqs. (11) and (12)].

Next, recombining wave packet (17) with the collision-induced perturbation of the wave function, as per Eqs. (15) and (16), and again making use of the fact that K_0 is large, we arrive at an expression for the net change of the wave function which is generated by the fast collision between the two solitons, in the first order of the perturbation theory:

$$\begin{aligned} \delta\Psi(\xi_1, \xi_2, \phi_1, \phi_2) &\approx \exp\left(-\frac{\rho^2}{4\Xi^2} - \frac{\theta^2}{4\Phi^2}\right) \\ &\times \frac{i\sqrt{2}\varepsilon\Phi}{K_0(4\Phi^4 + T^2)^{1/4}} \exp\left[i\left(K_0(r-r_0) - \frac{1}{2}K_0^2T\right)\right] \\ &\times \sum_{+,-} e^{\pm i\chi} \exp\left[-\frac{\Phi^2(\chi - \chi_0 \pm r/K_0)^2}{4\Phi^4 + T^2} + i\Omega_{\pm}\right], \end{aligned} \quad (18)$$

with phases shifts

$$\Omega_{\pm} \equiv \frac{1}{2} \left(\tan^{-1} \left(\frac{T}{2\Phi^2} \right) - \frac{T(\chi - \chi_0 \pm r/K_0)^2}{4\Phi^4 + T^2} \right). \quad (19)$$

A nontrivial feature of expression (18), which represents the *entanglement proper*, is the combination of two harmonics, $\exp(\pm i\chi)$, multiplied by the Gaussian factors, whose maxima are located along two spirals in the plane of (r, χ) : $r_{\text{max}} = \mp K_0(\chi - \chi_0)$. These maxima also determine the correlation between variables r and χ , which are an inherent part of the entanglement. The width of the maxima gradually spreads out with the growth of time (T), proportionally to $\sqrt{4\Phi^4 + T^2}$, as does any coherent state evolving in the free space.

It is relevant here to estimate a dephasing effect of a possible mismatch between amplitudes of the colliding solitons, $\Delta\eta$. It may be estimated through the respective change of the relative phase, $\Delta\chi \sim \eta\Delta\eta\Delta t_{\text{coll}} \sim \Delta\eta/c$, generated by the mismatch during the collision time, $\Delta t_{\text{coll}} \sim 1/(\eta c)$ [see Eqs. (1)-(3)]. The dephasing is

negligible if its size is small in comparison with the perturbation amplitude in expression (18), i.e.,

$$\Delta\eta/c \ll \varepsilon/K_0. \quad (20)$$

Currently available sophisticated experimental methods for the creation of solitons in BEC [18], as well as various theoretically elaborated schemes of matter-wave soliton lasers [52], make it possible to generate nearly identical solitons, with a sufficiently small difference between their amplitudes.

C. The initial phase-uniform state

Instead of initial wave packet (11), we can take one uniformly spread over phases ϕ_1 and ϕ_2 , which corresponds to an experimental situation in which the initial phases of solitons are not controlled. Then, modifying expression Eq. (12) accordingly, decomposition (17) is replaced by its simplified counterpart:

$$\begin{aligned} \Psi_0^{(\text{tot})}(\xi_1, \xi_2) &= \exp\left(-\frac{\rho^2}{4\Xi^2}\right) \\ &\times (\Xi/\sqrt{\pi}) \int_{-\infty}^{+\infty} dK e^{iK(r-r_0)} \exp\left[-\Xi^2(K-K_0)^2\right]. \end{aligned}$$

Recombining this with the result of the Born's approximation, as per Eq. (16), gives rise to the following expression for the collision-induced perturbation of the wave function:

$$\begin{aligned} \delta\Psi(\xi_1, \xi_2, \chi) &= \exp\left(-\frac{\rho^2}{4\Xi^2}\right) \cos\chi \\ &\times \frac{2i\varepsilon}{K_0} \exp\left[i\left(K_0(r-r_0) - \frac{1}{2}K_0^2T\right)\right] \end{aligned} \quad (21)$$

Although this result is much simpler than the one given by Eq. (18), in the case when the solitons' phases were initially allocated certain values, it is nontrivial too, demonstrating the dependence on relative phase χ of the perturbation generated, in the course of the collision, by the phase-independent initial wave function. Actually, this feature may be regarded as the simplest manifestation of the collision-induced entanglement. The correlation properties of the entanglement in this approximation are obvious, being, as a matter of fact, determined by factor $\cos\chi$.

IV. CONCLUSION

In this work, our aim was to present a proof of the principle that two initially uncorrelated quantum (actually, semi-classical) solitons may get entangled through the collision between them. To this end, we have restricted the analysis to macroscopic coordinates of the solitons, *viz.*, the position and phase, which may be justified for matter-wave solitons in BEC. In this sense, they

are regarded as Einstein-Podolsky-Rosen particles with the additional internal degree of freedom (the phase). We have presented simple analytic expressions characterizing the resulting entangled states, represented by collision-induced corrections to initial factorized wave functions of the soliton pair. The results were obtained by means of the Born's approximation, which is valid for the collision between fast soliton. The entanglement and quantum correlations are described in terms of the relative position and phase of the solitons, and the (conserved) total momentum and "angular momentum" (the latter is the canonical momentum conjugate to the total phase). A simple but noteworthy effect is that the collision of solitons with the completely uncertain (delocalized) relative phase leads to a partial phase localization. The predicted results are expected to be directly relevant to matter-wave solitons in BEC formed by atoms with attractive interactions and loaded into a nearly 1D trapping potential. A realization in terms of optical solitons may be possible too, in principle.

The entanglement predicted by the above analysis is weak, as the consideration was limited to the Born approximation, i.e., the first approximation of the perturbation theory. However, under more general conditions, the entanglement may reach significant levels (cf. Ref. [13] where the entanglement was analyzed by means of an exact quantum solution for NLS solitons interacting at zero velocity). What is then a possible advantage of using matter-wave solitons over individual particles, like photons or atoms? The obvious answer is that the solitons constitute macroscopic, or at least mesoscopic objects, and thus exhibit completely different properties in measurements. To some extent, they are similar to atomic ensembles [7], but, in contrast to the latter, the solitons represent coherent moving objects. In principle, the Bell's inequalities can be tested with entangled solitons, and even though such test would not be loophole-free, they would evidently exhibit different scaling and level of errors, in comparison with previously studied systems.

As said above, one may attain stronger correlations by going beyond the Born-approximation limit. However, in that regime the macroscopic (mean-field) description may cease to be quantitatively correct – in particular, because the deformation of the slowly colliding solitons makes the it irrelevant to use the description solely in terms of their collective degrees of freedom. Then, it may be appropriate to consider quantum solitons using the time-dependent GPE [or similar equation(s)] to describe the coherent soliton states proper, and the BdG (Bogoliubov-de Gennes) equations for quantum fluctuations around them [34], cf. the analysis of the stability of matter-wave solitons against quantum fluctuations performed in Refs. [30] and [31]. In particular, care should be taken of the collisional excitation of an IM (intrinsic mode), if the solitons support it. However, such a consideration is beyond the scope of this work.

Similar comments concern another important aspect of the studies of the entanglement, *viz.*, the decoherence.

In the present case, a part of the decoherence is due to the excitation of many-body degrees of freedom (such as BdG modes), rather than just quantum fluctuations of the solitons' positions and phases. Nevertheless, in the case of the collision between fast solitons we may restrict the attention to the collective degrees of freedom of the solitons, and look at the respective covariance matrix, similarly to the case of Gaussian states. Then, applying the entanglement criteria for Gaussian states [53, 54, 55] makes it possible to estimate the level of the entanglement of the present (ideal) state, as well as the maximum allowance of the contribution to the decoherence from the neglected many-body modes.

The estimate of the robustness of the entanglement is particularly simple for the solution corresponding to Eq. (21), because the degrees of freedom corresponding to the distance and phase shift between the solitons separate, hence the wave function of the two-soliton state can be written, in terms of dynamical variables (7) and (8), as [cf. Eqs. (18), (21)]

$$|\Psi(\rho, r, \chi)\rangle = |\Psi(\rho)\rangle|\Psi(r)\rangle \times [|0\rangle + \zeta(|1\rangle + |-1\rangle)], \quad (22)$$

where states of the relative phase (χ), $|0\rangle, |\pm 1\rangle$, correspond to harmonics with the respective numbers, and $\zeta \equiv \varepsilon/(2\sqrt{2}K_0)$. The phase part of state (22) is obviously entangled, since its partial transpose has three negative eigenvalues, namely, $-\zeta, -\zeta, -\zeta^2$ [56]. Therefore, the state remains entangled after adding arbitrary decoherence (noise) whose density matrix has an operator norm smaller than ζ . This actually means that one

can add “white noise” with norm $\zeta' < \zeta$ to state (22), and retain its non-positive partial transpose property. As concerns the dephasing due to a possible difference in the amplitudes of the colliding solitons, the estimate performed above demonstrates that it is negligible under condition (20).

A straightforward extension of the present analysis, still within the framework of the description based on the mean-field collective degrees of freedom and the Born's approximation, may be elaborated for the description of the generation of entanglement by collisions between two-dimensional solitons supported by a quasi-1D potential in the self-attractive BEC, as proposed (in the mean-field approximation) in Refs. [50]. Another extension, also possible in the framework of the quasi-particle approach, is to consider entanglement induced by collisions between gap solitons in a self-repulsive BEC loaded into an optical-lattice potential. It is known that such solitons may be mobile in 1D and 2D geometries, with a *negative* effective mass [51]. The interaction potential in the corresponding Hamiltonian is expected to be different from the above expression (6), namely, being $\sim \exp(-|r|) \cos(qr) \cos \chi$, with some wavenumber q , due to the oscillatory shape of tails of the gap solitons.

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