

Timelike and Spacelike Ricci Collineation Vectors in String Fluid

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Abstract

We study the consequences of the existence of timelike and spacelike Ricci collineation vectors (RCVs) for string fluid in the context of general relativity. Necessary and sufficient conditions are derived for a space-time with string fluid to admit a timelike RCV, parallel to u^a , and a spacelike RCV, parallel to n^a . In these cases, some results obtained are discussed.

Key Words: Ricci Collineation, String, String Fluid.

1. Introduction

The introduction of a symmetry (i.e. collineation) is most conveniently studied if the Lie derivative of the field equations is taken with respect to the symmetry vector. More specially, the Lie derivative of the Ricci and independently the energy momentum tensor can be computed by the symmetry. So, the field equations can be obtained as Lie derivatives along the symmetry vector of the dynamical variables.

Previous works on RCVs have been undertaken by several authors. Oliver and Davis, who gave necessary and sufficient conditions for a matter space-time to admit an RCV, $\eta^a = \eta u^a$, with $u^a = u_D^a$ where u_D^a is the dynamic four-velocity [1, 2]. Tsamparlis and Mason have considered Ricci collineation vectors (RCVs) in fluid space-times (perfect, imperfect and anisotropic) [3]. Duggal have also considered timelike Ricci inheritance vector in perfect fluid space-times [4]. Carot et al. have discussed space times with conformal Killing vectors [5].

The study of string fluid has aroused considerable interest as they are believed to give rise to density perturbations leading to the formation of galaxies [6]. The existence of a large scale network of strings in the early universe is not in contradiction with present day observations of the universe [7]. Also, the present of strings in the early universe can be explained using grand unified theories (GUTs) [6, 7]. Thus, it is interesting to study the symmetry features of string fluid.

Recently, work on symmetries of the string has been done by Yavuz and Yilmaz, and Yilmaz et al. who have considered inheriting conformal and special conformal Killing vectors, and also curvature inheritance symmetry in the string cosmology (string cloud and string fluids), respectively [8, 9]. Yilmaz has also considered timelike and spacelike Ricci collineations in the string cloud [10]. Baysal et al. have studied conformal collineation in the string cosmology [11].

The theory of spacelike congruences in general relativity was first formulated by Greenberg, who also considered applications to the vortex congruence in a rotational fluid [12]. The theory has been developed

and further applications have been considered by Mason and Tsamparlis, who also considered spacelike conformal Killing vectors and spacelike congruences [13].

A space-time admits a Ricci collineation vector (RCV) ξ^a if

$$\mathcal{L}_\xi R_{ab} = 0, \quad (1)$$

where R_{ab} is the Ricci tensor and \mathcal{L}_ξ denotes Lie derivative along ξ^a . A conservation law, valid for any RCV, was established by Collinson [14]. If ξ^a is an RCV, then it can be verified that

$$(R^{ab}\xi_b)_{;a} = 0, \quad (2)$$

and if Einstein's field equations

$$R^{ab} = T^{ab} - \frac{1}{2}Tg^{ab}, \quad (3)$$

are satisfied, then

$$\left[(T^{ab} - \frac{1}{2}Tg^{ab})\xi_b \right]_{;a} = 0. \quad (4)$$

Equation (4) plays an important role as one of necessary and sufficient conditions for a space-time to admit an RCV, ξ^a .

It is important to establish a connection between timelike or spacelike RCVs and material curves in the string fluid.

In this paper, we will investigate the properties of RCVs, $\eta^a = \eta u^a$, parallel to the string fluid unit four-velocity vector u^a :

$$u_a u^a = -1, \quad \eta = (-\eta_a \eta^a)^{1/2} > 0,$$

and spacelike RCVs, $\xi^a = \xi n^a$, orthogonal to u^a :

$$n_a n^a = +1, \quad n_a u^a = 0, \quad \xi = (\xi_a \xi^a)^{1/2} > 0.$$

We will express the necessary and sufficient conditions for string fluid space-time to admit a timelike RCV parallel to u^a and a spacelike RCV parallel to n^a in terms of the kinematic quantities of the timelike congruence of world-lines generated by u^a and the expansion, shear, and rotation of the spacelike congruence generated by n^a , respectively.

The energy-momentum tensor for a string fluid can be written as [8, 15]

$$T_{ab} = \rho_s (u_a u_b - n_a n_b) + q P_{ab}, \quad (5)$$

where ρ_s is string density and q is "string tension" and also "pressure".

The unit timelike vector u^a describes the fluid four-velocity and the unit spacelike vector n^a represents a direction of anisotropy, i.e., the string's directions. Also, note that

$$u_a u^a = -n_a n^a = -1 \quad \text{and} \quad u^a n_a = 0. \quad (6)$$

This paper may be outlined as follows. In section 2, necessary and sufficient conditions for string fluid space-time to admit a timelike RCV parallel to u^a are derived. In section 3, the basic aspects of the theory of spacelike congruences required in this paper are briefly reviewed. Also, necessary and sufficient conditions for a string fluid space-time to admit a spacelike RCV parallel to n^a are given. Finally, concluding remarks are made in section 4.

2. Timelike Ricci Collineation Vectors

If Einstein's field equation (3) are satisfied, then string fluid with energy-momentum tensor (5) admits an RCV, $\eta^a = \eta u^a$, if and only if

$$h_a^c h_b^d \dot{\gamma}_{cd} = -\frac{2}{3} [(2\rho_s - q)\sigma_{ab} - \gamma^{cd}\sigma_{cd}h_{ab} + \theta\gamma_{ab}] - 2\sigma_{c(a}\gamma_{b)}^c - 2\omega_{c(a}\gamma_{b)}^c, \quad (7)$$

$$q[\dot{u}_a - (\ln \eta)_{;a} - \theta u_a] = 0, \quad (8)$$

$$(\eta q u^a)_{;a} = 0, \quad (9)$$

where θ is the rate-of-expansion, σ_{ab} is the rate-of-shear tensor, ω_{ab} is the vorticity tensor of the timelike congruence generated by u^a , $h_{ab} = g_{ab} + u_a u_b$ and $\gamma_{ab} = (\rho_s + q) \left(\frac{1}{3}h_{ab} - n_a n_b\right)$.

Proof: From the definition of the Lie derivative it follows that

$$\mathcal{L}_{\eta u} R_{ab} = \eta \left\{ \dot{R}_{ab} + 2u^c R_{c(a}(\ln \eta)_{;b)} + 2R_{c(a} \dot{u}_{;b)}^c \right\} \quad (10)$$

which, using Einstein's field equation (3) for string fluid, may be rewritten as

$$\begin{aligned} \mathcal{L}_{\eta u} R_{ab} = \eta \left\{ \dot{q} u_a u_b + \frac{1}{3}(2\dot{\rho}_s - \dot{q})h_{ab} + \frac{4}{3}(\rho_s + q)\dot{u}_{(a} u_{b)} + \dot{\gamma}_{ab} \right. \\ \left. - 2q u_{(a}(\ln \eta)_{;b)} + \frac{2}{3}(2\rho_s - q)u_{(a;b)} + 2\gamma_{t(a} \dot{u}^{t;b)} \right\}. \end{aligned} \quad (11)$$

Suppose first that ηu^a is an RCV. Then (1) holds and the right-hand side of (11) vanishes. By contracting (11) in turn with $u^a u^b$, $u^a h_c^b$, h^{ab} , and $h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd}$ and by using the expansion

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\theta h_{ab} - \dot{u}_a u_b, \quad (12)$$

we obtain, respectively,

$$\dot{q} + 2q(\ln \eta)' = 0, \quad (13)$$

$$q[\dot{u}_a - (\ln \eta)_{;a} - (\ln \eta)' u_a] = 0, \quad (14)$$

$$2\dot{\rho}_s - \dot{q} + \frac{2}{3}(2\rho_s - q)\theta + 2\gamma^{ab}\sigma_{ab} = 0, \quad (15)$$

and Equation (7).

The energy momentum conservation equation will also be required. For string fluid, the momentum conservation equation, which follows from Einstein's field equations, is

$$\dot{\rho}_s = -\frac{2}{3}(\rho_s + q)\theta - \gamma^{ab}\sigma_{ab}. \quad (16)$$

(i) Condition (7) was derived directly in the decomposition of (11).

(ii) In order to determine condition (8), we first obtain an expression for $q(\ln \eta)'$ by eliminating $\dot{\rho}_s$ and \dot{q} from (13). Substituting from (16) for $\dot{\rho}_s$ into (15) gives

$$\dot{q} = -2q\theta, \quad (17)$$

and using (16) for $\dot{\rho}_s$ and (17) for \dot{q} , Equation (13) becomes

$$q(\ln \eta)' = q\theta. \quad (18)$$

Condition (8) is derived immediately from (14) and (18).

(iii) In order to derive condition (9), we observe that (13) may be written as

$$\dot{q} + q(\ln \eta)' + q(\ln \eta)' = 0. \quad (19)$$

If (18) is used to replace one of the terms $q(\ln \eta)'$ in (19), then (19) becomes

$$q_{,a}\eta u^a + q(\eta_{,a}u^a + \eta u^a_{;a}) = 0, \quad (20)$$

from which (9) follows directly. Conditions (7)- (9) are therefore necessary conditions if ηu^a is an RCV.

Conversely, suppose that conditions (7)- (9) are satisfied. Then if (7) for $\dot{\gamma}_{ab}$ is substituted into (11), and (12) is used to expand $u_{a;b}$ and $u_{t;b}$, (11) becomes

$$\mathcal{L}_{\eta u} R_{ab} = \eta \left\{ (\dot{q} + 2q\theta) u_a u_b + \frac{1}{3} \left[2\dot{\rho}_s - \dot{q} + \frac{2}{3}(2\rho_s - q)\theta + 2\gamma^{cd}\sigma_{cd} \right] h_{ab} \right\}. \quad (21)$$

Now, $\dot{\rho}_s$ is given by the energy conservation Equation (16). In order to obtain an expression for \dot{q} , we first observe that (9) can be expanded as

$$\dot{q} + q\theta + q(\ln \eta)' = 0. \quad (22)$$

But, by contracting (8) with u^a , (18) is again obtained and by eliminating $q(\ln \eta)'$ from (22), Equation (17) for \dot{q} is again derived.

By using (16) for $\dot{\rho}_s$ and (17) for \dot{q} it is easily verified that the coefficients of $u_a u_b$ and h_{ab} in (21) vanish and therefore ηu^a is an RCV.

3. Spacelike Ricci Collineation Vector

Before we discuss the calculation some general results can be presented for convenience on spacelike congruences that will be used in this work. Let $\xi^a = \xi n^a$ where n^a is a unit spacelike vector ($n_a n^a = +1$) normal to the four velocity vector u^a . The screen projection operator $P_{ab} = g_{ab} + u_a u_b - n_a n_b$ projects normally to both u^a and n^a . Some properties of this tensor are:

$$P^{ab}u_b = P^{ab}n_b = 0, \quad P_c^a P_b^c = P_b^a, \quad P_{ab} = P_{ba}, \quad P_a^a = 2.$$

The $n_{a;b}$ can be decomposed with respect to u^a and n^a as follows:

$$n_{a;b} = A_{ab} + \overset{*}{n}_a n_b - \dot{n}_a u_b + u_a \left[n^t u_{t;b} + (n^t \dot{u}_t) u_b - (n^t \overset{*}{u}_t) n_b \right], \quad (23)$$

where $\overset{*}{s} \dots = s \dots ;_a n^a$ and $A_{ab} = P_a^c P_b^d n_{c;d}$. We decompose A_{ab} into its irreducible parts

$$A_{ab} = S_{ab} + W_{ab} + \frac{1}{2} \overset{*}{\theta} P_{ab}, \quad (24)$$

where $S_{ab} = S_{ba}$, $S_a^a = 0$ is the traceless part of A_{ab} , $\overset{*}{\theta}$ is the trace of A_{ab} , and $W_{ab} = -W_{ba}$ is the rotation of A_{ab} . We have the relations:

$$S_{ab} = P_a^c P_b^d n_{(c;d)} - \frac{1}{2} \overset{*}{\theta} P_{ab}, \quad (25)$$

$$\begin{aligned} W_{ab} &= P_a^c P_b^d n_{[c;d]}, \\ \overset{*}{\theta} &= P^{ab} n_{a;b}. \end{aligned} \quad (26)$$

It is easy to show that in Equation (23) the u^a term in parenthesis can be written in a very useful form as follows:

$$-N_a + 2\omega_{tb} n^t + P_b^t \dot{n}_t,$$

where the vector

$$N_a = P_a^b(\dot{n}_b - \dot{u}_b) \quad (27)$$

is the Greenberg vector. Using (27), Equation (23) becomes

$$n_{a;b} = A_{ab} + \dot{n}_a n_b - \dot{n}_a u_b + P_b^c \dot{n}_c u_a + (2\omega_{tb} n^t - N_b) u_a. \quad (28)$$

The vector N^a is of fundamental importance in the theory of spacelike congruences. Geometrically the condition $N^a = 0$ means that the congruences u^a and n^a are two surface forming. Kinematically it means that the field n^a is "frozen in" along the observers u^a .

Having mentioned a few basic facts on the spacelike congruences we return to the computation of the Lie derivative of the Ricci tensor using the field equations.

If Einstein's field equation (3) are satisfied, then string fluid with energy-momentum tensor (5) admits an RCV, $\xi^a = \xi n^a$ if and only if

$$q\omega_{at} n^t = \frac{1}{2} \rho_s N_a, \quad (29)$$

$$\rho_s S_{ab} = 0, \quad (30)$$

$$q \left[\dot{n}_a + (\ln \xi)_{,a} - (n_t \dot{u}^t) n_a \right] = 0, \quad (31)$$

$$q \dot{\theta} = 0, \quad (32)$$

$$(\xi q n^a)_{;a} = 0. \quad (33)$$

Proof: From the definition of the Lie derivative it follows that

$$\mathcal{L}_{\xi n} R_{ab} = \xi \left[\dot{R}_{ab} + 2n^c R_{c(a} (\ln \xi)_{,b)} + 2R_{c(a} n^c_{;b)} \right] \quad (34)$$

which, using Einstein's field equation (3) for string fluid, may be rewritten as

$$\begin{aligned} \mathcal{L}_{\xi n} R_{ab} = \xi \left\{ \dot{q} (u_a u_b - n_a n_b) + \dot{\rho}_s P_{ab} - 2(\rho_s + q) \dot{n}_{(a} n_{b)} - 2q n_{(a} (\ln \xi)_{,b)} \right. \\ \left. + 2(\rho_s + q) \left[u_{(a} \dot{u}_{b)} - n_t u_{(a} \dot{u}_{;b)}^t \right] + 2\rho_s n_{(a;b)} \right\}. \end{aligned} \quad (35)$$

Suppose first that ξn^a is an RCV. Then Equation (1) is satisfied. The right-hand side of Equation (35) is therefore zero and by contracting it in turn with $u^a u^b$, $u^a n^b$, $u^a P^{bc}$, $n^a n^b$, $n^a P^{bc}$, P^{ab} , and $P^{ac} P^{bd} - \frac{1}{2} P^{ab} P^{cd}$ the following seven equations are derived:

$$\dot{q} + 2q n_a \dot{u}^a = 0, \quad (36)$$

$$q \left[(\ln \xi)_{,a} + \dot{n}_a u^a \right] = 0, \quad (37)$$

$$\rho_s P_a^b \dot{n}_b - (\rho_s + q) P_a^b \dot{u}_b + q P_a^b n^t u_{t;b} = 0, \quad (38)$$

$$\dot{q} + 2q (\ln \xi)^{\cdot} = 0, \quad (39)$$

$$q P_a^b \left[\dot{n}_b + (\ln \xi)_{,b} \right] = 0, \quad (40)$$

$$\dot{\rho}_s + \rho_s \dot{\theta} = 0, \quad (41)$$

$$\rho_s S_{ab} = 0. \quad (42)$$

The energy momentum conservation equation will also be required. For string fluid, the momentum conservation equation, which follows from Einstein's field equations, is

$$\dot{\rho}_s = -(\rho_s + q) \dot{\theta}. \quad (43)$$

(i) Condition (29) is derived from (38). We have

$$n^t u_{t;b} = 2n^t u_{[t;b]} + \overset{*}{u}_b = -2\omega_{bt} n^t - (n_t \dot{u}^t) u_b + \overset{*}{u}_b, \quad (44)$$

and by substituting from (44) into (38), (29) follows directly.

(ii) Condition (30) is given by Equation (42).

(iii) To derive condition (31), we first expand (40) and use (37); this gives

$$q \left[\overset{*}{n}_a + (\ln \xi)_{,a} - (\ln \xi)^* n_a \right] = 0. \quad (45)$$

If we subtract (39) from (36), then we have

$$q(\ln \xi)^* = q n_a \dot{u}^a. \quad (46)$$

If we substitute Equation (46) into equation (45), then we have condition (31).

(iv) To derive condition (32), we substitute equation (43) into (41), then we have condition (32).

(v) Consider the final condition (33). From (26), we have

$$n_a \dot{u}^a = n_{;a}^a - \overset{*}{\theta}. \quad (47)$$

Substitute (47) into (46); this gives

$$q(\ln \xi)^* = q(n_{;a}^a - \overset{*}{\theta}). \quad (48)$$

If one of the terms $q(\ln \xi)^*$ into Equation (39) is replaced by (48) and used condition (32), then (39) may be written as

$$q_{,a} \xi n^a + q(\xi_{,a} n^a + \xi n_{;a}^a) = 0, \quad (49)$$

from which (33) follows directly.

Hence, if $\xi^a = \xi n^a$ is an RCV then conditions (29)-(33) are satisfied.

Conversely, suppose that (29)-(33) are satisfied and Einstein's field equations hold. Using (28) for $n_{(a;b)}$, (30) and (31) for $q(\ln \xi)_{,a}$ Equation (35) becomes

$$\begin{aligned} \mathcal{L}_{\xi n} R_{ab} = \xi \left\{ \overset{*}{q} u_a u_b - (\overset{*}{q} + 2q n_t \dot{u}^t) n_a n_b + (\overset{*}{\rho}_s + \rho_s \overset{*}{\theta}) P_{ab} \right. \\ \left. + 2q \left[u_{(a} u_{b)} - n_t u_{(a} \dot{u}_{b)}^t \right] - 2\rho_s N_{(a} u_{b)} \right\}. \end{aligned} \quad (50)$$

Further, by using (44) for $n^t u_{t;b}$ and (29) for $q\omega_{at} n^t$ and by replacing $\overset{*}{\theta}$ by $n_t \dot{u}^t$ with the aid of (26), (50) reduces to

$$\mathcal{L}_{\xi n} R_{ab} = \xi \left\{ (\overset{*}{q} + 2q n_t \dot{u}^t) (u_a u_b - n_a n_b) + (\overset{*}{\rho}_s + \rho_s \overset{*}{\theta}) P_{ab} \right\}. \quad (51)$$

Now, $\overset{*}{\rho}_s$ is given Equation (43). To obtain $\overset{*}{q}$ in terms of $n_t \dot{u}^t$ we use the remaining condition (33), which may be expanded as

$$\overset{*}{q} + q(\ln \xi)^* + q n_{;a}^a = 0. \quad (52)$$

But if (31) is contracted with n^a , Equation (46) is again derived. Therefore (52) becomes

$$\overset{*}{q} + 2q n_t \dot{u}^t = 0. \quad (53)$$

It easily verified with the aid of (43), (53), and condition (32) that the right-hand side of (51) vanishes and therefore $\xi^a = \xi n^a$ is an RCV.

4. Results and Conclusions

In the case of timelike Ricci collineation vectors parallel to u^a , we have the following results:

- (a) In this case, it is easily verified that condition (9) is the conservation law (4) with $\xi_b = \eta u_b$.
- (b) Condition (8) may be rewritten alternatively either $q = 0$ or $\dot{u}_a = (\ln \eta)_{,a} + \theta u_a$. If, $q = 0$ the energy-momentum tensor reduces

$$T_{ab} = \rho_s(u_a u_b - n_a n_b)$$

which is pure string.

In the case of spacelike Ricci collineation vectors orthogonal to u^a , we have the following results:

- (a) In this case, it is easily verified that (33) is the conservation law (4) for the string fluid with $\xi_b = \xi n_b$.

- (b) From Equation (30), we have

$$\text{either } \rho_s = 0 \text{ or } S_{ab} = 0. \quad (54)$$

- (c) When $\omega = 0$, Equation (29) reduces to

$$\rho_s N_a = 0, \quad (55)$$

and hence either $\rho_s = 0$ or $N_a = 0$. When $N_a = 0$, the integral curves n^a are material curves and string fluid form two surface. When $\rho_s = 0$, strings disappear.

- (d) When $\omega \neq 0$, Equation (29) reduces to

$$q\omega_{at}n^t = \frac{1}{2}\rho_s N_a. \quad (56)$$

- (i) If $N_a = 0$, then Equation (56) reduces to

$$q\omega_{at}n^t = 0 \quad (57)$$

and hence if $q \neq 0$ then $\omega_{at}n^t = 0$ and since $\omega_{at} = \eta_{atrs}\omega^r u^s$ we find by contracting (57) with $\eta^{abcd}\omega_c u_d$ that

$$n^a = [(\omega_t n^t)/\omega^2]\omega^a. \quad (58)$$

Since both $n^a \neq 0$ and $\omega^a \neq 0$ it follows that $n^a = \pm\omega^a/\omega$.

- (ii) If the integral curves of n^a are material curves in the fluid then $N^a = 0$. Hence, since $q \neq 0$, from (46) $\omega_{at}n^t = 0$ and $n^a = \pm\omega^a/\omega$.
- (iii) If $n^a = \pm\omega^a/\omega$ and $q \neq 0$ then from Equation (56), $N^a = 0$ and the integral curves of n^a are material curves.

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