Inflation in a Flat Universe

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Abstract

We started the evolution of a flat universe from a nonsingular state called prematter which is governed by an inflationary equation of state $P = (\gamma - 1)\rho$, where γ represents the initial vacuum dominance of the universe. The evolution of the universe-except in the prematter era-is affected neither by the initial vacuum dominance nor by the initial expansion rate of the universe. On the other hand, present properties of the universe, such as the Hubble constant, age and density, are sensitive to the value of the temperature at the decoupling (T_m). Over a range between $3 \cdot 10^4$ and $5 \cdot 10^4$ K for T_m, our model predicts a value between 50 and 80 Km·s^{-1}·Mpc^{-1} for the present value of the Hubble constant (H_0). Assuming that the thermal history of the universe is independent from its geometry, the above range could be considered as a transition range for the decoupling temperature T_m.

1. Introduction

The standard model of hot big-bang cosmology exhibits difficulties stemming from the puzzling initial conditions. These difficulties are the well known singularity, flatness, horizon, causality, homogeneity and isotropy problems. During the past two decades, several models have been proposed to overcome these difficulties. One common feature in all of these works is that a new era called the "inflationary era" was added into the history of the universe before the radiation era with which the standard cosmological model of the universe begins. During this new era, the universe expands enormously from a finite size. This huge expansion is called "inflation" and arises due to the unusual characteristic of the equation of state ($P = (\gamma - 1)\rho$) used to describe the universe in this era. Inflation mechanism was first suggested as a rescue from the horizon and flatness problems [1] and then extended in such a way to construct models which are free from an initial singularity. The equation of state used in the inflationary era determines how much initially the universe is close to the vacuum state and this is done by adjusting the values of γ.

Therefore, the models constructed in such a way, are parametric universe models and evolution of these universes are also sensitive to the value of γ in the inflationary era [2-5]. In other words, the present properties of the universe such as the Hubble constant H and the age t_{now} are affected by the vacuum dominance of the early universe. Besides the form of the equation of state, initial expansion rate of the scale factor $\dot{a}$ also influences the present properties of the universe and is considered as a parameter of the universe model [5].

In the standard model, the flatness problem arises from the extreme fine tuning of the initial values of the energy density ρ and Hubble constant H, so that ρ is very close to $\rho_c \left(\equiv 3H^2/8\pi \right)$ [1]. This is necessary to produce a universe surviving $\sim 10^{10}$ years, an age prediction compatible with observational results. In the models of inflationary cosmology, without a fine tuning requirement, realistic universe models have been obtained. That is, the flatness problem is solved in these models. The solution to this problem could be attributed to the enormous expansion of the universe during the inflationary era which lasts a period on the order of the Planck scale, $\sim 10^{-44}$ sec. This carries the geometry of the universe to the flat one while the value of $\Omega(\rho/\rho_c)$ is driven toward one.

In this work, we construct a universe model having a flat space-time geometry with the same motivations as in the other inflationary universe models. Since the space-time geometry of this universe is already flat, there is no flatness problem in this universe. On the other hand, the universe is vacuum dominated due to the form of the equation of state used to describe the universe in the inflationary era. This would determine how long the universe would stay at the inflationary era before it enters the radiation era predicted by the standard model. In the other inflationary universe models, the present properties of the universe have been affected by the initial vacuum dominance of the early universe. In this work, we will study the effect of the initial vacuum dominance on the evolution of the universe having flat space-time geometry. This would give us an idea about the role of the inflation mechanism when there is no flatness problem. This would be interesting because the inflation mechanism was first suggested as a solution to the flatness problem [1] and the answer to the question "What would be the response of a universe having a flat space-time geometry when it is inflated through such a mechanism?" is not exactly known. Our aim in this paper would be to answer this question in a different context.

This paper is organized as follows. In the next section, we give the dynamical equations of our model and solve them analytically. In Sec. III, we give numerical results of the predictions of our model for the present properties of the universe. We present our discussions and conclusions in the final section.

2.   Dynamics of the Model

2.1.   Field Equations

The space-time geometry of the model describing a spatially homogenous and isotropic universe is determined by a RW line element:

$$ds^2 = dt^2 - a^2(t) \left[dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \tag{1}$$

where (t, r, θ, ϕ) are comoving coordinates, and $a(t)$ is the scale factor which represents the size of the universe.

For this line element, Einstein's gravitational field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} \tag{2}$$

give

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 = -8\pi P, \tag{3}$$

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} \rho, \tag{4}$$

where P and ρ are the energy density and pressure, respectively, in the universe filled with a cosmological fluid well-approximated by a perfect fluid, and a dot denotes differentiation with respect to the cosmic time t.

Equations (3) and (4) could be written as a single equation as

$$\frac{\ddot{a}}{a} + 4\pi \left(\gamma - \frac{2}{3} \right) \rho = 0, \tag{5}$$

where we have made use of the general form of the equation of state given as $P = (\gamma - 1)\rho$, and γ is assumed to be a constant parameter during each era in the history of the universe.

From Eqs. (4) and (5), we could obtain an equation involving only the scale factor $a(t)$ as

$$\left(\frac{\ddot{a}}{a}\right) + \left(\frac{3}{2}\gamma - 1\right)\left(\frac{\dot{a}}{a}\right)^2 = 0. \tag{6}$$

This equation could be solved for any γ if we define conformal time η as

$$dt = a(\eta)\,d\eta. \tag{7}$$

In conformal time η, Eqs. (4), (5) and (6) become

$$\left(\frac{a'}{a^2}\right)^2 = \frac{8\pi}{3}\rho, \tag{8}$$

$$\frac{a''}{a^3} - \left(\frac{a'}{a^2}\right)^2 + 4\pi\left(\gamma - \frac{2}{3}\right)\rho = 0, \tag{9}$$

$$\frac{a''}{a} + \left(\frac{3}{2}\gamma - 2\right)\left(\frac{a'}{a}\right)^2 = 0, \tag{10}$$

where a prime denotes differentiation with respect to the conformal time. If we make the change of variable

$$u \equiv \frac{a'}{a} = \frac{d\ln a}{d\eta}, \tag{11}$$

Eq. (10) could be brought into a much simpler form as

$$u' + cu^2 = 0, \tag{12}$$

where

$$c = \frac{3}{2}\gamma - 1. \tag{13}$$

Equation (12) is a kind of Riccati equation and, since we will consider vacuum like state for the inflationary era $(c \neq 0)$, it could be solved by setting

$$u = \frac{1}{c}\frac{w'}{w} = \left[\ln\left(w^{1/c}\right)\right]'. \tag{14}$$

This new change of variable leads us to write

$$\frac{w''}{w} = 0, \tag{15}$$

which gives the solution in the form

$$a(\eta) = a_0(\eta + b)^{1/c}, \tag{16}$$

where a_0 and b are integration constants.

Due to the unusual characteristic of the equation of state used in the inflationary era, the temperature of the universe rises in this era although the expansion of the universe is assumed to be adiabatic. This could be seen if one considers the following relation derived from the first law of thermodynamics:

$$\frac{T'}{T} + 3\frac{a'}{a}(\gamma - 1) = 0 \tag{17}$$

(corresponding to Eq. (33) in [5]). Our modelling of the universe lies on a thermodynamic assumption that the expansion of the universe continues until the Planck temperature T_{pl} which is assumed to be the maximum allowed temperature. Then the equation of state takes the form of that of the isotropic radiation. In other words, the universe starts behaving as predicted in the standard model of the universe. Therefore, we give an outline of our toy model as

a) **The inflationary (prematter) era** $(0 \leq \eta \leq \eta_r)$: The equation of state is given as $P = (\gamma - 1)\rho$ where $\gamma = \gamma_p \simeq 10^{-3}$. The constitution and behavior of the material substance filling the universe in this era is not exactly known.

b) **The radiation era** $\eta_r \leq \eta \leq \eta_m)$: The universe is assumed to be composed of isotropic radiation for which the equation of state is known to be $P = \rho/3$. This corresponds to $\gamma = \gamma_r = 4/3$ in the general form of the equation of state.

c) **The matter era** $(\eta_m \leq \eta)$: The era that we live in. Due to the large intergalactic distance and small relative motions of the intergalactic objects, it would be assumed that the universe is assumed to be filled with zero pressure dust such that $P = 0$, $\gamma = \gamma_m = 1$.

These eras are connected by first order phase transitions occurring at some specific temperatures T_{pl} and $T_m \cdot T_m$ is the temperature at which radiation and matter are decoupled. There is not a generally accepted value for this temperature other than the ones which are speculated [6,7]. Due to the lack of information, we will not assign a numerical value to T_m until we find expressions about the present properties of the universe.

2.2. Boundary conditions and solutions for the scale factor

To eliminate the initial singularity in our model, we start the evolution of the universe from a limiting density called the Planck density ρ_{pl} corresponding to finite size for the scale factor of the universe. To solve for the dynamics of the universe, we then specify the initial expansion rate of the universe as

$$a'(0) = v, \tag{18}$$

where v is a positive constant.

Writing Eq. (8) at $\eta = 0$, we get

$$a(0) = \sqrt{\frac{v}{\sqrt{d}}}, \tag{19}$$

which reflects the singularity-free character of our cosmological model. The solutions for the scale factor in different eras are

$$a(\eta) = \begin{cases} a_0^{(p)}(\eta + b_p)^{1/c_p} & 0 \leq \eta \leq \eta_r, \\ a_0^{(r)}(\eta + b_r)] & \eta_r \leq \eta \leq \eta_m, \\ a_0^{(m)}(\eta + b_m)^2 & \eta_m \leq \eta. \end{cases} \tag{20}$$

We next impose that the scale factor and its derivative are continuous at points (η_r, η_m) where the phase transitions take place. This leads us to determine the integration constants:

$$a_0^{(p)} = \frac{\sqrt{\frac{v}{H(0)}}}{b_p^{\frac{1}{c_p}}}, \tag{21}$$

$$b_p = \frac{1}{c_p \sqrt{vH(0)}}, \tag{22}$$

$$a_0^{(r)} = \frac{a_0^{(p)}}{c_p}(\eta_r + b_p)^{\frac{1}{c_p}-1}, \tag{23}$$

$$b_r = c_p b_p + (c_p - 1)\eta_r, \tag{24}$$

$$a_0^{(m)} = \frac{a_0^{(r)}}{4(\eta_m + b_r)}, \tag{25}$$

$$b_m = \eta_m + 2b_r, \tag{26}$$

where $H(0)$ is the initial value of the Hubble constant which is given as

$$H(0) = \frac{a'(0)}{a^2(0)} = \sqrt{d}, \tag{27}$$

2.3. Physical properties of the model

During the radiation era, the universe is assumed to consist of pure radiation which is in thermal equilibrium. Therefore, its energy density could be expressed in the blackbody form which is given as

$$\rho_{\text{blackbody}} = \frac{8\pi^5 (kT_{pl})^4}{15}, \tag{28}$$

where $k = 1.38 \cdot 10^{-16}\text{K}^{-1}$ is the Boltzmann constant.

As mentioned before, the universe heats up while expanding during the inflationary era. We end the inflationary era with a thermodynamical constraint which is the attainment of the Planck temperature T_{pl}. Knowing that the scale factor of the universe evolves with the energy density according to the Einstein's equations, we consider Eq. (8) together with Eqs. (20), (21), (27) and (28), and end up with

$$\eta_r = b_p \left[(1.5201)^{\frac{c_p}{2(c_p+1)}} - 1 \right], \tag{29}$$

which is the conformal time corresponding to the phase transition between the inflationary and radiation eras.

With the help of the Eq. (17), we could match the evolution of the scale factor to that of the temperature. To this end, we have to specify γ for each era. It is already mentioned that γ equals 4/3 for the radiation era. Then we write

$$\frac{a(\eta_r)}{a(\eta_{m)}} = \frac{T_m}{T_{pl}}, \tag{30}$$

which, when combined with Eq. (24), gives

$$\eta_m = c_p b_p (1.5201)^{\frac{c_p}{2(c_p+1)}} \frac{T_{pl}}{T_m} - b_r, \tag{31}$$

which is the conformal time associated to the phase transition between radiation and matter eras.

Before the emergence of matter as the dominant constituent of the energy density in the universe, radiation decouples from matter and basic constituents of matter (electrons, protons, neutrons etc.) start to form. Then they combine for the first time in a process called "recombination" to form the more complex form of matter in the universe. At this point, it is to be noted that after the decoupling between radiation and matter eras, radiation still behaves as a perfect fluid responsible for the temperature of the universe. As Kolb and Turner [6], we will consider decoupling as the beginning of the matter era. With these in mind, we get from Eq. (17)

$$\frac{a\left(\eta_m\right)}{a\left(\eta_{\text{now}}\right)} = \frac{T_{\text{now}}}{T_m}, \tag{32}$$

where we have chosen integration limits to be the conformal times corresponding to the second phase transition and the present time. This equation could be solved for η_{now} as

$$\eta_{\text{now}} = \sqrt{\frac{T_m}{T_{\text{now}}}}\eta_m + \left(\sqrt{\frac{T_m}{T_{\text{now}}}} - 1\right) b_m \tag{33}$$

In order to find comoving times corresponding to conformal times, we first consider the definition given by Eq.(7) and assume that $t = 0$ at $\eta = 0$. Then we get from Eq. (16)

$$t\left(\eta\right) = a_0 \int_0^\eta \left(\eta' + b\right)^{1/c} d\eta'. \tag{34}$$

Since c takes different values for each era in the history of the universe, this integral has to be computed separately for each era. For the inflationary $(c = c_p)$, radiation $(c_r = 1)$ and matter $(c_m = 1/2)$ eras, Eq. (34) can be integrated to yield analytical expressions as

$$t_r = \frac{\sqrt{1.5201} - 1}{\left(c_p + 1\right)\sqrt{d}}, \tag{35}$$

$$t_m = t_r + \frac{\sqrt{1.5201}}{2\sqrt{d}}\left[\left(\frac{T_{pl}}{T_m}\right)^2 - 1\right], \tag{36}$$

$$t_{\text{now}} = t_m + \frac{2}{3}\sqrt{\frac{1.5201}{d}}\left(\frac{T_{pl}}{T_m}\right)^2\left[\left(\frac{T_m}{T_{\text{now}}}\right)^{3/2} - 1\right]. \tag{37}$$

We make use of the definition given as

$$H\left(\eta\right) \equiv \frac{a'\left(\eta\right)}{a^2\left(\eta\right)}, \tag{38}$$

to find the Hubble constant at η_r, η_m and η_{now}. They are

$$H\left(\eta_r\right) = \frac{a'\left(\eta_r\right)}{a^2\left(\eta_r\right)} = 1.3436 \cdot 10^{63} \quad \text{Km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}, \tag{39}$$

$$H\left(\eta_m\right) = \frac{a'\left(\eta_m\right)}{a^2\left(\eta_m\right)} = 6.6925 \cdot 10^{-2} \cdot T_m^2 \quad \text{Km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}, \tag{40}$$

$$H\left(\eta_{\text{now}}\right) = \frac{a'\left(\eta_{\text{now}}\right)}{a^2\left(\eta_{\text{now}}\right)} = 0.2969 \cdot \sqrt{T_m} \quad \text{Km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}. \tag{41}$$

The energy density evolves with the scale factor of the universe as described by Eq. (4) which when combined with Eq. (5) gives

$$\dot{\rho} + 3\gamma\frac{\dot{a}}{a}\rho = 0. \tag{42}$$

In terms of conformal time η, this equation becomes

$$\rho' + 3\gamma\frac{a'}{a}\rho = 0, \tag{43}$$

which could be solved as

$$\frac{\rho\left(\eta_f\right)}{\rho\left(\eta_i\right)} = \left(\frac{a\left(\eta_i\right)}{a\left(\eta_f\right)}\right)^{3\gamma}, \tag{44}$$

where η_i and η_f mark the initial and final instants respectively of any conformal time interval in a given era. When η_i and η_f are chosen as $(\eta_i, \eta_f) = (0, \eta_r), (\eta_r, \eta_m), (\eta_m, \eta_{\text{now}})$ we obtain, respectively

$$\rho\left(\eta_r\right) = 3.3923 \cdot 10^{93} gr \cdot cm^{-3}, \tag{45}$$

$$\rho\left(\eta_m\right) = 8.4166 \cdot 10^{-36} \cdot T_m^4 \; gr \cdot cm^{-3}, \tag{46}$$

$$\rho\left(\eta_{\text{now}}\right) = 1.6566 \cdot 10^{-34} \cdot T_m \; gr \cdot cm^{-3}. \tag{47}$$

3.   Numerical Results

The results that we have found for the physical properties of the universe during its evolution clearly indicates that our model is sensitive to the temperature at the last phase transition T_m. On the other hand, except the lifetime of the inflationary era, the time periods are insensitive to the choice of parameter γ_p. Since we do not have enough information about what the value of T_m should be, we will try to assign some numerical results in the light of the recent observational results for the present properties of the universe such as Hubble constant, age and density. While doing this, as mentioned previously, we keep in mind that γ_p is no longer a parameter which affects the evolution of the universe. Therefore, T_m could be regarded as the only parameter of our model.

Recent observations suggest a value approximately between 9 and 15 Gyr for the present age of the universe (t_{now}) [8,9] and a value in the range of $\sim 50 - 80$ Km $\cdot$ s^{-1} $\cdot$ Mpc^{-1} for the present value of the Hubble constant H_0 [10-16]. In our model, T_m should take a value falling roughly into the range of $3 \cdot 10^4 - 5 \cdot 10^4$ K so that the predictions of the model for H_0 and t_{now} agree with observations. Since γ_p has no effect on the evolution of the universe, we fix it to $1.9000 \cdot 10^{-3}$ and we present the results about the present properties of the universe and those at the end of each era for $T_m = 3 \cdot 10^4, 3.5 \cdot 10^4, 4 \cdot 10^4, 4.5 \cdot 10^4, 5.0 \cdot 10^4$ in tables 1-5.

Table 1. $T_m = 3 \cdot 10^4$K

$H(\frac{km}{s \cdot mpc})$	$t(yr)$	$\rho(\frac{gr}{cm^3})$
$1.3436 \cdot 10^{63}$	$4.8238 \cdot 10^{-50}$	$3.3923 \cdot 10^{93}$
$6.0233 \cdot 10^{7}$	$8.1164 \cdot 10^{3}$	$6.8174 \cdot 10^{-18}$
51.4246	$1.2675 \cdot 10^{10}$	$4.9698 \cdot 10^{-30}$

Table 2. $T_m = 3.5 \cdot 10^4 \text{K}$

$H(\frac{km}{s \cdot mpc})$	$t(yr)$	$\rho(\frac{gr}{cm^3})$
$1.3436 \cdot 10^{63}$	$4.8238 \cdot 10^{-50}$	$3.3923 \cdot 10^{93}$
$8.1983 \cdot 10^7$	$5.9631 \cdot 10^3$	$1.2630 \cdot 10^{-17}$
55.5449	$1.1735 \cdot 10^{10}$	$5.7981 \cdot 10^{-30}$

Table 3. $T_m = 4 \cdot 10^4 \text{K}$

$H(\frac{km}{s \cdot mpc})$	$t(yr)$	$\rho(\frac{gr}{cm^3})$
$1.3436 \cdot 10^{63}$	$4.8238 \cdot 10^{-50}$	$3.3923 \cdot 10^{93}$
$1.0708 \cdot 10^8$	$4.5655 \cdot 10^3$	$2.1546 \cdot 10^{-17}$
59.3800	$1.0977 \cdot 10^{10}$	$6.6264 \cdot 10^{-30}$

Table 4. $T_m = 4.5 \cdot 10^4 \text{K}$

$H(\frac{km}{s \cdot mpc})$	$t(yr)$	$\rho(\frac{gr}{cm^3})$
$1.3436 \cdot 10^{63}$	$4.8238 \cdot 10^{-50}$	$3.3923 \cdot 10^{93}$
$1.3552 \cdot 10^8$	$3.6073 \cdot 10^3$	$3.4513 \cdot 10^{-17}$
62.9820	$1.0349 \cdot 10^{10}$	$7.4547 \cdot 10^{-30}$

Table 5. $T_m = 5 \cdot 10^4 \text{K}$

$H(\frac{km}{s \cdot mpc})$	$t(yr)$	$\rho(\frac{gr}{cm^3})$
$1.3436 \cdot 10^{63}$	$4.8238 \cdot 10^{-50}$	$3.3923 \cdot 10^{93}$
$1.6731 \cdot 10^8$	$2.9219 \cdot 10^3$	$5.2604 \cdot 10^{-17}$
66.3889	$9.8178 \cdot 10^9$	$8.2830 \cdot 10^{-30}$

4.   Discussion

In this work, we try to explore the connection between the flatness problem and the inflation mechanism. We do this by constructing a flat universe model by adding a new era called the inflationary era into the history of the universe. In our model, the universe starts its journey with a vacuum like state and undergoes a first order phase transition into the radiation era which is predicted by the standard model as the era with which the universe first starts to expand.

It is an already known fact that inflation mechanism carries different types of space-time geometries (closed, open) to a flat one by causing a huge expansion in a time period at the order of Planck-Scale. Unlike other models in which an initial vacuum-like structure of the universe affected the present properties of the universe, in this model the radiation and matter eras are not affected by the vacuum dominance of the early universe. The vacuum dominance of the early universe is important when the universe is in the inflationary era. If the universe starts with a pure vacuum state $(\gamma = 0)$, it cannot exit from inflation. This could be seen from the singular behavior of t_r for $\gamma = 0$. As the initial state of the universe approaches the vacuum state, the time spent by the universe in the inflationary era increases. After the universe exits from inflation, it behaves as predicted in the standard model in which there is no initial vacuum dominance.

The present properties of the universe, such as the Hubble constant, age and density, are not affected by the initial vacuum dominance. Instead, they are highly sensitive to the temperature at the last phase transition (T_m). In that case, this temperature can be considered as a parameter. Therefore, the model we construct is a one-parameter universe model. Considering the fact that the thermal history of the universe is independent of the geometry of the universe, we may argue that the range considered for T_m could be seen as a transition range for the decoupling temperature.

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