Turk J Phys
25 (2001) , 481 – 487.

On the Statistical Properties of a Q-Oscillator

Abdullah ALGIN
Physics Department, Osmangazi University, Meşelik, Eskişehir-TURKEY

Received 09.11.2000

Abstract

Several statistical properties of the Coon-Baker-Yu (CBY) q-oscillator algebra are discussed. Particular emphasis is given to a careful derivation of second quantized particle statistics obtained via the CBY q-oscillator. It is outlined that, beside other limiting cases, the limit $q = 0$ of the CBY q-oscillator gives the Cuntz oscillator and the Fock space properties of both of these oscillators lead to various interesting results.

In the past decade, q-deformed oscillator systems have yielded new developments in the context of statistical physics [1-4]. These developments are mainly given in the framework of unusual statistics rather than conventional Bose or Fermi statistics. In the meantime, the concept of unusual statistics has shown properties completely different from those of conventional statistics.

As is well known, the earliest example outside of Bose or Fermi statistics was introduced by the work of Green [5]. In this article, Green proved the existence of generalized statistics of identical particles and particularly showed that spin-half fields can be quantized in such a way that an arbitrary finite number of particles may occupy a single quantum state. According to this remarkable result, at most p particles with half-integer spin may be allowed to occupy a given state. The state can even be completely symmetric under particle interchange. In the literature, this type of statistics is known as parastatistics where in identical particles are referred to as parabosons and parafermions or, simply, called parons in a unified terminology [1]. However, the algebras obeyed by parabosons and parafermions have the form of trilinear oscillator commutation relations [5].

Beside parastatistics, other deformed oscillator algebras are often used in anyonic statistics which is related to non-local particles defined on two-dimensional space [6,7] and q-deformed statistics [8]. In this study, we aim to show that starting from the structural properties of the algebra and of the representations of a q-oscillator, Bose-Einstein, Fermi-Dirac or Maxwell-Boltzmann statistics arises.

The earliest multidimensional q-oscillator was postulated by Coon, Baker and Yu (CBY) [9] and is expressed as follows:

$$a_\mu a_\nu^+ - q a_\nu^+ a_\mu = \delta_{\mu\nu}, \qquad a_\mu |0\rangle = 0, \qquad \mu, \nu = 1, 2, ..., d\,, \tag{1}$$

where a_μ and a_μ^+ are annihilation and creation operators, respectively, and q is a real parameter. This model was used to derive an operator formalism for dual resonance model amplitudes with nonlinear trajectories [9]. Though its SU(d)-invariance was known before development of quantum groups, the most important property of the CBY q-oscillator is its invariance under the group SU(d).

Quantum states belonging to the q-oscillator given in Eq. (1) can be created by applying the creation operators on the ground state defined by $a_\mu |0\rangle = 0$. For instance, consider two and three particle states as follows:

$$\begin{aligned} |\mu\nu\rangle &= a_\mu^+ a_\nu^+ |0\rangle\,, \\ |\mu\nu\lambda\rangle &= a_\mu^+ a_\nu^+ a_\lambda^+ |0\rangle\,, \end{aligned} \tag{2}$$

and their hermitian conjugates

$$\begin{aligned} \langle\mu\nu| &= \langle 0| a_\nu a_\mu, \\ \langle\mu\nu\lambda| &= \langle 0| a_\lambda a_\nu a_\mu\;. \end{aligned}$$

n-particle states can be created by successively applying the creation operators on the ground state n-times. However, it has been proved that the CBY q-oscillator does not definitely contain any commutation relation between any two creation (or annihilation) operators [10]. It is now possible to make a different interpretation to this proof. The norms and scalar products can be calculated for two particle states with $\mu \neq \nu$ given in Eq. (2) as

$$\begin{aligned} \langle\mu\nu \mid \mu\nu\rangle &= \langle\nu\mu \mid \nu\mu\rangle = 1, \\ \langle\mu\nu \mid \nu\mu\rangle &= \langle\nu\mu \mid \mu\nu\rangle = q, \end{aligned} \tag{3}$$

whereas these two particle states are orthogonal to all other states and the angle between the states $|\mu\nu\rangle$ and $|\nu\mu\rangle$ is

$$\cos\theta = q. \tag{4}$$

Hence it turns out that deformation parameter q should be in the interval $-1 \le \cos\theta \le +1$. For the end points of this interval, the CBY q-oscillator gives a bosonic oscillator as follows:

$$a_\mu a_\nu^+ - a_\nu^+ a_\mu = \delta_{\mu\nu} \qquad \text{for} \qquad q = +1\,. \tag{5}$$

Thereby it characterizes Bose statistics. Secondly, this oscillator also gives a fermionic oscillator:

$$a_\mu a_\nu^+ + a_\nu^+ a_\mu = \delta_{\mu\nu} \qquad \text{for} \qquad q = -1\,. \tag{6}$$

Thereby it characterizes Fermi statistics. In the intermediate points $(-1 < q < +1)$, one may think that this q-oscillator leads to an interesting statistical interpretation between Bose and Fermi statistics: From Eq. (4), it is seen that the states $|\mu\nu\rangle$ and $|\nu\mu\rangle$ are linearly independent and, in this sense, there is no limitation related to occupancy of the states created by applying successive creation operators on the ground state. In fact, there is also no such limitation in the case of the group constructed by the representations of creation operators of a state such as $(a_{\mu_1}^+ a_{\mu_2}^+ ... a_{\mu_n}^+)\,|0\rangle \equiv |\mu_1\mu_2...\mu_n\rangle$, i.e. the permutation group S_n. Namely, the representations $S(a_\mu^+)$ may involve any state. Multiple occupancy of the quantum states is also allowed. Therefore, the states $S(a_\mu^+)\,|0\rangle$ can be considered as consisting of bosonic and fermionic states at the same time. However, it can be called as "Maxons" due to satisfying the Maxwell-Boltzmann statistics, if the particles hold the limit $-1 < q < +1$. Therefore, there will be d^n n-particle states. These states can also be involved in mixed symmetric states and they will obey the Boltzmann distribution function as $e^{-\beta(\epsilon_r - \alpha)}$, where α is the chemical potential, $\beta = 1/k_B T$ and ϵ_r is the energy of state r. From this remarkable result, it is impossible to say that this can be called quantum Boltzmann statistics [11] since the CBY q-oscillator does not possess quantum group invariance. Now, it should be mentioned that in the literature, quon algebra [3] is completely rooted through the SU(d)-invariant CBY q-oscillator algebra. To the particles represented by this q-oscillator algebra it seems as a different algebra due to an incomplete statistical point of view.

It is interesting to investigate the limit $q = 0$ of the CBY q-oscillator in addition to the $q = \pm 1$ limiting cases:

$$a_\mu a_\nu^+ = \delta_{\mu\nu} \qquad \text{for} \qquad q = 0, \qquad \mu, \nu = 1, 2, ..., d. \tag{7}$$

This shows the Cuntz oscillator introduced by Cuntz [12]. In the one dimensional case, the Fock space representation of this oscillator is as follows:

$$\begin{aligned} a\,|0\rangle &= 0, \\ a\,|n\rangle &= |n-1\rangle\,, \qquad n = 1, 2, ... \\ a^+\,|n\rangle &= |n+1\rangle\,, \qquad n = 0, 1, 2, ... \end{aligned} \tag{8}$$

and from these representations,

$$a^+ a\,|n\rangle = \begin{cases} 0, & n = 0 \\ |n\rangle\,, & n > 0, \end{cases}$$

$$(a^+)^2 a^2\,|n\rangle = \begin{cases} 0, & n \le 1 \\ |n\rangle\,, & n > 1, \end{cases}$$

$$...............,$$

$$\left(a^{+}\right)^{n}(a)^{n}|n\rangle = |n\rangle .$$

Since the number operator of the Cuntz oscillator satisfies the relation $N|n\rangle = n|n\rangle$, it turns out that

$$N = \sum_{n=1}^{\infty}\left(a^{+}\right)^{n}a^{n}. \tag{9}$$

For the multidimensional case, the general number operator of the Cuntz oscillator is

$$N = \sum_{n=1}^{\infty}\sum_{\mu_1,\mu_2,...,\mu_n=1}^{d} a^{+}_{\mu_1}a^{+}_{\mu_2}...a^{+}_{\mu_n}a_{\mu_n}a_{\mu_{n-1}}...a_{\mu_1}. \tag{10}$$

On the other hand, for the case $q \neq 0$, the same notion can also be thought for the number operator of the CBY q-oscillator. For this purpose, it is useful to use the permutations, for example

$$\begin{aligned} P_1 &= a^{+}_{\mu_1}a_{\mu_1}, \\ P_{12} &= a^{+}_{\mu_2}a^{+}_{\mu_1}a_{\mu_1}a_{\mu_2}, \\ P_{21} &= a^{+}_{\mu_2}a^{+}_{\mu_1}a_{\mu_2}a_{\mu_1}, \end{aligned}$$

which can be represented as

$$\begin{aligned} P_{k_1} &= a^{+}_{\mu_1}a_{\mu_{k_1}}, \\ P_{k_1k_2} &= a^{+}_{\mu_2}a^{+}_{\mu_1}a_{\mu_{k_1}}a_{\mu_{k_2}}, \end{aligned}$$

where k_1, k_2 is a permutation of $1, 2$. The general expression for such permutations is

$$P_{k_1k_2...k_n} = a^{+}_{\mu_n}a^{+}_{\mu_{n-1}}...a^{+}_{\mu_1}a_{\mu_{k_1}}a_{\mu_{k_2}}...a_{\mu_{k_n}}, \tag{11}$$

where $k_1, k_2, ..., k_n$ is a permutation of $1, 2, ..., n$. Hence, the number operator of the CBY q-oscillator can be written as

$$N = \sum_{n=1}^{\infty}\sum_{permutation\{k_1,k_2,...,k_n\}} C_{k_1k_2...k_n}P_{k_1k_2...k_n} \tag{12}$$

where $P_{k_1k_2...k_n}$ is defined by Eq. (11). It can be verified that

$$C_1 = 1, \qquad C_{12} = \frac{1+q^2}{1-q^2}, \qquad C_{21} = \frac{-2q}{1-q^2}$$

by applying N on the states $|0\rangle$, $|\mu\rangle$, $|\mu_1\mu_2\rangle$. The other coefficients can be successively found by applying N on the states $|\mu_1\mu_2\mu_3\rangle$, which will give 3! equations of the six coefficients $C_{k_1k_2k_3}$, and extending this procedure to higher number of indices step by step.

It can be seen from Fock space properties of the CBY q-oscillator that the norm of states $S(a_\mu^+)\,|0\rangle$ given before are always positive-definite. For instance,

$$\langle \mu\nu \mid \mu\nu\rangle = \langle 0|\, a_\nu a_\mu a_\mu^+ a_\nu^+\, |0\rangle > 0,$$

and the general form of this vacuum-to-vacuum matrix element can be written as

$$\langle 0|\, a_{\mu_n} a_{\mu_{n-1}} ... a_{\mu_1} a_{\mu_1}^+ a_{\mu_2}^+ ... a_{\mu_n}^+\, |0\rangle > 0. \tag{13}$$

This result can also be represented as $\left\| S(a_\mu^+)\,|0\rangle \right\| > 0$. One can now return to investigate the limit $q = 0$ of the CBY q-oscillator such that

$$a_\mu a_\nu^+\, |0\rangle = \delta_{\mu\nu}\, |0\rangle \qquad \text{for} \quad q = 0. \tag{14}$$

Equation (14) holds for the Cuntz oscillator. Moreover, it is not necessary to use the vacuum condition $a_\mu\,|0\rangle = 0$ in calculation of vacuum-to-vacuum matrix elements for the Cuntz oscillator states. The Cuntz oscillator also leads to the Maxwell-Boltzmann statistics due to both Eq. (10) and Eq. (13), which state that any matrix element for the representation of the group $S(a_\mu^+)$ will be positive. This important result is uniquely based on the limit $q = 0$ of the CBY q-oscillator invariant under the ordinary SU(d) group instead of quantum group invariant oscillator. These concluding remarks are similar to that of [11] which found these results by using a commutation relation between number and creation operators.

Now let us again consider the states belonging to the CBY q-oscillator. It is assumed that a linear combination of two particle states $|12\rangle$ and $|21\rangle$ is the state $|\chi\rangle = |12\rangle - \eta\,|21\rangle$, where η is some parameter. This new state can be rewritten as

$$|\chi\rangle = \left(a_1^+ a_2^+ - \eta a_2^+ a_1^+\right) |0\rangle\, . \tag{15}$$

In quantum mechanics, all linear combinations of nonzero vectors $|\chi\rangle$ should have positive norms. By using Eqs. (1) and (15), it is straightforward to show that norm of the state $|\chi\rangle$ is

$$\langle \chi \mid \chi\rangle = 1 + \eta^2 - 2\eta q. \tag{16}$$

It is not possible to say anything about parameter η due to the states $a_1^+ a_2^+ |0\rangle$ and $a_2^+ a_1^+ |0\rangle$ being linearly independent for SU(d)-invariant CBY q-oscillator (Eq. (4)). However, for $-1 \leq q \leq +1$, $\langle \chi \mid \chi \rangle = 0$ only for $\eta = q$ and $q^2 = 1$. Then $|\chi\rangle = 0$. If one chooses $\eta = q$, we then see that

$$\langle \chi \mid \chi \rangle = 1 - q^2. \tag{17}$$

Hence, an interpretation to this equation can be stated as follows:

1. $\langle \chi \mid \chi \rangle = 1$ for $q = 0$, is physically consistent,

2. $|\chi\rangle = 0$ for $q = \pm 1$, is physically not allowed,

3. $\langle \chi \mid \chi \rangle > 0$ for $-1 < q < +1$, is physically consistent and is exactly relevant for the Maxwell-Boltzmann statistics.

In this paper, we have studied the algebraic and the representative properties of both the CBY q-oscillator and the Cuntz oscillator. The different limiting cases of the deformation parameter q of the CBY q-oscillator are discussed for the second quantized particle statistics. In this respect, the following interesting results can be summarized: The CBY q-oscillator in the range $-1 < q < +1$ coincides with the Cuntz oscillator, in that both lead to Maxwell-Boltzmann statistics. For the multidimensional case, the number operators for both of these oscillators are explicitly constructed. In particular, the Cuntz oscillator has a number operator of infinite degree as shown in Eq. (10), which is based upon Fock space properties of the oscillator. Moreover, the linear combinations of the quantum states of the CBY q-oscillator, such as $|\chi\rangle$ in Eq. (15), have a positive norm and lead to the Maxwell-Boltzmann statistics only for the range $-1 < q < +1$.

As a final remark, from all calculations above the commutation relation which leads to the Maxwell-Boltzmann statistics does not have a connection with quantum groups. Thus, in any case, the Maxwell-Boltzmann statistics can not be worked with quantum group invariant oscillators. This statistics is completely based on a the limit case of the ordinary SU(d) group invariant CBY q-oscillator. Moreover, in connection with this type of statistics, it may be important to note that some observables of the Maxwell-Boltzmann statistics field, which are represented by operators, do not have local commutativity property. With the above in mind, this non-locality property does not seem related to quantum group invariant oscillators.

Acknowledgement

I thank Professor Metin Arık for our many useful discussions and I thank the Feza Gürsey Institute for its hospitality.

References

[1] O. W. Greenberg and R. N. Mohapatra, Phys. Rev. D, 39 (1989) 2032.

[2] R. N. Mohapatra, Phys. Lett. B, 242 (1990) 407.

[3] O. W. Greenberg, Phys. Rev. D, 43 (1991) 4111.

[4] O. W. Greenberg, R. C. Hilborn, Phys. Rev. Lett., 83 (1999) 4460.

[5] H. S. Green, Phys. Rev., 90 (1953) 270.

[6] A. Lerda, Anyons: Quantum Mechanics of Particles with Fractional Statistics (Springer-Verlag, Berlin, Germany, 1992).

[7] M. Frau, M. A. R. Monteiro, S. Sciuto, J. Phys. A, 27 (1994) 801.

[8] C. R. Lee, J. P. Yu, Phys. Lett. A, 150 (1990) 63.

[9] D. D. Coon, M. Baker, S. Yu, Phys. Rev. D, 5 (1972) 1429.

[10] M. Arık, G. Ünel, M. Mungan, Phys. Lett. B, 321 (1994) 385.

[11] O. W. Greenberg, Phys. Rev. Lett., 64 (1990) 705.

[12] J. Cuntz, Commun. Math. Phys., 57 (1977) 173.