Turk J Phys
25 (2001) , 393 – 398.

Coproduct of SU$_q$(2), Coherent States and Four Point Function with Logarithmic Regge Trajectories

Metin ARIK
Department of Physics, Boğaziçi University,
Bebek, İstanbul-TURKEY
Ayşe PEKER-DOBIE
Department of Mathematics, İstanbul Technical University,
Maslak, İstanbul-TURKEY

Received 29.03.2000

Abstract

The representation of the operators belonging to a 2×2 $SU_q(2)$ matrix yield a Hilbert space H. The coproducts of these operators define a Hilbert space $H^{(2)}$ isomorphic to H and canonically embedded in $H \otimes H$. The four-point function obtained by taking the scalar product of the ground state of $H^{(2)}$ with the coherent states in $H \otimes H$ is uniquely defined, is meromorphic and has Regge behaviour.

Key Words: Dual amplitudes, Quantum groups, Regge behaviour, Hopf algebras, Coherent states.

Dual amplitudes with logarithmic trajectories are the widest class of dual amplitudes having simple analyticity structure that is unchanged under a linear transformation of the Mandelstam variables s and t. They were discovered in the 1970's [1, 2] and were rediscovered [3] in 1990 after the discovery of quantum groups [4, 5]. Considering the scattering amplitude as a function of two variables σ and τ, where σ and τ have a linear dependence on s and t, respectively, then the most general meromorphic dual four-point function with Regge behaviour is given by [6]

$$M(\sigma,\tau) = \sum_{m,n=0}^{\infty} h_{mn}\ A(q^m\ \sigma, q^n\ \tau), \tag{1}$$

where

$$A(\sigma,\tau) = \frac{G_q(\sigma,\tau)}{G_q(\sigma)\ G_q(\tau)} = \sum_{m,n=0}^{\infty} \frac{\sigma^m\ q^{mn}\ \tau^n}{f_m(q)\ f_n(q)} \tag{2}$$

is the Coon-Baker [1] four-point function,

$$\begin{aligned} G_q(x) &= \prod_{l=0}^{\infty}(1-q^l\ x) = \sum_{l=0}^{\infty}(-1)^l\ \frac{q^{l(l-1)/2}}{f_l(q)}\ x^l, \\ f_l(q) &= (1-q)\cdots(1-q^l) = \frac{G_q(q)}{G_q(q^{l+1})} \end{aligned} \tag{3}$$

and h_{mn} are entire function coefficients, i.e. the function

$$H(w,z) = \sum_{m,n=0}^{\infty} h_{mn}\ w^m\ z^n \tag{4}$$

is an entire function of both variables. Equation (1) can also be expressed in terms of the Cremmer-Nuyts [2] four-point function which itself does not have Regge behaviour for $\tau < q$:

$$\begin{aligned} a(\sigma,\tau) &= \sum_{m,n=0}^{\infty}\sigma^m\ q^{mn}\ \tau^n \\ &= \sum_{n=0}^{\infty}\frac{\tau^n}{1-q^n\ \sigma} \end{aligned} \tag{5}$$

by

$$A(\sigma,\tau) = \sum_{m,n=0}^{\infty} C_{mn}\ a(q^m\ \sigma, q^n\ \tau), \tag{6}$$

where

$$C_{mn} = (-1)^{m+n}\ \frac{q^{m(m+1)/2}}{f_m(q)}\ \frac{q^{n(n+1)/2}}{f_n(q)}. \tag{7}$$

The four-point function in (1) has a great deal of arbitrariness associated with the arbitrariness of the coefficients h_{mn}. In particular, taking any finite number of the coefficients h_{mn} to be nonzero defines a meromorphic four-point function with Regge behaviour. Thus a physical or mathematical guiding principle is needed to determine the coefficients h_{mn} in (1). In this paper, we propose such a principle using the Hilbert space associated with the operators defined by the Hopf algebra generated by the matrix elements of $SU_q(2)$ [5]. The well-known $SU_q(2)$ quantum matrix group is composed of the matrices

$$U = \begin{pmatrix} a & -qb \\ b^* & a^* \end{pmatrix}.$$

Let $\mathtt{A}$ be the Hopf algebra over the complex numbers generated by the elements a, a^*, b and b^*, satisfying the Hermiticity conditions $(a^*)^* = a$, $(b^*)^* = b$ and the commutation relations

$$
\begin{aligned}
a\ a^* + q^2 bb^* &= 1 \\
a^* a + b^* b &= 1 \\
ab &= qba \\
ab^* &= qb^* a \\
bb^* &= b^* b.
\end{aligned}
\tag{8}
$$

The coproduct
$\Delta : \mathtt{A} \to \mathtt{A} \otimes \mathtt{A}$, the antipode $S : \mathtt{A} \to \mathtt{A}$ and the counit $\varepsilon : \mathtt{A} \to C$ are defined by

$$
\begin{aligned}
\Delta(a) &= a \otimes a - qb \otimes b^* \\
\Delta(b) &= a \otimes b + b \otimes a^* \\
\Delta(a^*) = (\Delta(a))^* \quad &, \quad \Delta(b^*) = (\Delta(b))^*
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon(a) &= \varepsilon(a^*) = 1 \\
\varepsilon(b) &= \varepsilon(b^*) = 0
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
S(a) &= a^* \\
S(a^*) &= a \\
S(b) &= -q^{-1} b \\
S(b^*) &= -qb^*.
\end{aligned}
$$

It can easily be seen that the defining relations (8) of $\mathtt{A}$ are invariant under $b \leftrightarrow b^*$. Therefore, there exists a related second coproduct with

$$
\begin{aligned}
\Delta(a) &= a \otimes a - qb^* \otimes b \\
\Delta(b^*) &= a \otimes b^* + b^* \otimes a^* \\
\Delta(a^*) = (\Delta(a))^* \quad &, \quad \Delta(b) = (\Delta(b^*))^*.
\end{aligned}
\tag{10}
$$

We look for a representation of $\mathtt{A}$ on a Hilbert space such that b is invertible. If b is not invertible, then its zero eigenvalue subspace is a trivial irreducible representation where $b = 0$ and a is any unitary operator. If b is invertible, then a^*a and the phase of b form a commuting set. a^* and a act as creation and annihilation operators, respectively. In other words, we have

$$\begin{aligned} a \mid n, \alpha\rangle &= (1-q^{2n})^{1/2} \mid n-1, \alpha\rangle \\ a^* \mid n, \alpha\rangle &= (1-q^{2n+2})^{1/2} \mid n+1, \alpha\rangle \\ b \mid n, \alpha\rangle &= q^n e^{i\alpha} \mid n, \alpha\rangle \\ b^* \mid n, \alpha\rangle &= q^n e^{-i\alpha} \mid n, \alpha\rangle, \end{aligned} \tag{11}$$

where $\alpha \,\epsilon\, [0, 2\pi)$ and we have chosen the normalization $\langle n, \alpha \mid m, \alpha\rangle = \delta_{nm}$ for a fixed α. We denote the Hilbert space spanned by $\mid n, \alpha\rangle$ by H_α. The algebra generated by $\Delta(a)$ and $\Delta(b)$ has a unique representation on $H_\alpha \otimes H_\alpha$. In other words, there is only one $|| \, 0, \alpha\rangle\rangle \,\epsilon\, H_\alpha \otimes H_\alpha$ such that the following relations hold:

$$\begin{aligned} \Delta(a) \, || \, 0, \alpha\rangle\rangle &= 0 \\ \Delta(b) \, || \, 0, \alpha\rangle\rangle &= e^{i\alpha} \, || \, 0, \alpha\rangle\rangle, \end{aligned} \tag{12}$$

where $|| \, 0, \alpha\rangle\rangle = \sum_{n,m} C_{nm} \ \mid n, \alpha\rangle \otimes \mid m, \alpha\rangle$. Note that $|| \, n, \alpha\rangle\rangle$ is given by:

$$|| \, n, \alpha\rangle\rangle = \frac{(\Delta(a^*))^n}{\sqrt{f_n(q^2)}} \, || \, 0, \alpha\rangle\rangle. \tag{13}$$

Using the coproduct relations (9) together with (12), one finds the following recursion relations for $C_{n,m}$:

$$\begin{aligned} C_{n,m} &= q^m \ (1-q^{2(n+1)})^{1/2} \ C_{n+1,m} + q^n \ (1-q^{2m})^{1/2} \ C_{n,m-1} \\ C_{n+1,m+1} &= q^{n+m+1} (1-q^{2(n+1)})^{-1/2} \ (1-q^{2(m+1)})^{-1/2} \ C_{n,m}. \end{aligned} \tag{14}$$

Setting $C_{0,0} = 1$, these recursion equations have the unique solution

$$C_{n,m} = \frac{q^{nm}}{\sqrt{f_n(q^2)} \ \sqrt{f_m(q^2)}}. \tag{15}$$

This can easily be verified by putting $C_{n,m} = \frac{q^{nm}}{\sqrt{f_n(q^2)} \ \sqrt{f_m(q^2)}} \ A_{n,m}$ into (14) which leads to $A_{n,m}$ being independent of n and m. Thus, considering

$$| n, \alpha\rangle \equiv \frac{(a^*)^n}{\sqrt{f_n(q^2)}} \mid 0, \alpha\rangle \tag{16}$$

and (15) one obtains

$$|| 0, \alpha\rangle\rangle = \sum_{n,m=0}^{\infty} \frac{q^{nm}}{f_n(q^2)\ f_m(q^2)} \ ((a^*)^n \otimes (a^*)^m) \ (| 0, \alpha\rangle \otimes | 0, \alpha\rangle). \tag{17}$$

Let $| \sigma, \alpha\rangle$ be the coherent state of the annihilation operator a; that is

$$a \ \ | \sigma, \alpha\rangle = \sigma \ \ | \sigma, \alpha\rangle, \tag{18}$$

then

$$| \sigma, \alpha\rangle = \sum_{n=0}^{\infty} \frac{\sigma^n}{\sqrt{f_n(q^2)}} \ \ | n, \alpha\rangle. \tag{19}$$

Recall that $|| 0, \alpha\rangle\rangle$ is the ground state of $\Delta(a)$ where Δ is the Hopf algebra coproduct of $SU_q(2)$ defined in (9). Consider the inner product in $H_\alpha \otimes H_\alpha$ defined by

$$M(\sigma, \tau) = \langle\langle 0, \alpha \,|| \ \ (| \sigma, \alpha\rangle \otimes \ \ | \tau, \alpha\rangle) \tag{20}$$

where $| \sigma, \alpha\rangle$ and $| \tau, \alpha\rangle$ are the coherent states of a satisfying (18). Then $M(\sigma, \tau)$ in (20) is the transition amplitude from the ground state of the coproduct of a to the tensor product of the coherent states of a. We will show that the function $M(\sigma, \tau)$ in (20) defines a unique, Regge behaved, meromorphic four-point function. Using (3) enables us to express $f_m(q^2)$ in the form

$$f_m(q^2) = f_m(q) \ \frac{G_q(-q)}{G_q(-q^{m+1})}.$$

If we consider the expression above together with (17) and (20), then we get

$$M(\sigma, \tau) = \frac{1}{G_q^2(-q)} \sum_{k,l=0}^{\infty} \frac{q^{k(k+1)/2} \ q^{l(l+1)/2}}{f_k(q) \ f_l(q)} \ A(q^k \sigma, q^l \tau), \tag{21}$$

where $A(\sigma, \tau)$ is given by (2). The function in (4) then becomes

$$\begin{aligned} H(w, z) &= \textstyle\sum_{k,l=0}^{\infty} \frac{q^{k(k+1)/2} \ q^{l(l+1)/2}}{f_k(q) \ f_l(q)} \ w^k \ z^l \\ &= G_q(-qw) \ G_q(-qz). \end{aligned}$$

Since $H(w, z)$ is an entire function of both variables, then $M(\sigma, \tau)$ in (20) is Regge behaved. The nicest feature of our derivation of the four-point function (21) is its uniqueness. We have shown that quantum group considerations and in particular the $SU_q(2)$

Hopf algebra coproduct can be used to construct Regge behaved, meromorphic scattering amplitudes. A deeper physical understanding of our construction and construction of multiparticle scattering amplitudes are worth further investigation.

References

[1] D.D. Coon, *Phys. Lett.,* **29B** (1969) 669.
M. Baker and D.D. Coon, *Phys. Rev.,* **D2** (1970) 2349.

[2] E. Cremmer and J. Nuyts, *Nucl. Phys.,* **B26** (1971) 151.

[3] L. Romans, *"q-strings" in "Strings '89"*, R. Arnowitt et al., World Scientific (1990).

[4] L.D. Faddeev, N.Y. Reshetikin and L.A. Takhtajan, *Algebr. Anal.* **1**, (1989) 178 (in Russian); V.G. Drinfeld, *Quantum groups, Proc. Intern. Congress of Mathematicians Vol.1* (Berkeley, CA) (1986) pp.798-820; M. Jimbo, *Lett. Math. Phys.*, **11** (1986) 247.

[5] S.L. Woronowicz, Commun. *Math. Phys.,* **111** (1987) 613.

[6] M. Arik, *Phys. Rev.*, **D9** (1974) 3467.