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# Covariant Two Fermion Equations with Anomalous Magnetic Moments*

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**Abstract**

We derive covariant equations for two-fermion systems, taking into account the anomalous magnetic moments of the particles.

## 1. Introduction

Relativistic dynamics of two interacting fermions is the basic problem of the test of quantum electrodynamics in low energy bound state problems.

Altough 4-dimensional Bethe-Salpeter equation (BSE) [1] provides an exact formalism, one has to resort to an appropriate 3-dimensional exactly solvable wave equation for practical calculations. Different kinds of 3-dimensional reductions of BSE are available in the literature [2, 3, 4].

A new approach, called the self energy formulation of quantum electrodynamics, to the bound state problem has been formulated by Barut and his collaborators [5, 6] and applied to the spectra of hydrogen, muonium and positronium [7, 8] There is also a generalization of the results of this approach to the relativistic N-body problem in Ref. [9].

Since it is easy to boost a system when the theory is fully covariant, we are going to derive the covariant equation of two fermions interacting with their charges and anomalous magnetic moments in the framework of the self-energy QED.

*Dedicated to the memory of Prof. A. O. Barut

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## 2. Derivation of the two-fermion equation

We start from the action of two (distinct) fermion fields $\psi_1(x),\ \psi_2(x)$ interacting with the electromagnetic field $A_\mu$:

$$A = \int dx \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_j \left[ \bar{\psi}_j(x) \left( i\gamma_\mu \partial^\mu - m_j \right) \psi_j(x) \right.\right. \tag{1}$$
$$\left.\left. -\ e_j \bar{\psi}_j(x) \gamma_\mu \psi_j(x) A^\mu(x) - a_j \bar{\psi}_j(x) \sigma_{\mu\nu} \psi_j(x) F^{\mu\nu} \right] \right\}.$$

The fermions have electric charges $e_j$ and anomalous magnetic moments $a_j$ and spin matrices $\gamma^j;\ j = 1,2$. The equations of motion obtained from (1) are

$$\partial^\nu F_{\mu\nu} = -j_\mu^{tot} = -\sum_j \left[ e_j \bar{\psi}_j \gamma_\mu \psi_j + 2a_j \partial^\nu \left( \bar{\psi}_j \sigma_{\mu\nu} \psi_j \right) \right] \tag{2}$$

and,

$$\left[ (i\gamma^\mu \partial_\mu - m_j) - e_j \gamma^\mu A_\mu - a_j \sigma^{\mu\nu} F_{\mu\nu} \right] \psi_j = 0. \tag{3}$$

The total current in (2) satisfies the continuity equation $\partial_\mu J^\mu = 0$ as required by the Maxwell equations and charge conservation. $A_\mu$ is obtained from $\partial_\nu \partial^\nu A_\mu = j_\mu$ in the gauge $\partial_\mu A^\mu = 0$ , and the general solution is

$$A_\mu(x) = \int dy D(x-y) j_\mu^{tot}(y), \tag{4}$$

and from (2) we have

$$A_\mu(x) = \int dy D(x-y) \sum_j \left[ e_j \bar{\psi}_j(y) \gamma_\mu \psi_j(y) + 2a_j \partial_y^\nu \left( \bar{\psi}_j(y) \sigma_{\mu\nu} \psi_j(y) \right) \right]. \tag{5}$$

The last term in the action (1) is $2a_j \bar{\psi}_j \sigma^{\mu\nu} \psi_j \partial_\nu A_\mu$ , or by partial integration,

$$-a_j \int dx \bar{\psi}_j(x) \sigma_{\mu\nu} \psi_j(x) F^{\mu\nu} = -2a_j \int dx \left[ \partial_\nu \left( \bar{\psi}_j \sigma^{\mu\nu} \psi_j \right) \right] A_\mu.$$

Also the free Lagrangian of the Maxwell field is equivalent to $\frac{1}{2} \int dx j^\mu A_\mu$. Consequently the action can be written as

$$A = \int dx \left[ \sum_j \bar{\psi}_j(x) \left( i\gamma^\mu \partial_\mu - m_j \right) \psi_j(x) \right. \tag{6}$$
$$- \sum_{j,k} \frac{1}{2} e_j e_k \int dy \bar{\psi}_j(x) \gamma^\mu \psi_j(x) D(x-y) \bar{\psi}_k(y) \gamma_\mu \psi_k(y)$$

$$
\begin{aligned}
&- \sum_{j,k} e_j a_k \int dy \bar{\psi}_j(x)\gamma^\mu \psi_j(x) \partial^\lambda D(x-y) \bar{\psi}_k(y) \sigma_{\mu\lambda} \psi_k(y) \\
&- \sum_{j,k} a_j e_k \int dy \bar{\psi}_j(x)\sigma^{\mu\nu} \psi_j(x) \partial_\mu D(x-y) \bar{\psi}_k(y) \gamma_\nu \psi_k(y) \\
&- \left. \sum_{j,k} 2a_j a_k \int dy \bar{\psi}_j(x)\sigma^{\mu\nu} \psi_j(x) \partial_\nu \partial^\lambda D(x-y) \bar{\psi}_k(y) \sigma_{\lambda\mu} \psi_k(y) \right] .
\end{aligned}
$$

The diagonal terms $j = k$ correspond to self energy, the interaction of the particle's current with itself. Here, we are interested in the mutual interaction of two different particles, hence in terms with $j \neq k$.

For the mutual interaction of two particles we use the retarded Green's function $D^{\mathrm{ret}}(x-y)$. Now for the $e_1 e_2$-interaction there are two terms in (6) with coefficents $e_1 e_2$ and $e_2 e_1$. In the second term we interchange $x$ and $y$ and use the identity

$$
D^{\mathrm{ret}}(y-x) = D^{\mathrm{adv}}(x-y). \tag{7}
$$

This is equivalent to writing the interaction term as

$$
-\sum_{j<k} e_j e_k \int dx dy \bar{\psi}_j(x)\gamma^\mu \psi_j(x) \bar{D}(x-y) \bar{\psi}_k(y) \gamma_\mu \psi_k(y).
$$

where

$$
\bar{D} = \frac{1}{2}(D^{\mathrm{ret}} + D^{\mathrm{adv}}).
$$

Similarly, for the other interaction terms, we note that

$$
\partial^\lambda D^{\mathrm{ret}}(x-y) = -\partial^\lambda D^{\mathrm{adv}}(x-y),
$$

which allows as to combine various terms to obtain the action in (6) as

$$
\begin{aligned}
A = \int dx & \left[ \sum_j \bar{\psi}_j(x) \left( i\gamma^\mu \partial_\mu - m_j \right) \psi_j(x) \right. \qquad (8) \\
&- \sum_{j<k} e_j e_k \int dy \bar{\psi}_j(x)\gamma^\mu \psi_j(x) \bar{D}(x-y) \bar{\psi}_k(y) \gamma_\mu \psi_k(y) \\
&- \sum_{j<k} 2e_j a_k \int dy \bar{\psi}_j(x)\gamma^\mu \psi_j(x) \partial^\lambda \bar{D}(x-y) \bar{\psi}_k(y) \sigma_{\mu\lambda} \psi_k(y) \\
&- \sum_{j<k} 2a_j e_k \int dy \bar{\psi}_j(x)\sigma^{\mu\nu} \psi_j(x) \partial_\mu \bar{D}(x-y) \bar{\psi}_k(y) \gamma_\nu \psi_k(y) \\
&- \left. \sum_{j<k} 4a_j a_k \int dy \bar{\psi}_j(x)\sigma^{\mu\nu} \psi_j(x) \partial_\nu \partial^\lambda \bar{D}(x-y) \bar{\psi}_k(y) \sigma_{\lambda\mu} \psi_k(y) \right] .
\end{aligned}
$$

Now we define the composite or bilocal field $\phi(x,y)$, which is a 16-component spinor, by

$$\phi(x,y) = \psi_1(x) \otimes \psi_2(y) \tag{9}$$

so that the spin algebra is a direct product of two Dirac algebras. And from now on the first term of the direct product will always refer to particle 1, and the second term to particle 2.

In order to vary the action with respect to this composite field we have to rewrite the action in terms of it. The interaction terms in the action already contain the composite fields. The free part of the action is a sum of terms each containing one-field only. In order to write this part in terms of composite fields we multiply the free particle part of one particle with the general relativistic integral of the other particle, e.g.,

$$\int d\vec{y}\bar{\psi}_2(y)\gamma \cdot n\psi_2(y) = 1 \tag{10}$$

from which we get the usual normalization integral $\int d\vec{y}\psi_2^\dagger\psi_2 = 1$ for $n = (1000)$.

Then the action in terms of composite fields $\phi$ becomes

$$\begin{aligned} A = &\int dxd\vec{y}\bar{\phi}(x,y)\left[(i\gamma^\mu\partial_\mu - m_1)\otimes\gamma\cdot n + \gamma\cdot n\otimes(i\gamma^\mu\partial_\mu - m_2)\right]\phi(x,y) \\ &+ \int dxdy\bar{\phi}(x,y)\left[-e_1e_2\gamma^\mu\otimes\gamma_\mu\bar{D}(x-y) - 2e_1a_2\gamma^\mu\otimes\sigma_{\mu\lambda}\partial^\lambda\bar{D}(x-y)\right. \\ &- \left. 2a_1e_2\sigma^{\mu\nu}\otimes\gamma_\nu\partial_\mu\bar{D}(x-y) - 4a_1a_2\sigma^{\nu\mu}\otimes\sigma_{\mu\lambda}\partial_\nu\partial^\lambda\bar{D}(x-y)\right]\phi(x,y). \end{aligned} \tag{11}$$

In this form it is not yet convenient to vary the action. We have to rewrite it with all terms under the same 7-fold integral sign. To do that we decompose the Green's function $\bar{D}(x-y)$ in the second integral-term, as

$$\begin{aligned} \bar{D}(x-y) &= \frac{1}{2}\frac{1}{4\pi}\delta\left[(x-y)^2\right] \\ &= \frac{1}{2}\frac{1}{4\pi}\frac{\delta\left[(x-y)\cdot n - r_{12\perp}\right] + \delta\left[(x-y)\cdot n + r_{12\perp}\right]}{r_{12\perp}}, \end{aligned} \tag{12}$$

where we introduce the relativistic distance $r_{12\perp}$ as

$$r_{12\perp} = \left[\left((x-y)\cdot n\right)^2 - (x-y)^2\right]^{1/2}, \tag{13}$$

which for $n = (1000)$ is just the 3-dimensional $r_{12}$. Now $y^0$ integration can be performed. But in the terms involving the derivative of $\bar{D}(x-y)$, we perform partial integrations with respect to $x$ before $y^0$ integration to avoid the derivatives of the Green's function. The result is:

$$A = \int dxd\vec{y}\bar{\phi}(x,y)\left[(i\gamma^\mu\partial_\mu - m_1)\otimes\gamma\cdot n + \gamma\cdot n\otimes(i\gamma^\mu\partial_\mu - m_2)\right.$$

$$+\frac{1}{4\pi}\left(-e_1e_2\frac{1}{r_{12\perp}}\gamma^\mu\otimes\gamma_\mu-2e_1a_2\left(\partial^\lambda\frac{1}{r_{12\perp}}\right)\gamma^\mu\otimes\sigma_{\mu\lambda}\right.$$
$$\left.-\;2a_1e_2\left(\partial_\mu\frac{1}{r_{12\perp}}\right)\sigma^{\mu\nu}\otimes\gamma_\nu-4a_1a_2\left(\partial_\nu\partial^\lambda\frac{1}{r_{12\perp}}\right)\sigma^{\nu\mu}\otimes\sigma_{\mu\lambda}\right)\Bigg]\,\phi(x,y). \tag{14}$$

In (14) the composite field $\phi(x,y)$ refers to $\frac{1}{2}(\phi_{\rm ret}+\phi_{\rm adv})$, that is the combination of the two contributions coming from retarded and advanced points. For simplicity, we shall write only $\Phi(x,y)$ for this combination.

Now the variation of the action with respect to composite field $\Phi$ gives the equation of motion:

$$[(i\gamma^\mu\partial_\mu-m_1)\otimes\gamma\cdot n+\gamma\cdot n\otimes(i\gamma^\mu\partial_\mu-m_2)$$
$$+\frac{1}{4\pi}\left(-e_1e_2\frac{1}{r_{12\perp}}\gamma^\mu\otimes\gamma_\mu-2e_1a_2\left(\partial^\lambda\frac{1}{r_{12\perp}}\right)\gamma^\mu\otimes\sigma_{\mu\lambda}\right.$$
$$\left.-\;2a_1e_2\left(\partial_\mu\frac{1}{r_{12\perp}}\right)\sigma^{\mu\nu}\otimes\gamma_\nu-4a_1a_2\left(\partial_\nu\partial^\lambda\frac{1}{r_{12\perp}}\right)\sigma^{\nu\mu}\otimes\sigma_{\mu\lambda}\right)\Bigg]\,\Phi(x,y)=0.. \tag{15}$$

Finally writing the composite field $\Phi$ as a column matrix

$$\Phi=\begin{pmatrix}\Phi_{11}\\ \Phi_{12}\\ \Phi_{21}\\ \Phi_{22}\end{pmatrix}, \tag{16}$$

in which $\Phi_{11},\ldots,\Phi_{22}$ are four-component spinors, and inserting the explicit form of Dirac matrices in the covariant Weyl representation, with

$$\gamma^\mu=\begin{pmatrix}0 & \tilde\sigma^\mu\\ \sigma^\mu & 0\end{pmatrix},\qquad \tilde\sigma^\mu=\sigma_\mu=(1,\vec\sigma)$$

where $\vec\sigma$ are Pauli matrices, we obtain four coupled covariant equations satisfied by four 4-component spinors:

$$\left(\tilde\sigma\cdot p_1\otimes\tilde\sigma\cdot n+\tilde\sigma\cdot n\otimes\tilde\sigma\cdot p_2-e_1e_2\frac{1}{r_{12\perp}}\tilde\sigma^\mu\otimes\tilde\sigma_\mu\right)\Phi_{22}$$
$$-\left[m_2\tilde\sigma\cdot n\otimes I+2ie_1a_2\partial^\lambda\frac{1}{r_{12\perp}}\left(\tilde\sigma^\mu\otimes\tilde\sigma_\mu\sigma_\lambda-\tilde\sigma_\lambda\otimes I\right)\right]\Phi_{21}$$
$$-\left[m_1I\otimes\tilde\sigma\cdot n+2ia_1e_2\partial_\mu\frac{1}{r_{12\perp}}\left(\tilde\sigma^\mu\sigma^\nu\otimes\tilde\sigma_\nu-I\otimes\tilde\sigma^\mu\right)\right]\Phi_{12}$$
$$+\,4a_1a_2\left(\partial_\nu\partial^\lambda\tilde\sigma^\nu\sigma^\mu\otimes\tilde\sigma_\mu\sigma_\lambda-\partial_\nu\partial_\mu\tilde\sigma^\nu\sigma^\mu\otimes I\right.$$
$$\left.-\;\partial^\mu\partial^\lambda I\otimes\tilde\sigma_\mu\sigma_\lambda+\partial^2I\otimes I\right)\frac{1}{r_{12\perp}}\Phi_{11}=0 \tag{17}$$

$$\begin{aligned}
&\left(\tilde{\sigma}\cdot p_1\otimes\sigma\cdot n+\tilde{\sigma}\cdot n\otimes\tilde{\sigma}\cdot p_2-e_1e_2\frac{1}{r_{12\perp}}\tilde{\sigma}^{\mu}\otimes\sigma_{\mu}\right)\Phi_{21} \\
&-\left[m_2\tilde{\sigma}\cdot n\otimes I+2ie_1a_2\partial^{\lambda}\frac{1}{r_{12\perp}}\left(\tilde{\sigma}^{\mu}\otimes\sigma_{\mu}\tilde{\sigma}_{\lambda}-\tilde{\sigma}_{\lambda}\otimes I\right)\right]\Phi_{22} \\
&-\left[m_1 I\otimes\sigma\cdot n+2ia_1e_2\partial_{\mu}\frac{1}{r_{12\perp}}\left(\tilde{\sigma}^{\mu}\sigma^{\nu}\otimes\sigma_{\nu}-I\otimes\sigma^{\mu}\right)\right]\Phi_{11} \\
&+4a_1a_2\left(\partial_{\nu}\partial^{\lambda}\tilde{\sigma}^{\nu}\sigma^{\mu}\otimes\sigma_{\mu}\tilde{\sigma}_{\lambda}-\partial_{\nu}\partial_{\mu}\tilde{\sigma}^{\nu}\sigma^{\mu}\otimes I\right. \\
&\left.-\,\partial^{\mu}\partial^{\lambda}I\otimes\sigma_{\mu}\tilde{\sigma}_{\lambda}+\partial^2 I\otimes I\right)\frac{1}{r_{12\perp}}\Phi_{12}=0
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\left(\sigma\cdot p_1\otimes\tilde{\sigma}\cdot n+\sigma\cdot n\otimes\tilde{\sigma}\cdot p_2-e_1e_2\frac{1}{r_{12\perp}}\sigma^{\mu}\otimes\tilde{\sigma}_{\mu}\right)\Phi_{12} \\
&-\left[m_2\sigma\cdot n\otimes I+2ie_1a_2\partial^{\lambda}\frac{1}{r_{12\perp}}\left(\sigma^{\mu}\otimes\tilde{\sigma}_{\mu}\sigma_{\lambda}-\sigma_{\lambda}\otimes I\right)\right]\Phi_{11} \\
&-\left[m_1 I\otimes\tilde{\sigma}\cdot n+2ia_1e_2\partial_{\mu}\frac{1}{r_{12\perp}}\left(\sigma^{\mu}\tilde{\sigma}^{\nu}\otimes\tilde{\sigma}_{\nu}-I\otimes\tilde{\sigma}^{\mu}\right)\right]\Phi_{22} \\
&+4a_1a_2\left(\partial_{\nu}\partial^{\lambda}\sigma^{\nu}\tilde{\sigma}^{\mu}\otimes\tilde{\sigma}_{\mu}\sigma_{\lambda}-\partial_{\nu}\partial_{\mu}\sigma^{\nu}\tilde{\sigma}^{\mu}\otimes I\right. \\
&\left.-\,\partial^{\mu}\partial^{\lambda}I\otimes\tilde{\sigma}_{\mu}\sigma_{\lambda}+\partial^2 I\otimes I\right)\frac{1}{r_{12\perp}}\Phi_{21}=0
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\left(\sigma\cdot p_1\otimes\sigma\cdot n+\sigma\cdot n\otimes\sigma\cdot p_2-e_1e_2\frac{1}{r_{12\perp}}\sigma^{\mu}\otimes\sigma_{\mu}\right)\Phi_{11} \\
&-\left[m_2\sigma\cdot n\otimes I+2ie_1a_2\partial^{\lambda}\frac{1}{r_{12\perp}}\left(\sigma^{\mu}\otimes\sigma_{\mu}\tilde{\sigma}_{\lambda}-\sigma_{\lambda}\otimes I\right)\right]\Phi_{12} \\
&-\left[m_1 I\otimes\sigma\cdot n+2ia_1e_2\partial_{\mu}\frac{1}{r_{12\perp}}\left(\sigma^{\mu}\tilde{\sigma}^{\nu}\otimes\sigma_{\nu}-I\otimes\sigma^{\mu}\right)\right]\Phi_{21} \\
&+4a_1a_2\left(\partial_{\nu}\partial^{\lambda}\sigma^{\nu}\tilde{\sigma}^{\mu}\otimes\sigma_{\mu}\tilde{\sigma}_{\lambda}-\partial_{\nu}\partial_{\mu}\sigma^{\nu}\tilde{\sigma}^{\mu}\otimes I\right. \\
&\left.-\,\partial^{\mu}\partial^{\lambda}I\otimes\sigma_{\mu}\tilde{\sigma}_{\lambda}+\partial^2 I\otimes I\right)\frac{1}{r_{12\perp}}\Phi_{22}=0.
\end{aligned} \tag{20}$$

## 3. Conclusion

After the elimination of the electromagnetic field in the interaction of two charged fermions $\psi_1, \psi_2$ we wrote the action in terms of a composite field $\phi(x,y)=\psi_1(x)\otimes\psi_2(y)$. By varying the action with respect to $\Phi$ we obtained the covariant linear equations in configuration space of the system in Weyl representation.

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