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Analysis on the 2–Dim Quantum Poincaré Group at Roots of Unity*

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Abstract

2–Dim quantum Poincaré Group $E_q(1,1)$ at roots of unity, its dual $U_q(e(1,1))$ and some of its homogeneous spaces are introduced. Invariant integrals on $E_q(1,1)$ and its invariant discrete subgroup $E(1,1 \mid p)$ are constructed. $*$–Representations of the quantum algebra $U_q(e(1,1))$ constructed in the homogeneous space $SO(1,1 \mid p)$ are integrated to the pseudo–unitary representations of $E_q(1,1)$ by means of the universal T–matrix. $U_q(e(1,1))$ is realized on the quantum plane $E_q^{(1,1)}$ and the eigenfunctions of the complete set of observables are obtained in the angular momentum and momentum basis. The matrix elements of the pseudo–unitary irreducible representations are given in terms of the cut off q–exponential and q–Bessel functions whose properties we also investigate.

1. Introduction

Finite dimensional representations of the quantum algebra $U_q(g)$ for real q is very similar to the representations of the universal enveloping algebra $U(g)$ where g is the complex simple Lie algebra [17, 19, 23, 24]. Theory of the algebraic quantum group G_q which is the Hopf algebra of the quantized polynomials on the Lie group G is essentially the same as that of G too (see [26] and references therein). Matrix elements of the irreducible representations of G_q are expressed in term of the q–special functions which are the generalization of the ones related to the Lie group G. There also exist an invariant distance [1], an invariant integral and Peter–Weyl approximation theorem [27] on the compact quantum group G_q and its symmetric spaces.

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On the other hand the quantum algebra $U_q(g)$ at roots of unity admits finite dimensional irreducible representations which have no classical analogs [6, 10, 11, 20, 22]. Because of the peculiar algebraic structure of these representations quantum algebras at roots of unity have found interesting applications, especially in determining knot invariants [21] and in the quantum Hall effect [13]. Unlike the case of real q theory of the dual space G_q at roots of unity is not well established :

(i) what is the structure of the quantum group G_q at roots of unity ?

(ii) what are the q–special functions related to G_q at roots of unity?

(iii) are there invariants (integral, distance) on G_q at roots of unity?

For the quantum group $SL_q(2,C)$ at roots of unity some aspects of this programm was developed in the series of papers [2, 9, 16]. Quantum groups at roots of unity appear to be a natural generalization of the usual supersymmetry to the fractional one (FSUSY) which replaces the Z_2–grading of the SUSY algebra with a Z_p–graded algebra in such a way that the FSUSY transformation mix elements of all grades [3] (see also [12] and references therein).

The purpose of this paper is to solve the problems (i), (ii) and (iii) for the 2–dim quantum Poincaré group $E_q(1,1)$ at $q^p = 1$. This group is the Z_p–graded product of the p^3–dimensional invariant $E(1,1 \mid p)$ and translation R^2 subgroups. We define the invariant integral on $E_q(1,1)$ and demonstrate that all the methods of representation theory available at generic q can be extended on this group.

In Section 2 we define the quantum Poincaré group $E_q(1,1)$ at roots of unity, its homogeneous spaces $E(1,1 \mid p)$, $SO(1,1 \mid p)$, $M^{(1,1)}$, $E_q^{(1,1)}$ and the dual space $U_q(e(1,1))$. Section 3 is devoted to the construction of the invariant integral on $E_q(1,1)$ and its invariant discrete subgroup $E(1,1 \mid p)$. The irreducible $*$–representation of $U_q(e(1,1))$ constructed in Section 4 are integrated to the pseudo–unitary irreducible representations of $E_q(1,1)$ by means of the universal T–matrix in Section 5. The matrix elements of these representations and some of their properties are investigated in Section 5 also. In Section 6 we realize the quantum algebra $U_q(e(1,1))$ on the quantum plane $E_q^{(1,1)}$ and obtain the eigenfunctions of the complete set of commuting elements of $U_q(e(1,1))$ in the angular momentum and momentum basis.

2. 2–Dim Quantum Poincaré Group $E_q(1,1)$ at Roots of Unity

Let us start by reviewing the principal facts of the 2–dimensional complex quantum Euclidean group $E_q(2,C)$ and its dual $U_q(e(2,C))$ [4].

The quantum group $E_q(2,C)$ is the Hopf algebra $A(E_q(2,C))$ generated by $\eta_\pm$ and $\delta^{\mp 1}$ satisfying the relations

$$\eta_-\eta_+ = q^2\eta_+\eta_-, \quad \eta_\pm\delta = q^2\delta\eta_\pm \tag{1}$$

and

$$\begin{gathered}\Delta(\eta_\pm) = \eta_\pm \otimes 1_A + \delta^{\pm 1} \otimes \eta_\pm, \qquad \Delta(\delta) = \delta \otimes \delta, \\ \varepsilon(\delta^{\pm 1}) = 1, \quad \varepsilon(z_\pm) = 0, \quad S(\delta^{\pm 1}) = \delta^{\mp 1}, \quad S(\eta_\pm) = -\delta^{\mp 1}\eta_\pm.\end{gathered} \tag{2}$$

The quantum algebra $U_q(e(2,C))$ is the Hopf algebra generated by $p_\pm$ and $\kappa^{\pm 1}$ satisfying the relations

$$p_+p_- = p_-p_+, \qquad p_\pm\kappa = q^{\mp 1}\kappa p_\pm \tag{3}$$

and

$$\begin{gathered}\Delta(p_\pm) = p_\pm \otimes \kappa + \kappa^{-1} \otimes p_\pm, \qquad \Delta(\kappa) = \kappa \otimes \kappa, \\ \varepsilon(p_\pm) = 0, \quad \varepsilon(\kappa^{\pm 1}) = 1, \quad S(p_\pm) = -q^{\pm 1}p_\pm, \quad S(\kappa^{\pm 1}) = \kappa^{\mp 1}.\end{gathered} \tag{4}$$

The duality pairings between $A(E_q(2,C))$ and $U_q(e(2,C))$ are given by

$$\langle \kappa^j, \delta^{j'} \rangle = q^{jj'}, \qquad j,\ j' \in Z \tag{5}$$

and

$$\langle p_\pm^n, \eta_\pm^m \rangle = i^n q^{\pm \frac{n}{2}} [n]! \delta_{nm}, \qquad n,\ m \in N, \tag{6}$$

where

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad [n]! = [1][2]\cdots[n].$$

Since Δ is a homomorphism (2) implies that

$$\Delta(\eta_\pm^n) = \sum_{m=0}^{n} \left[\begin{array}{c} n \\ m \end{array} \right]_\pm \eta_\pm^{n-m} \delta^{\pm m} \otimes \eta_\pm^m, \tag{7}$$

where

$$\left[\begin{array}{c} n \\ m \end{array} \right]_\pm = q^{\pm m(m-n)} \frac{[n]!}{[n-m]![m]!}.$$

The Hopf algebra $A(E_q(2,C))$ has two real forms $A(E_q(2))$ and $A(E_q(1,1))$ defined by the involutions

$$\delta^* = \delta^{-1}, \qquad \eta_\pm^* = \eta_\mp \quad \text{for} \quad q \in R$$

and

$$\delta^* = \delta, \qquad \eta_\pm^* = \eta_\pm \quad \text{for} \quad |\, q \,| = 1 \tag{8}$$

respectively. The 2–dimensional quantum Euclidean group $E_q(2)$ which is the $*$–Hopf algebra $A(E_q(2))$ was treated in detail in [25, 28, 5]. $A(E_q(1,1))$ is the 2–dimensional quantum Poincaré group $E_q(1,1)$. The Hopf algebra $U_q(e(2,C))$ has two real forms $U_q(e(2))$ and $U_q(e(1,1))$ defined by the involutions

$$p_\pm^* = p_\mp, \qquad \kappa^* = \kappa \quad \text{for} \quad q \in R$$

and

$$p_\pm^* = p_\pm, \qquad \kappa^* = \kappa \quad \text{for} \quad |\, q \,| = 1 \tag{9}$$

respectively.

For future convenience we would like to introduce the convolution product $\diamond$. Let $\xi : A \to V$ be the homomorphic map of a Hopf algebra A to a linear space V. We set

$$\xi \diamond f = (id \otimes \xi)\Delta(f), \quad f \diamond \xi = (\xi \otimes id)\Delta(f), \quad \xi \diamond \xi = (\xi \otimes \xi)\Delta.$$

Clearly $\xi \diamond f$ and $f \diamond \xi$ belong to $A \otimes V$ and $V \otimes A$ respectively; $\xi \diamond \xi$ is homomorphic map of $A \otimes A$ into $V \otimes V$.

When q is a root of unity $q^p = 1$ (we deal with odd p) the duality relations (5) and (6) become degenerate. To get rid of these degeneracies we have to redefine the $*$–Hopf algebras $A(E_q(1,1))$ and $U_q(e(1,1))$.

To remove the degeneracy in (5) we put

$$\delta^p = 1_A \tag{10}$$

and

$$\kappa^p = 1_U. \tag{11}$$

Instead of (5) we then have

$$\langle \kappa^n, \zeta(m) \rangle = \delta_{nm}, \qquad n,\ m \in [0, p-1], \tag{12}$$

where

$$\zeta(m) = \frac{1}{p} \sum_{n=0}^{p-1} q^{-nm} \delta^n, \quad m \in [0,\ p-1],$$

which satisfies the periodicity property $\zeta(m + pj) = \zeta(m)$, $j \in Z$.

To remove the degeneracy in (6) we put

$$\eta_\pm^p = 0 \tag{13}$$

such that new variables $z_\pm$

$$z_\pm = \lim_{q^p=1} (-1)^{\frac{p+1}{2}} \frac{\eta_\pm^p}{[p]!} \tag{14}$$

are well defined. The above limiting process stems from the work De Concini, Kac and collaborators, and Lusztig which also appears in two recent monographs [7], [15], from which it can be traced back to the original references. The expression (6) now reads

$$\langle p_\pm^n, \eta_\pm^m \rangle = i^n q^{\pm \frac{n}{2}} [n]! \delta_{nm}, \quad n,\ m \in [0,\ p-1] \tag{15}$$

and

$$\langle P_\pm^n, z_\pm^m \rangle = i^n n! \delta_{nm}, \quad n,\ m \in N, \tag{16}$$

where $P_\pm = p_\pm^p$. Inspecting (1) and (14) we conclude that the new variables $z_\pm$ commute with $\eta_\pm$ and δ. By the virtue of (7) and (14) we obtain

$$\Delta(z_\pm) = z_\pm \otimes 1_A + 1_A \otimes z_\pm + (-1)^{\frac{p+1}{2}} \sum_{n=1}^{p-1} \frac{q^{\pm n^2}}{[p-n]![n]!} \eta_\pm^{p-n} \delta^{\pm n} \otimes \eta_\pm^n.$$

Moreover, we have

$$S(z_\pm) = -z_\pm, \quad \varepsilon(z_\pm) = 0, \quad z_\pm^* = z_\pm.$$

At this point we would like to introduce the short hand notation

$$\Delta(z) = Z + B,$$

where $z = (z_+,\ z_-)$, $Z = (Z_+,\ Z_-)$, $B = (B_+,\ B_-)$ and

$$Z_\pm = z_\pm \otimes 1_A + 1_A \otimes z_\pm, \qquad B_\pm = (-1)^{\frac{p+1}{2}} \sum_{n=1}^{p-1} \frac{q^{\pm n^2}}{[p-n]![n]!} \eta_\pm^{p-n} \delta^{\pm n} \otimes \eta_\pm^n.$$

Since $B_\pm^2 = 0$ for any function f from the space $C^\infty(R^2)$ of all infinitely differentiable functions on R^2 we have

$$\Delta(f(z)) = f(Z) + \frac{df}{dz_+} |_{z=Z}\ B_+ + \frac{df}{dz_-} |_{z=Z}\ B_- + \frac{d^2 f}{dz_+ dz_-} |_{z=Z}\ B_+ B_-. \tag{17}$$

We can also define the antipode, counite and involution on $C^\infty(R^2)$. They are given by

$$S(f(z)) = f(-z), \quad \varepsilon(f(z)) = f(0), \quad (f(z))^* = \overline{f(z)}, \tag{18}$$

where the bar denotes the usual complex conjugation.

Let $A(E(1,1 \mid p))$ be the space of polynomials of $\eta_\pm$ and δ. The restrictions (10), (13) together with (1),(2) and (8) imply that it is finite $*$–Hopf algebra with dimension p^3. We call it reduced quantum Poincaré group and denote by $E(1,1 \mid p)$.

Definition 1 *Quantum Poincaré group $E_q(1,1)$ at roots of unity is the $*-$ algebra $A(E_q(1,1)) = A(E(1,1 \mid p)) \times C^\infty(R^2)$ with a Hopf algebra structure given by (2), (17) and (18).*

Let us define the homomorphism $\xi_C : A(E_q(1,1)) \to C^\infty(R^2)$, such that

$$\xi_C(\eta_\pm) = 0, \quad \xi_C(\delta) = 1, \quad \xi_C(z_\pm) = z_\pm.$$

From (17) we get

$$\xi_C \diamond \xi_C(f(z)) = f(Z). \tag{19}$$

The operations (18) and (19) define a Hopf algebra structure on $C^\infty(R^2)$. The transformation law

$$\xi_C \diamond \xi_C(z_\pm) = z_\pm \otimes 1 + 1 \otimes z_\pm$$

implies that the $*$–Hopf algebra $C^\infty(R^2)$ is the space of all infinitely differentiable functions on the translation group R^2. The quantum Poincaré group $E_q(1,1)$ at roots of unity contains the invariant discrete $E(1,1 \mid p)$ and translation R^2 subgroups. Using the group multiplication law (17) and analogies with the supersymmetry theory we call $E_q(1,1)$ Z_p–graded product of $E(1,1 \mid p)$ and R^2.

The quantum group $E(1,1 \mid p)$ contains p-dimensional invariant subgroup $SO(1,1 \mid p)$, which is the $*$–Hopf algebra $A(SO(1,1 \mid p))$ of polynomials of δ subject to the restriction (10). The right sided coset $M^{(1,1)} = E(1,1 \mid p)/SO(1,1 \mid p)$ is the subspace $A(M^{(1,1)})$ of $A(E(1,1 \mid p))$ defined as

$$A(M^{(1,1)}) = \{a \in A(E(1,1 \mid p)) : \quad \xi_S \diamond a = a \otimes 1\},$$

where ξ_S be the homomorphic map of $A(E(1,1 \mid p))$ into $A(SO(1,1 \mid p))$, such that

$$\xi_S(\eta_\pm) = 0, \quad \xi_S(\delta) = \delta.$$

One can show that

$$\xi_S \diamond \eta_+^n \eta_-^m \delta^k = \eta_+^n \eta_-^m \delta^k \otimes \delta^k$$

which implies that $\eta_+^n \eta_-^m$, $n,\ m \in [0, p-1]$, form a basis of $A(E_p^{(1,1)})$. The elements

$$e_{nm}^{\pm} = \frac{\eta_+^{p-1-n} \eta_-^{p-1-m} \pm \eta_+^n \eta_-^m}{\sqrt{q^{2n+1} + q^{-2n-1}}}, \qquad n, m \in [0, p-1] \tag{20}$$

also form a basis in $M^{(1,1)})$ which are independent in the range

$$n \in [0, n_0 - 1], \quad m \in [0, 2n_0] \quad \text{and} \quad n = n_0, \quad m \in [0, n_0],$$

where $p = 2n_0 + 1$. The number of independent vectors e_{nm}^+ and e_{nm}^- are $\frac{p^2+1}{2}$ and $\frac{p^2-1}{2}$ respectively.

The quantum plane $E_q^{(1,1)} = E_q(1,1)/SO(1,1 \mid p)$ is the subspace $A(E_q^{(1,1)})$ of $A(E_q(1,1))$ defined as

$$A(E_q^{(1,1)}) = A(M_p^{(1,1)}) \times C^\infty(R^2).$$

Definition 2 *The quantum algebra $U_q(e(1,1))$ at roots of unity is the $*$–Hopf algebra generated by $p_\pm$ and κ subject to condition (11). The monomials*

$$P_+^t P_-^s p_+^n p_-^m \kappa^k, \quad n,\ m,\ k \in [0,\ p-1], \quad t,\ s \in N, \tag{21}$$

where $P_\pm = p_\pm^p$, form a basis of $U_q(e(1,1))$. The $*$–Hopf algebra structure of $U_q(e(1,1))$ is given by (3), (4), (9) and

$$\Delta(P_\pm) = P_\pm \otimes 1 + 1 \otimes P_\pm, \quad S(P_\pm) = -P_\pm, \quad \varepsilon(P_\pm) = 0, \quad P_\pm^* = P_\pm.$$

The $*$–Hopf algebra $U(r^2)$ generated by $P_\pm$ forms the invariant $*$–sub–Hopf algebra of $U_q(e(1,1))$, which is dual to the Hopf algebra $C^\infty(R^2)$. More precisely due to the Schwartz theorem $U(r^2)$ is isomorphic to the subspace of distributions on $C^\infty(R^2)$ with support at the unit element $(0,\ 0) \in R^2$.

The homomorphism $\xi'_C : U_q(e(1,1)) \to U(e(1,1 \mid p))$ given by

$$\xi'_C(p_\pm) = p_\pm, \quad \xi'_C(\kappa) = \kappa, \quad \xi'_C(P_\pm) = 0$$

defines another sub–Hopf algebra of $U_q(e(1,1))$, which is generated by the elements $p_\pm$ and κ subject to the conditions

$$p_\pm^p = 0, \quad \kappa^p = 1_U.$$

$U(e(1,1 \mid p))$ is in non–degenerate duality with $A(E(1,1 \mid p))$.

3. Invariant Integral on $E_q(1,1)$

Theorem 1 *The linear functional $\mathcal{I}$ on $A(E(1,1 \mid p))$ such that*

$$\mathcal{I}(\eta_+^n \eta_-^m \delta^k) = q^{-1}\delta_{n,p-1}\delta_{m,p-1}\delta_{k,0(\mathrm{mod}\ p)}$$

defines the unique invariant integral on the reduced quantum Poincaré group $E(1,1 \mid p)$.

Proof. Let us find the linear functional $\mathcal{I}'$ on $A(E(1,1 \mid p))$ which for any element a from $A(E(1,1 \mid p))$ satisfies the left

$$\mathcal{I}' \diamond a = \mathcal{I}'(a)1_A$$

and right

$$a \diamond \mathcal{I}' = \mathcal{I}'(a)1_A$$

invariance conditions. By the virtue of (7) for $a = \eta_+^n \eta_-^m \delta^k$ the left invariance condition reads

$$\sum_{t,s=0}^{n,m} \begin{bmatrix} n \\ t \end{bmatrix}_+ \begin{bmatrix} m \\ s \end{bmatrix}_- q^{2t(s-m)} \eta_+^{n-t} \eta_-^{m-s} \delta^{k-s+t} \mathcal{I}'(\eta_+^t \eta_-^s \delta^k) = 1_A \mathcal{I}'(\eta_+^n \eta_-^m \delta^k)$$

which implies

$$\mathcal{I}' \diamond (\eta_+^t \eta_-^s \delta^k) = 0 \quad \text{for} \quad t \in [0, n-1],\ s \in [0, m-1]$$

and

$$k + n - m = 0(\mathrm{mod}\ p). \tag{22}$$

If $n,\ m \in [0, p-2]$ we can employ the above reasoning for the element $a = \eta_+^{n+1}\eta_-^{m+1}\delta^k$ and obtain

$$\mathcal{I}'(\eta_+^n \eta_-^m \delta^k) = 0 \quad \text{for} \quad n,\ m \in [0, p-2]. \tag{23}$$

(22) and (23) imply that the linear functional $\mathcal{I}'$ satisfies the left invariance condition if

$$\mathcal{I}'(\eta_+^n \eta_-^m \delta^k) = \omega\delta_{n,p-1}\delta_{m,p-1}\delta_{k,0(\mathrm{mod}\ p)},$$

where ω is an arbitrary complex number. In a similar fashion one can show that the right invariance implies the same condition on $\mathcal{I}'$. Thus every linear functional on $A(E(1,1 \mid p))$ satisfying the left and right invariance conditions is proportional to $\mathcal{I}$. $\square$

Define the bilinear form $(\cdot,\cdot)_p$ on $E(1,1\mid p)$ by

$$(a,b)=\mathcal{I}(ab^*). \tag{24}$$

Because of the property

$$\mathcal{I}(a^*)=\overline{\mathcal{I}(a)}$$

this bilinear form is Hermitian. The vectors $e^{\pm}_{nm}$ spanning the basis of the coset space $A(M^{(1,1)})$ are orthonormal with respect to the above form

$$(e^{\pm}_{nm},e^{\pm}_{n'm'})=\pm\delta_{nn'}\delta_{mm'},\quad (e^{\pm}_{nm},e^{\mp}_{n'm'})=0. \tag{25}$$

Thus $A(M^{(1,1)}_p)$ equipped with the Hermitian form (24) is the pseudo–Euclidean space with $\frac{p^2+1}{2}$ positive and $\frac{p^2-1}{2}$ negative signatures.

Let $\mathcal{I}_C$ be the linear functional on the space $C^\infty(R^2)$ of all infinitely differentiable functions with finite support in R^2 given by

$$\mathcal{I}_C(f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dz_+dz_-f(z_+,z_-) \tag{26}$$

and let $A_0(E_q(1,1))$ be the subspaces

$$C_0^\infty(R^2)\times A(E(1,1\mid p))$$

of $A(E_q(1,1))$ whose any element F is the finite sum

$$F=\sum_n a_nf_n,$$

where $f_n\in C_0^\infty(R^2)$ and $a_n\in A(E(1,1\mid p))$. It is clear that $\mathcal{I}_C$ is the invariant integral on the translation group satisfying the properties

$$(\mathcal{I}_C\otimes id)(\xi_C\diamond\xi_C)(f)=\zeta(f),\quad (id\otimes\mathcal{I}_C)(\xi_C\diamond\xi_C)(f)=\zeta(f) \tag{27}$$

for any $f\in C_0^\infty(R^2)$.

Theorem 2 *The linear functional $\mathcal{I}_E$ on $A_0(E_q(1,1))$ given by*

$$\mathcal{I}_E(F)=\sum_n\mathcal{I}(a_n)\mathcal{I}_C(f_n)$$

defines the unique invariant integral on the quantum Poincaré group $E_q(1,1)$.

Proof. By the virtue of (17) and (19) for $G=af$ we have

$$\begin{aligned}\mathcal{I}_E\diamond G &= (id\otimes\mathcal{I}_E)[\Delta(a)\{(\xi_C\diamond\xi_C)(f)+B_+(\xi_C\diamond\xi_C)(\frac{df}{dz_+})\\ &\quad +B_-(\xi_C\diamond\xi_C)(\frac{df}{dz_-})+B_+B_-(\xi_C\diamond\xi_C)(\frac{d^2f}{dz_+dz_-})\}].\end{aligned}$$

By making use of (27) we get

$$\begin{aligned}\mathcal{I}_E \diamond G &= 1_A \mathcal{I}(a)\mathcal{I}_C(f) + (id \otimes \mathcal{I})[\Delta(a)\{B_+\mathcal{I}_C(\frac{df}{dz_+}) + B_-\mathcal{I}_C(\frac{df}{dz_-})\}] \\ &\quad + (id \otimes \mathcal{I})(\Delta(a)B_+B_-)\mathcal{I}_C(\frac{d^2 f}{dz_+ dz_-}).\end{aligned}$$

Using the properties

$$\zeta_C(\frac{df}{dz_\pm}) = 0, \quad \zeta_C(\frac{d^2 f}{dz_+ dz_-}) = 0$$

satisfied by the functions $f \in C_0^\infty(R^2)$ we arrive at

$$\mathcal{I}_E \diamond G = 1_A \mathcal{I}(a)\mathcal{I}_C(f) = 1_A \mathcal{I}_E(G),$$

which together with the linearity of the functional $\mathcal{I}_E$ implies

$$\mathcal{I}_E \diamond F = 1_A \mathcal{I}_E(F)$$

for any $F \in A_0(E_q(1,1))$. We have proved the left invariance condition. In a similar fashion one can prove the right invariance condition. The uniqueness of the invariant integral $\mathcal{I}_E$ follows from the uniqueness of the invariant integrals $\mathcal{I}$ and $\mathcal{I}_C$. $\square$

By means of the invariant integral we define in $E_q(1,1)$ the bilinear form by

$$(F, G)_E = \mathcal{I}_E(FG^*), \tag{28}$$

where F, $G \in A_0(E_q(1,1))$. Because of the property

$$\mathcal{I}_E(F^*) = \overline{\mathcal{I}_E(F)}$$

this bilinear form is Hermitian.

Let $A_0(E_q^{(1,1)})$ be the subspace

$$C_0^\infty(R^2) \times A(M^{(1,1)}).$$

of $A(E_q^{(1,1)})$ whose any element X is the finite sum

$$X = \sum_{nm} f_{nm}^+ e_{nm}^+ + \sum_{nm} f_{nm}^- e_{nm}^-,$$

where $e_{nm}^\pm$ form a basis of $A(M^{(1,1)})$ and $f_{nm} \in C_0^\infty(R^2)$. By the virtue of (25) we get

$$(X, X)_E = \sum_{nm} \mathcal{I}_C(f_{nm}^+ \overline{f_{nm}^+}) - \sum_{nm} \mathcal{I}_C(f_{nm}^- \overline{f_{nm}^-}), \tag{29}$$

which implies that $A_0(E_q^{(1,1)})$ equipped with the Hermitian form (28) is the pseudo–Euclidean space.

4. Irreducible $*$–Representations of $U_q(e(1,1))$

The homomorphism $\mathcal{L}^\lambda : U_q(e(1,1)) \to \mathrm{Lin}\ A(SO(1,1 \mid p))$ given by

$$\mathcal{L}^\lambda(p_\pm)\delta^m = \lambda_\pm \delta^{m\pm 1}, \quad \mathcal{L}^\lambda(\kappa)\delta^m = q^m \delta^m \tag{30}$$

for $\lambda = (\lambda_+,\ \lambda_-) \neq (0,\ 0)$ defines p–dimensional irreducible representation of the quantum algebra $U_q(e(1,1))$ in the linear space $A(SO(1,1 \mid p))$. Since $\delta^p = 1_A$ for any $a \in A(SO(1,1 \mid p))$ we have

$$\mathcal{L}^\lambda(P_\pm)a = \lambda_\pm^p a$$

This representation is cyclic. For $\lambda = (0,\ 0)$ we have one dimensional representation

$$\mathcal{L}^{(m)}(p_\pm)\delta^m = 0, \quad \mathcal{L}^{(m)}(\kappa)\delta^m = q^m \delta^m \tag{31}$$

with the weight $m \in [0,\ p-1]$. The homomorphisms $\mathcal{L}^\lambda$ and $\mathcal{L}^{(m)}$ exhaust all irreducible representations of the quantum algebra $U_q(e(1,1))$. This is rather trivial consequence of the general theory presented in [11], to which we refer for proof and details. Representations of the quantum algebra $U_q(e(1,1))$ is also considered in [8]. However the quantum algebra studied in [8] differs because there the restriction (10) is not considered.

Let us find out when the homomorphism $\mathcal{L}^\lambda$ defines $*$–representation of the quantum algebra $U_q(e(1,1))$, that is when for any $\phi \in U_q(e(1,1))$ we have

$$(\mathcal{L}^\lambda(\phi))^* = \mathcal{L}^\lambda(\phi^*) \tag{32}$$

For this purpose we define in $A(SO(1,1 \mid p))$ the Hermitian form

$$(a, b)_S = \mathcal{I}_S(a^* b), \tag{33}$$

where $\mathcal{I}_S$ is the invariant integral on $SO(1,1 \mid p)$ given by

$$\mathcal{I}_S(\delta^m) = \delta_{m,0(\mathrm{mod}\ p)}.$$

For n, $m \in [0, p-1]$ we have

$$(\delta^n, \delta^m)_S = \delta_{m+n,0} + \delta_{m+n,p}, \tag{34}$$

which implies that the vectors

$$e_m^\pm = \frac{1}{\sqrt{2}}(\delta^m \pm \delta^{p-m}), \qquad m \in [0, \frac{p-1}{2}]$$

are orthonormal with respect to the Hermitian form (33)

$$(e_m^\pm, e_k^\pm)_S = \pm\delta_{mk}, \quad (e_m^\mp, e_k^\pm)_S = 0.$$

The $*$–Hopf algebra $A(SO(1,1 \mid p))$ equipped with the Hermitian form (33) is pseudo–Euclidean space with $\frac{p+1}{2}$ positive and $\frac{p-1}{2}$ negative signatures.

The adjoint $(\mathcal{L}^\lambda(\phi))^*$ of the linear operator $\mathcal{L}^\lambda(\phi)$ is defined as

$$(\mathcal{L}^\lambda(\phi)a, b)_S = (a, (\mathcal{L}^\lambda(\phi))^* b)_S,$$

where $a,\ b$ are arbitrary elements from $A(SO(1,1 \mid p))$. Using the representation formula (30) and the involution (9) we conclude that when $\lambda_\pm$ are real numbers the homomorphism $\mathcal{L}^\lambda$ defines $*$–representation of the quantum algebra $U_q(e(1,1))$. The homomorphism $\mathcal{L}^{(m)}$ also defines $*$–representation of $U_q(e(1,1))$.

5. Pseudo–Unitary Irreducible Representations of $E_q(1,1)$

Let us briefly recall the construction and the main properties of universal $T-$ matrix [14]. Consider two Hopf algebras $A(G)$ and $U(g)$ in non–degenerate duality. Let $\{x_a\}$ and $\{X^b\}$ be dual linear basis of $A(G)$ and $U(g)$ respectively, with a and b running in an appropriate set of indices, so that $\langle x_a, X^b\rangle = \delta_{ab}$. We define the element $T \in U(g) \otimes A(G)$ as

$$T = \sum_a x_a \otimes X^a.$$

The universal T–matrix is a resolution of the identity which maps the Lie group G into itself. Moreover, if we choose the representation of $U(g)$ we correspondingly obtain the corepresentation of $A(G)$ or representation of G.

The elements $z_+^t z_-^s \eta_+^n \eta_-^m \zeta(k)$ and (21) defines the linear basis in $A(E_q(1,1))$ and $U_q(e(1,1))$ respectively. Introducing the cut off q–exponential

$$e_\pm^x = \sum_{m=0}^{p-1} \frac{q^{\pm\frac{m(m-1)}{2}}}{[m]!} x^m. \tag{35}$$

by the direct calculation we arrive at the following result.

Proposition 1 *We have the duality relations*

$$\begin{aligned}\langle P_+^t P_-^s p_+^n p_-^m \kappa^k, z_+^{t'} z_-^{s'} \eta_+^{n'} \eta_-^{m'} \zeta(k')\rangle \quad &= \quad i^{n+m+t+l} q^{\frac{n-m}{2} - nm} t! s! [n]! [m]! \\ &\quad\quad \delta_{nn'} \delta_{mm'} \delta_{tt'} \delta_{ll'} \delta_{k+t+l,k'},\end{aligned}$$

which implies that the universal T–matrix in $U_q(e(1,1)) \otimes A(E_q(1,1))$ has the form

$$T = e^{-iP_+\otimes z_+ - iP_-\otimes z_-} e_+^{i\epsilon_+\otimes\eta_+} e_-^{i\epsilon_-\otimes\eta_-} D(\kappa, \delta),$$

where

$$\epsilon_\pm = -q^{\mp\frac{1}{2}} p_\pm \kappa^{-1}$$

and

$$D(\kappa, \delta) = \frac{1}{p} \sum_{m,k=0}^{p-1} q^{-mk} \kappa^m \otimes \delta^k$$

The universal T–matrix satisfies the properties

$$[(*\otimes *)T]\cdot T = 1_U\otimes 1_A, \quad T\cdot[(*\otimes *)T] = 1_U\otimes 1_A \tag{36}$$

and

$$(id\otimes\Delta)T = (T\otimes 1_A)(id\otimes\sigma)(T\otimes 1_A), \tag{37}$$

where $\sigma(F\otimes G) = G\otimes F$, F, $G\in A(E_q(1,1))$ is the permutation operator.

Define the linear map $T^\lambda : A(SO(1,1\mid p))\to A(SO(1,1\mid p))\otimes A(E_q(1,1))$, such that

$$T^\lambda a = e^{-i\mathcal{L}^\lambda(P_+)\otimes z_+ - i\mathcal{L}^\lambda(P_-)\otimes z_-} e_+^{i\mathcal{L}^\lambda(\epsilon_+)\otimes\eta_+} e_-^{i\mathcal{L}^\lambda(\epsilon_-)\otimes\eta_-} D(\mathcal{L}^\lambda(\kappa),\delta)(a\otimes 1). \tag{38}$$

Due to (37) and the irreducibility of the representation $\mathcal{L}^\lambda$ we conclude that the above linear map defines p–dimensional irreducible representations of the quantum Poincaré group in the linear space $A(SO(1,1\mid p))$. Let us extend the Hermitian form (33) to the form $\{\cdot,\cdot\}_S$ by setting

$$\{a\otimes F, b\otimes G\}_S = F^* G(a,b)_S, \tag{39}$$

where F, $G\in A(E_q(1,1))$ and a, $b\in A(SO(1,1\mid p))$. When $\lambda_\pm$ are real numbers due to (36) we get

$$\{T^\lambda a, T^\lambda b\}_S = (a,b)_S 1_A. \tag{40}$$

Thus the irreducible representation T^λ of the quantum group $E_q(1,1)$ in the pseudo–Euclidean space $A(SO(1,1\mid p))$ is pseudo–unitary when $\lambda_\pm\in R$.

By the virtue of the representation formula (38) and the relation (34) we obtain the integral representation for the matrix elements of the irreducible pseudo–unitary representations T^λ

$$D^\lambda_{mn} = \{\delta^{p-m}\otimes 1_A, T^\lambda\delta^n\}_S. \tag{41}$$

After lengthily but straightforward calculations we have the following result.

Proposition 2 *The matrix elements of the pseudo–unitary irreducible representations of* $E_q(1,1)$ *are*

$$D^\lambda_{mn} = e^{-i\lambda_+^p z_+ - i\lambda_-^p z_-}[\sum_{k=0}^{p-1-n+m}\frac{(-\lambda^2)^k q^{-k(m+n)}}{[k]![k+n-m]!}\xi^k(-iq^{(\frac{1}{2}-n)}\lambda_-\eta_-)^{n-m}\delta^n$$

$$+(-iq^{(-\frac{1}{2}-n)}\lambda_+\eta_+)^{p+m-n}\delta^n\sum_{k=0}^{n-m}\frac{(-\lambda^2)^k q^{k(m+n)}}{[k]![k+p+m-n]!}\xi^k] \quad for \quad n\geq m$$

and

$$D^\lambda_{mn} = e^{-i\lambda_+^p z_+ - i\lambda_-^p z_-}[\sum_{k=0}^{m-n}\frac{(-\lambda^2)^k q^{-k(m+n)}}{[k]![k+p+n-m]!}\xi^k(-iq^{(\frac{1}{2}-n)}\lambda_-\eta_-)^{p+n-m}\delta^n$$

$$+(-iq^{(-\frac{1}{2}-n)}\lambda_+\eta_+)^{m-n}\delta^n\sum_{k=0}^{p-1-m+n}\frac{(-\lambda^2)^k q^{k(m+n)}}{[k]![k+m-n]!}\xi^k] \quad for \quad m\geq n,$$

where $\xi = q\eta_+\eta_-$ *and* $\lambda^2 = \lambda_+\lambda_-$.

For the special case D^λ_{m0} we have the explicit formula

$$D^\lambda_{m0} = e^{-i\lambda_+^p z_+ - i\lambda_-^p z_-}[\mathcal{J}_{p-m}(\lambda^2\xi)(-iq^{\frac{1}{2}}\lambda_-\eta_-)^{p-m} + (-iq^{-\frac{1}{2}}\lambda_+\eta_+)^m \mathcal{J}_m(\lambda^2\xi)], \quad (42)$$

where $m \in [0,\ p-1]$ and

$$\mathcal{J}_m(x) = \sum_{k=0}^{p-1-m} \frac{(-1)^k}{[k]![k+m]!}(q^m x)^k. \quad (43)$$

The pseudo–unitarity condition (40) implies

$$(D^\lambda_{0m})^* D^\lambda_{0n} + \sum_{k=1}^{p-1}(D^\lambda_{km})^* D^\lambda_{p-kn} = (\delta^m, \delta^n)_S 1_A. \quad (44)$$

Special cases are

$$(D^\lambda_{00})^* D^\lambda_{00} + \sum_{k=1}^{p-1}(D^\lambda_{k0})^* D^\lambda_{p-k0} = 1_A$$

and

$$(D^\lambda_{0s})^* D^\lambda_{0p-s} + \sum_{k=1}^{p-1}(D^\lambda_{ks})^* D^\lambda_{p-kp-s} = 1_A,$$

where $s \in [1, p-1]$. Moreover, we have the addition theorem

$$\Delta(D^\lambda_{nm}) = \sum_{k=0}^{p-1} D^\lambda_{nk} \otimes D^\lambda_{km}. \quad (45)$$

The pseudo–unitary representation $T^{(m)}$ of the quantum Poincaré group corresponding to the $*$–representation $\mathcal{L}^m$ is given by

$$T^{(m)}\delta^m = \delta^m \otimes \delta^m,$$

where $m \in [0,\ p-1]$.

Remarks. (i) Recall that the Hahn–Exton q–Bessel functions $J_m(x)$ related to the unitary irreducible representations of the quantum Euclidean group $E_q(2)$ are [18]

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]![k+m]!}(q^m x)^k.$$

Comparing (43) to the above expression we conclude that the matrix elements of the pseudo–unitary irreducible representations of the quantum Poincaré group are the cut off Hahn–Exton q–Bessel function.

(ii) Inspecting (38) we observe that irreducible representations of $E_q(1,1)$ are induced by the irreducible representations of the translation subgroup R^2.

(iii) The linear map $T^{(m)}$ defines the one dimensional pseudo–unitary representations of the invariant subgroup $SO(1,1 \mid p) \in E_q(1,1)$.

6. Quasi–Regular Representation

The comultiplication

$$\Delta : A_0(E_q^{(1,1)}) \to A_0(E_q(1,1)) \otimes A_0(E_q^{(1,1)}) \tag{46}$$

defines the left quasi–regular representation of the quantum Poincaré group $E_q(1,1)$ in the vector space $A_0(E_q^{(1,1)})$. Let us extend the Hermitian form (28) to the form $\{\cdot,\cdot\}_E$ by setting

$$\{F \otimes X, G \otimes Y\}_E = FG^*(X,Y)_E,$$

where X, $Y \in A_0(E_q^{(1,1)})$ and F, $G \in A_0(E_q(1,1))$. Since the Hermitian form $(\cdot,\cdot)_E$ is defined by means of the invariant integral we have

$$\{\Delta(X), \Delta(Y)\}_E = 1_A(X,Y)_E, \tag{47}$$

which implies that the left quasi–regular representation (46) is pseudo–unitary.

The right representation $\mathcal{R}$ of the quantum algebra $U_q(e(1,1))$ corresponding to the left quasi–regular representation (46) is given by

$$\mathcal{R}(\phi)F = F \diamond \phi.$$

We have

$$\mathcal{R}(p_\pm)\eta_\pm^k = iq^{\pm\frac{1}{2}}[k]\eta_\pm^{k-1}, \quad \mathcal{R}(p_\pm)\eta_\mp^k = 0, \quad \mathcal{R}(\kappa)\eta_\pm^k = q^{\pm k}\eta_\pm^k \tag{48}$$

and

$$\mathcal{R}(p_\pm)f = iq^{\pm\frac{1}{2}}\frac{(-1)^{\frac{p+1}{2}}}{[p-1]!}\eta_\pm^{p-1}\frac{df}{dz_\pm}, \quad \mathcal{R}(P_\pm)f = i\frac{df}{dz_\pm}, \quad \mathcal{R}(\kappa)f = f, \tag{49}$$

where $f \in C_0^\infty(R^2)$. Using the following relations satisfied by the right representation $\mathcal{R}$

$$\mathcal{R}(\phi\phi') = \mathcal{R}(\phi')\mathcal{R}(\phi),$$

$$\mathcal{R}(p_\pm)(XY) = \mathcal{R}(p_\pm)X\mathcal{R}(\kappa)Y + \mathcal{R}(\kappa^{-1})X\mathcal{R}(p_\pm)Y,$$
$$\mathcal{R}(\kappa)(XY) = \mathcal{R}(\kappa)X\mathcal{R}(\kappa)Y$$

we can define the action of an arbitrary operator $\mathcal{R}(\phi)$ on any function from $A_0(E_q^{(1,1)})$. Due to the identity

$$\overline{\langle \phi, F^* \rangle} = \langle (S(\phi))^*, F \rangle, \qquad F \in A_0(E_q(1,1))$$

and the pseudo–unitarity condition (47) for any $\phi \in U_q(e(1,1))$ we have

$$(\mathcal{R}(\phi)X, Y)_E = (X, \mathcal{R}(\phi^*)Y)_E$$

Thus the antihomomorphism $\mathcal{R} : U_q(e(1,1)) \to \mathrm{Lin}\ A_0(E_q^{(1,1)})$ defines $*$–representation of the quantum algebra $U_q(e(1,1))$ in the pseudo–Euclidean space $A_0(E_q^{(1,1)})$.

The quantum algebra $U_q(e(1,1))$ has three Casimir elements $P_\pm$ and p_+p_- with one restriction

$$P_+P_- = (p_-p_+)^p.$$

Therefore irreducible representations of $U_q(e(1,1))$ will be labelled by two indices. We construct the irreducible representations of the quantum algebra $U_q(e(1,1))$ in the pseudo–Euclidean space $A_0(E_q^{(1,1)})$ by diagonalizing the complete set of commuting elements of $U_q(e(1,1))$ in $A_0(E_q^{(1,1)})$.

(i) *The angular momentum states.* Choose the following complete set of observables : $\mathcal{R}(P_\pm)$, $\mathcal{R}(p_+p_-)$, $\mathcal{R}(\kappa)$. Inspecting (48) and (49) we observe that the functions

$$X = e^{-i\lambda_+^p z_+ - i\lambda_-^p z_-}[X_1(\xi)\eta_-^{p-m} + \eta_+^m X_2(\xi)],$$

with $X_1(\xi)$ and $X_2(\xi)$ being some polynomials, are eigenstates of the linear operators $\mathcal{R}(P_\pm)$ and $\mathcal{R}(\kappa)$ with eigenvalues $\lambda_\pm^p$ and q^m respectively. The eigenvalue equation

$$\mathcal{R}(p_+p_-)X = \lambda^2 X$$

is solved by

$$X = D^\lambda_{m0},$$

where $\lambda^2 = \lambda_+\lambda_-$ and D^λ_{m0} are the matrix elements (42). By direct calculations we arrive at the following results.

Proposition 3 *The right representation of $U_q(e(1,1))$ on the matrix elements D^λ_{m0} is given by*

$$\mathcal{R}(p_+)D^\lambda_{m0} = \lambda_+ D^\lambda_{m-1,0}, \qquad m \in [1, p-1],$$

$$\mathcal{R}(p_-)D^\lambda_{m0} = \lambda_- D^\lambda_{m+1,0}, \qquad m \in [0, p-2]$$

and

$$\mathcal{R}(p_+)D^\lambda_{00} = \lambda_+ D^\lambda_{p-10}, \qquad \mathcal{R}(p_-)D^\lambda_{p-1,0} = \lambda_- D^\lambda_{00}.$$

Proposition 4 *The matrix elements of the irreducible pseudo–unitary representation satisfy the orthogonality condition*

$$(D^\lambda_{n0}, D^{\lambda'}_{m0})_E = \Lambda\delta(\lambda_+ - \lambda'_+)\delta(\lambda_- - \lambda'_-)\delta_{n+m,0(\text{mod } p)},$$

where

$$\Lambda = \frac{2\pi}{p^2}\sum_{k=0}^{p-1}\frac{1}{([k]![p-1-k]!)^2}.$$

is the normalization constant.

(ii) *The Plane wave states*. We choose the following complete set of observables: $\mathcal{R}(P_\pm)$, $\mathcal{R}(p'_\pm)$, where

$$p'_+ = q^{-\frac{1}{2}} p_+ \kappa^{-1}, \qquad p'_- = q^{-\frac{1}{2}} p_- \kappa.$$

Due to the relation $P_\pm = -(p'_\pm)^p$ it is sufficient to solve the eigenvalue equations

$$\mathcal{R}(p'_\pm) Y = \chi_\pm Y. \tag{50}$$

Proposition 5 *The eigenfunctions of (50) are*

$$Y = e_+^{-i\chi_+\eta_+} e_+^{-iq\chi_-\eta_-} e^{i\chi_+^p z_+} e^{i\chi_-^p z_-},$$

where e_+^x is the cut off exponential (35).

Proof. Substituting

$$Y = e^{i\chi_+^p z_+} e^{i\chi_-^p z_-} Y_+(\eta_+) Y_-(\eta_-)$$

in (50) we get

$$[\mathcal{R}(p'_+) - q\frac{(-1)^{\frac{p+1}{2}}}{[p-1]!}\chi_+^p \eta_+^{p-1}] Y_+ = \chi_+ Y_+$$

and

$$[\mathcal{R}(p'_-) - \frac{(-1)^{\frac{p+1}{2}}}{[p-1]!}\chi_-^p \eta_-^{p-1}] Y_- = \chi_- Y_-,$$

which imply the desired result. □

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