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Quark and lepton masses from fundamental algebra with exact color symmetry*

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Abstract

We propose a deformed algebra of creation and annihilation operators relating quark-lepton states. A diagonal mass operator with three parameters together with the three deformation parameters of the algebra gives a reasonable fit to observed quark-lepton masses.

The concept of broken symmetries is a customary one in particle physics. A well known example is the $SU(3)$-flavour symmetry which is broken by the relatively high mass of the strange quark. Historically this led to the Gell-Mann-Okubo [2] mass formulas for hadrons. In this scheme the hadrons are classified according to representations of $SU(3)$-flavour. The mass operator, however, is taken not invariant under $SU(3)$-flavour which is broken in the direction of flavour hypercharge. This theory has evolved into the standard model where the elementary fermions are the $SU(3)$-color singlet leptons and $SU(3)$-color triplet quarks.

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A perusal of the Review of Particle Physics [3] reveals that the masses of quarks span a range of five orders of magnitude. If all quarks somehow formed a single symmetry multiplet, it is obvious that a Gell-Mann-Okubo type of approach would not work due to this large spread of masses. The view we will adopt in this paper is that a deformed symmetry approach will work. We assume that the symmetry is broken, albeit in the simplest way, in the algebra which describes the fundamental fermions. The algebra we will adopt for this purpose is a deformation of $\mathcal{F}^3 \otimes SU(2)$ where $\mathcal{F}^3$ is the 3-dimensional fermion algebra and $SU(2)$ is the familiar Lie algebra of angular momentum where the third component now describes the different families. We now discuss how this algebra arises.

As is popularly known a single family of quarks and leptons can be mapped onto the corners of a cube with sides: $\vec{R} = (0, 0, 1)$, $\vec{G} = (0, 1, 0)$, $\vec{B} = (1, 0, 0)$, where R, G, B stand for Red, Green and Blue. With these basis vectors, the first family fundamental fermions become:

	$e^- = \vec{R} + \vec{B} + \vec{G}$	
$\bar{u}_R = \vec{G} + \vec{B}$	$\bar{u}_G = \vec{R} + \vec{B}$	$\bar{u}_B = \vec{R} + \vec{G}$
$d_R = \vec{R}$	$d_G = \vec{G}$	$d_B = \vec{B}$
	$\bar{\nu}_e = \vec{0}$	

It should be noted that in this sequence, the electric charge increases as one goes from bottom to top.

With this as a motivation, one can try to associate creation/annihilation operators in the direction/reverse-direction of the basis vectors of the cube. These operators should, however, be fermionic so that they reproduce only and exactly the states we require. Thus one can write:

$$a_i a_j + a_j a_i = 0 \quad , \tag{1}$$

where i =Red, Green, Blue, which we will be referring to as $1, 2, 3$. The commutation relation between the a_i's and the $a_j^\dagger$'s is free except that the resulting algebra should be invariant under $SU(3)$-color acting on the indices i, j, k. One can try to choose it as the ordinary fermion algebra:

$$a_i a_j^\dagger + a_j^\dagger a_i = \delta ij \quad . \tag{2}$$

Then we have the states

States	Multiplicities
$a_i^\dagger a_j^\dagger a_k^\dagger \mid 0 \rangle = \mid ijk \rangle$	1
$a_i^\dagger a_j^\dagger \mid 0 \rangle = \mid ij \rangle$	3
$a_i^\dagger \mid 0 \rangle = \mid i \rangle$	3
$\mid 0 \rangle$	1

which are in one-to-one correspondence with the ones in the table above. Thus, every operator $a_i^\dagger$ increases the electric charge of a state by one-third of the electron charge. One can try to approximate lepton masses using these operators and using $N = \sum a_i^\dagger a_i$

as the mass operator. This, however, fails because the masses in the first family, in the order we have placed them, do not increase linearly.

Thus we are led to look for a suitable deformation of the simple 3-dimensional fermion algebra, $\mathcal{F}^3$. However, we want the $SU(3)$-color invariance of (1) and (2) to survive the deformation.

The three dimensional fermionic Newton oscillator algebra, which we will call $\mathcal{F}_q^3$, is given by:

$$a_i a_j^\dagger + q a_j^\dagger a_i = H\delta ij \quad , \tag{3}$$

$$a_i H = q H a_i \quad , \tag{4}$$

and

$$a_i a_j + a_j a_i = 0 \quad . \tag{5}$$

This algebra is invariant under the $SU(3)$-color Lie group by the foregoing arguments. The spectrum of the operator:

$$N_q = \sum a_i^\dagger a_i \tag{6}$$

on the states of this algebra is:

State	Eigenvalue of N_q
$\mid ijk\,\rangle$	$3q^2h$
$\mid ij\,\rangle$	$2qh$
$\mid i\,\rangle$	h
$\mid 0\,\rangle$	0

where h is defined as the eigenvalue of the operator H on the state $\mid 0\,\rangle$ and where $\mid ijk\,\rangle \sim a_i^\dagger a_j^\dagger a_k^\dagger \mid 0\,\rangle$.

To deform the $SU(2)$ whose third component describes the different families, we can consider a q-deformation of $SU(2)$. In order that the deformation be group-like, it should be a Hopf algebra [1]. The standard q-deformed $SL_q(2)$ algebra does not have an acceptable unitary (hermitian) structure since the antipode is not unitary, i.e. $S(b^\dagger) \neq (S(b))^\dagger$ where b and $b^\dagger$ are the usual ladder operators. The following deformation which is actually the $q_1 \to q_2$ limit of a two parameter deformation has all the desired properties:

$$bb^\dagger - sb^\dagger b = -Ms^{M-1} \quad , \tag{7}$$

$$bM = (M+1)b \tag{8}$$

The Hopf algebra structure is provided that the coproduct, the antipode and the counit are defined by

$$\Delta(b) = b \otimes s^{\frac{M}{2}} + s^{\frac{M}{2}} \otimes b \quad , \tag{9}$$

$$\Delta(b^\dagger) = b^\dagger \otimes s^{\frac{M}{2}} + s^{\frac{M}{2}} \otimes b^\dagger \quad , \tag{10}$$

$$\begin{aligned}
\Delta(M) &= M\otimes 1 + 1\otimes M \quad , & (11)\\
S(b) &= -s^{-M-\frac{1}{2}}b \quad , & (12)\\
S(b^\dagger) &= -b^\dagger s^{-M-\frac{1}{2}} \quad , & (13)\\
S(M) &= -M \quad , & (14)\\
\epsilon(b) &= \epsilon(b^\dagger) \quad = \quad \epsilon(M) \quad = \quad 0 \quad , & (15)
\end{aligned}$$

and it can be shown that Δ is an algebra homomorphism. We will denote the q-parameter of this algebra as s and call it $SU_s(2)$.

We will only be interested in the 3-dimensional representations of this algebra for the simple reason that there probably exists only three families. This representation is characterized by the three eigenvalues of the operator $b^\dagger b$ and M:

$$\begin{aligned}
M \mid m\,\rangle &= (m-1) \mid m\,\rangle & (16)\\
b^\dagger b \mid m\,\rangle &= \beta_m \mid m\,\rangle \quad m = 0,1,2 \quad . & (17)
\end{aligned}$$

Note that we have denoted the eigenvalues of M as $m-1$ so that the lowest value of m is 0. Of these three eigenvalues, the one corresponding to the lowest state should be zero and the remaining two should be determined in terms of the parameters of the algebra. The general equation for β_m in terms of the algebra parameters is

$$\beta_{m+1} - s\beta_m = (1-m)s^{m-2} \quad , \tag{18}$$

which has the solution

$$\beta_m = As^m + Bms^m + Cm^2 s^m \quad . \tag{19}$$

Here A, B and C depend on the dimension of the representation. For the three dimensional representation,

$$\beta_m = \frac{m(3-m)}{2}s^{m-3} \quad . \tag{20}$$

This gives for β_1 and β_2

$$\begin{aligned}
\beta_1 &= s^{-2} \quad , & (21)\\
\beta_2 &= s^{-1} \quad . & (22)
\end{aligned}$$

If we use the deformed number operator $N_q = \sum a_i^\dagger a_i$ of $\mathcal{F}_q^3$ as being proportional to the mass operator for the first family of fundamental fermions of ν_e, d, u and e, then a fit for the values of q for the known masses of d and e gives a result of $q \approx 0.17$ and $\frac{m_u}{m_d} \approx \frac{1}{3}$.

Obviously, the deformation gives us a valid mass spectrum while keeping the essential aspects of the model intact. The creation operator still raises the charge of a state by one-third of the electron charge. This algebra is limited, however, to only the first family of quarks and leptons. One should tensor multiply it with another algebra so that one can get the mass spectrum of the other families of quarks.

The proposed algebra to use for this purpose is the $SU_s(2)$ algebra with parameter s. Thus, $b^\dagger$ and b would be raising and lowering operators on the family number and M

would be the operator acting on the states to give us a function of the family number. Thus our states would be:

First Family	Second Family	Third Family
e $a_i^\dagger a_j^\dagger a_k^\dagger \mid 0 \rangle$	μ $a_i^\dagger a_j^\dagger a_k^\dagger b^\dagger \mid 0 \rangle$	τ $a_i^\dagger a_j^\dagger a_k^\dagger (b^\dagger)^2 \mid 0 \rangle$
u $a_i^\dagger a_j^\dagger \mid 0 \rangle$	c $a_i^\dagger a_j^\dagger b^\dagger \mid 0 \rangle$	t $a_i^\dagger a_j^\dagger (b^\dagger)^2 \mid 0 \rangle$
d $a_i^\dagger \mid 0 \rangle$	s $a_i^\dagger b^\dagger \mid 0 \rangle$	b $a_i^\dagger (b^\dagger)^2 \mid 0 \rangle$
ν_e $\mid 0 \rangle$	ν_μ $b^\dagger \mid 0 \rangle$	ν_τ $(b^\dagger)^2 \mid 0 \rangle$

Now, in order to get an increase in mass as one moves across the families, one should assume the commutation relations of a's and b's to be:

$$a_i b = r\ b a_i \tag{23}$$

$$a_i^\dagger b = r\ b a_i^\dagger \tag{24}$$

This way, the r-deformed product algebra will be able to explain the near equality of the ratios of $\frac{m_s}{m_d}$ and $\frac{m_b}{m_s}$.

One can again try to use $N_q = \sum a_i^\dagger a_i$ as the mass operator but looking at the quark masses one can see that this naive choice would fail to reflect the fact that $m_d < m_u$ but $m_c > m_s$. Thus, one should add more terms to the mass operator that would fit the observed spectrum of quark and lepton masses.

A mass operator one can write using the elements of the $\mathcal{F}_q^3 \otimes_r SU_s(2)$ algebra in which the first family masses and the $\frac{1}{3}$-charge quark masses are solely given by the first term is:

$$M = \mu_0 a_i^\dagger a_i + \frac{1}{2!}\mu_1\ a_i^\dagger a_j^\dagger\ b^\dagger b\ a_j a_i + \frac{1}{3!}\mu_2\ a_k^\dagger a_j^\dagger a_i^\dagger\ {b^\dagger}^2 b^2\ a_i a_j a_k \quad . \tag{25}$$

The spectrum of this operator on the states $\mid n, m \rangle \sim (a^\dagger)^n (b^\dagger)^m \mid 0, 0 \rangle$, where n stands for a string of (n_i, n_j, n_k) such that $n = n_i + n_j + n_k$ $(n_1, n_2, n_3 = 0, 1)$, and similarly $(a^\dagger)^n$ stands for such a string of $a_i^\dagger a_j^\dagger a_k^\dagger$, is given by:

$$\begin{aligned} M = \mu_0 h n q^{n-1} r^{-2m} + \mu_1 h^2 \frac{n(n-1)}{2} q^{2n-3} r^{-4m} \beta_m \\ + \mu_2 h^3 \frac{n(n-1)(n-2)}{6} q^{3n-6} r^{-6m} \beta_m \beta_{m-1} \end{aligned}$$

where $n = 0, 1, 2, 3$; $m = 0, 1, 2$. Neutrino masses, obtained with $n = 0$, vanish. We need to choose 6 parameters in order to calculate the remaining 9 mass values. We define these parameters to be as follows:

$$\begin{aligned}
m_d &= \mu_0 h \ , \\
p_1 &= \frac{\mu_1 h \beta_1}{\mu_0} \ , \\
p_2 &= \frac{\mu_2 h^2 {\beta_1}^2}{\mu_0} \ , \\
s &= \frac{\beta_2}{\beta_1} \ , \\
q &= \sqrt{\frac{m_e}{3m_d}} \ , \\
r &= \left(\frac{m_d}{m_b}\right)^{1/4} \ .
\end{aligned}$$

For a given value of the down quark mass m_d, q and r are adjusted to give the electron and bottom quark masses. The value for s is calculated from top quark mass. The parameters p_1, p_2 are determined using the known values of muon and tau lepton masses respectively:

$$\begin{aligned}
p_1 &= \frac{r^2}{q}\left(\frac{m_\mu r^2}{3m_d q^2} - 1\right) \ , \\
s &= \frac{2r^4}{p_1}\left(\frac{m_t r^4}{3m_d q} - 1\right) \ , \\
p_2 &= \frac{r^8}{qs}\left(\frac{m_\tau r^4}{m_d q^2} - 3 - \frac{3p_1 qs}{r^4}\right) \ .
\end{aligned}$$

The limits for m_d are given to be 3 - 9 MeV [3]. In Table 1, we present the nine mass values fitted, using $m_d = 6$ MeV. This table shows that we have been able to devise an algebraic scheme which fits the quark and lepton masses with reasonable accuracy in terms of the six parameters, m_d, q, r, s, p_1, p_2. We have used q-deformations of familiar algebras such that the symmetries $SU(3)$-color invariance acting on the indices i of the operators a_i and $U(1)$-electromagnetic invariance acting on the phases of the a_i remain exact. The ground state of the oscillators of our algebra is ν_e, the electron neutrino, state; hence a_i, $a_i^\dagger$, b, $b^\dagger$ are boson operators, speaking in the conventional sense. However, the a_i belong to an $SU(3)$-color triplet and obey fermion-like commutations. As unconventional as this might seem, we believe that such a scheme can be implemented in the context of quantum field theory or a generalization of such.

Table 1. Fitted mass values in MeV with $m_d = 6$ MeV, $p_1 = 1.509$, $p_2 = -5.78 \times 10^{-3}$, $s = 0.225$, $q = 0.168$, $r = 0.194$. The ranges specified in Review of Particle Physics[3] are given in parantheses.

m_e .511 (.511)	m_μ 105.7 (105.7)	m_τ 1777 (1777)
m_u 2.0 (1.5 - 5)	m_c 1135 (1100 - 1400)	m_t 173800 (173800 ± 5000)
m_d 6 (3 - 9)	m_s 160 (60 - 170)	m_b 4250 (4100 - 4400)

There are twelve fundamental fermions in the three families. Our mass formula is chosen such that the neutrino masses are zero. The six free parameters q, r, s, p_1, p_2, m_d are used to fit the nine nonvanishing quark-lepton masses. q, r and s are dimensionless deformation parameters which in a field theoretical model will be associated with coupling constants with formulas such as $q = e^{-Ce^2}$ where e is the coupling [9]. The resulting values of our fit indicates a relation which is satisfied $r = \sqrt{sq}$ with an accuracy of 0.2%. This, in terms of the couplings, results a relation of the form

$$e_r^2 = \frac{1}{2}(e_q^2 + e_s^2) \quad . \tag{26}$$

These speculations will hopefully yield a standard model of fundamental fermions.

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