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Product Integral Representations of Wilson Lines and Wilson loops and Non-Abelian Stokes Theorem*

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Abstract

We make use of product integrals to provide an unambiguous mathematical representation of Wilson line and Wilson loop operators. Then, drawing upon various properties of product integrals, we discuss such properties of Wilson lines and Wilson loops as approximating them with partial sums, their convergence, and their behavior under gauge transformations. We also obtain a surface product integral representation for the Wilson loop operator. The result can be interpreted as the non-abelian version of Stokes theorem.

1. Introduction

The notion of Wilson loop [1, 2] provides a systematic method of obtaining gauge invariant observables in gauge theories. Its applications range over such diverse fields as phenomenology and lattice gauge theories on the practical side and topological gauge theories [3] and string theory [4] on the purely theoretical side. The importance of the Wilson line as a parallel transport operator in the gauge independent formulation of gauge theories has been emphasized by Mandelstam [5], and further developed by Wu and Yang [6]. More recently, in the context of the AdS/CFT correspondence [7], an interesting connection between Wilson loops in supersymmetric gauge theories and membranes in supergravity theories has been suggested [8]. In view of all these developments, it is imperative that Wilson lines and Wilson loops be described within a well defined

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mathematical framework. The main purpose of this work is to provide such a representation by means of product integrals. This will permit us to give, among other things, two unambiguous proofs of the non-abelian version of Stokes theorem.

The product integral formalism has been used extensively in the theory of differential equations and of matrix valued functions [9]. In the latter context, it has a built-in feature for keeping track of the *order* of the matrix valued functions involved. As a result, product integrals are ideally suited for the description of *path ordered* quantities such as Wilson lines and Wilson loops. In fact, they make precise what one means by these concepts as well as what one means by the notion of path ordering in general. Moreover, since the theory of product integrals is well developed independently of particular applications, we can be confident that the properties of Wilson lines and Wilson loops which we establish using this method are correct and unambiguous.

Among the important advantages of the product integral representation of Wilson lines, one is the manner in which it deals with convergence issues. In the physics literature, the exponential of an operator such as a Wilson line is defined formally in terms of its power series expansion. In such a representation, it will be difficult, without any further elaboration, to establish whether the series converges and if so how. In contrast to this, it is a straight forward matter to establish the criteria for the convergence of the Wilson line in its product integral representation. This is because in such a framework the Banach space structure of the corresponding matrix valued functions is already built into the formalism.

Another important advantage of the product integral formulation of the Wilson loop is that, at least for orientable surfaces, it permits a 2-surface representation for it. Based on the central role of Stokes theorem in physics and in mathematics, it is not surprising that the non-abelian version of this theorem has attracted a good deal of attention in the physics literature [10]- [20]. The central features of the earlier attempts[10]- [16] have been reviewed and improved upon in a recent work [17]. Other recent works on non-abelian Stokes theorem [18, 19, 20] focus on specific problems such as confinement [19], zig-zag symmetry [20] suggested Polyakov [21], etc. With one exception [17], the authors of these works seem to have been unaware of a 1927 work in the mathematical literature by Schlessinger [22] which bear strongly on the content of this theorem. Modern non-abelian gauge theories did not exist at the time, and Schlesinger's work dealt with integrals of matrix valued functions and their ordering problems. Its relevance to Wilson lines and Wilson loops is tied to the fact that in non-abelian gauge theories, the connection and curvature are matrix valued functions. As a result, Schlesinger's work amounts to establishing the non-abelian Stokes theorem in two (target space) dimensions. By an appropriate extension and reinterpretation of his results, we show that the product integral approach to the proof of this theorem is valid in any target space dimension.

This work is organized as follows: To make this manuscript self- contained, we review in Section 2 the main features of product integration [9] and state without proof a number of theorems which will be used in the proof of the non-abelian Stokes theorem and other properties. In Section 3, we express Wilson lines and Wilson loops in terms of product integrals. In Section 4, we turn to the proof of the non-abelian Stokes theorem for orientable surfaces. In section 5, we give a variant of this proof. Section 6 is devoted to convergence issues for Wilson lines and Wilson loops. Section 7 deals with two observables from the Wilson loop operator. In Section 8, we study behavior of Wilson lines and Wilson loops under gauge transformations. This is another instance in which the significant advantage of the product integral representation of these operators becomes transparent.

2. Some properties of product integrals

The method of *product integration* has a long history, and its origin can be traced to the works of Volterra[9]. The justification for its name lies in the property that the product integral is to the product what the ordinary (additive) integral is to the sum. One of the most common applications of product integrals is to the solution of systems of linear differential equations. To see how this comes about, let us consider an evolution equation of the type

$$\mathbf{Y}'(\mathbf{s}) = A(s)\mathbf{Y}(\mathbf{x}), \quad \mathbf{Y}(\mathbf{s_0}) = \mathbf{Y_0}. \tag{1}$$

where s is a real parameter, and prime indicates differentiation. When the quantities Y and A are ordinary functions, and Y_0 is an ordinary number, the solution is given by an ordinary integral. On the other hand, if these quantities are matrix valued functions arising from a system of, say, n linear differential equations in n unknowns, then the solution will be a product integral.

To motivate a more precise formulation of product integration, we start with a simple example which exhibits its main features. Let us suppose that all the matrix valued functions that appear in the above equation are continuous on the real interval $[a, b]$. Then, given the value of $\mathbf{Y}$ at the point a, i.e., given $\mathbf{Y}(a)$, we want to find $\mathbf{Y}(b)$. One can obtain an approximate value for $\mathbf{Y}(b)$ using a variant of Euler's tangent-line method. Let $P = \{s_0, s_1, \ldots, s_n\}$ be a partition of the interval $[a, b]$, and let $\Delta s_k = s_k - s_{k-1}$ for all $k = 1, \ldots, n$. In the interval $[s_0, s_1]$, we approximate $A(s)$ by the constant value $A(s_1)$, solve the differential equation with initial value $\mathbf{Y}(a)$ and get the approximate solution for $\mathbf{Y}$ at s_1:

$$\mathbf{Y}(s_1) \approx e^{A(s_1)\Delta s_1}\mathbf{Y}(a).$$

In the next interval $[s_1, s_2]$, using the above approximate value as input, and replacing $A(s)$ by $A(s_2)$, one finds

$$\mathbf{Y}(s_2) \approx e^{A(s_2)\Delta s_2} e^{A(s_1)\Delta s_1}\mathbf{Y}(a).$$

Proceeding in this manner we obtain the following approximate value for $\mathbf{Y}(b)$:

$$\mathbf{Y}(b) \approx e^{A(s_n)\Delta s_n} \ldots e^{A(s_1)\Delta s_1}\mathbf{Y}(a) = \Pi_p(A)\mathbf{Y}(a), \tag{2}$$

where

$$\Pi_p(A) = \prod_{k=1}^{n} e^{A(s_k)\Delta s_k}.$$

We stress that the order of the exponentials on the right hand side of this equation is important since the corresponding matrices do not commute in general.

Since A is continuous on the compact interval $[a, b]$, it follows that A is uniformly continuous in that interval. This implies that for all $k = 1, \ldots, n$ the value $A(s_k)$ will be close to the values of $A(s)$ on $[s_{k-1}, s_k]$. It is thus reasonable to suppose that if the mesh $\mu(P)$ of the partition P (the length of the longest subinterval) is small, the above calculation results in a good approximation to $\mathbf{Y}(b)$. Then, we expect that the exact value of $\mathbf{Y}(b)$ is given by the natural limiting procedure

$$\mathbf{Y}(b) = \lim_{\mu(P)\to 0} \Pi_P(A)\mathbf{Y}(a) \equiv \prod_a^x e^{A(s)ds}\mathbf{Y}(a). \tag{3}$$

Having identified the main ingredients which characterize the above construction, we proceed to give a precise definition of the product integral [9]. We begin with the

Definition 1 *Let $A : [a, b] \to \mathbf{C}_{n\times n}$ be a function with values in the space of complex $n\times n$ matrices. Let $P = \{s_0, s_1, \ldots, s_n\}$ be a partition of the interval $[a, b]$, with $\Delta s_k = s_k - s_{k-1}$ for all $k = 1, \ldots, n$.*

(i) A is called a step function *iff there is a partition P such that A is constant on each open subinterval (s_{k-1}, s_k) for all $k = 1, \ldots, n$.*

(ii) The point value approximant *A_P corresponding to the function A and partition P is the step function taking the value $A(s_k)$ on the interval $(s_{k-1}, s_k]$ for all $k = 1, \ldots, n$.*

(iii) If A is a step function, then we define the function $E_A : [a, b] \to \mathbf{C}_{n\times n}$ by $E_A(x) := e^{A(s_k)(x-s_{k-1})} \ldots e^{A(s_2)\Delta s_2} e^{A(s_1)\Delta s_1}$ for any $x \in (s_{k-1}, s_k]$, for all $k = 1, \ldots, n$ and $E_A(a) := I$.

After a number of intermediate developments, one arrives at the following fundamental theorem which can be taken as the starting point of product integration:

Definition-Theorem 1 *Given a continuous function $A : [a, b] \to \mathbf{C}_{n\times n}$ and a sequence of step functions $\{A_n\}$, which converge to A in the sense of $L^1([a, b])$, then the sequence $\{E_{A_n}(x)\}$ converges uniformly on $[a, b]$ to a matrix called the* product integral *of A over $[a, b]$.*

More explicitly, we have:

$$\textit{The product integral of } A \textit{ over } [a, b] \quad = \quad \prod_a^b e^{A(s)ds}. \tag{4}$$

Now we are in position to enumerate some of the basic properties of product integrals. The proofs of these assertions are given in reference [9]. Let $A : [a, b] \to \mathbf{C}_{n\times n}$ be a continuous function, and for any $x \in [a,b]$ let

$$F(x,a) := \prod_a^x e^{A(s)ds} \tag{5}$$

denote the product integral from a to x. Then, F satisfies the following integral equation:

$$F(x,a) = 1 + \int_a^x ds\, A(s) F(s,a), \tag{6}$$

where $I = I_{n\times n}$ is the $n \times n$ unit matrix. The quantity F is also a solution of the following initial value problem:

$$\frac{d}{dx} F(x,a) = A(x) F(x,a), \quad F(a,a) = I. \tag{7}$$

Although product integrals can formally be defined for singular matrices, the above definition makes sense if they are non-singular. This is true, e.g., when the matrices form a group. Then the determinant of the product integral is given by the following theorem:

Theorem 1 *Given the continuous function $A : [a,b] \to \mathbf{C}_{n\times n}$, then for every $x \in [a,b]$, the product integral $\prod_a^x e^{A(s)ds}$ is non-singular and the following formula holds:*

$$\det\left(\prod_a^x e^{A(s)ds}\right) = e^{\int_a^x \mathrm{Tr\,A(s)ds}}, \tag{8}$$

where "Tr" stands for trace.

When the set of matrices $\{A(s) : s \in [a,b]\}$ is commutative, i.e. $[A(s), A(s')] = 0 \quad \forall s, s' \in [a,b]$, it is easy to show that

$$\prod_a^x e^{A(s)ds} = e^{\int_a^x A(s)ds}. \tag{9}$$

It is convenient to define the product integral $\prod_a^b e^{A(s)ds}$ also in the case when $a \geq b$. It will be recalled that for ordinary (additive) integrals $\int_a^b A(s)ds = -\int_b^a A(s)ds$. To obtain the analog of this for product integrals, we merely replace the "additive" property with the corresponding "multiplicative" property:

$$\prod_a^b e^{A(s)ds} := \left(\prod_b^a e^{A(s)ds}\right)^{-1}. \tag{10}$$

The additive property of ordinary integrals also provides a composition rule for them: $\int_a^c A(s)ds + \int_c^b A(s)ds = \int_a^b A(s)ds$. For product integrals, we have an analogous composition rule [9]:

$$\prod_a^b e^{A(s)ds} = \prod_c^b e^{A(s)ds} \prod_a^c e^{A(s)ds}. \tag{11}$$

Another well known property of ordinary integrals is the differentiation rule with respect to the endpoints: $\frac{\partial}{\partial b}\left(\int_a^b A(s)ds\right) = A(b)$ and $\frac{\partial}{\partial a}\left(\int_a^b A(s)ds\right) = -A(a)$. The following theorem gives the corresponding rule for product integrals.

Theorem 2 *Let $A : [a,b] \to \mathbf{C}_{n\times n}$ be continuous. For any $x, y \in [a,b]$ we have:*

$$\frac{\partial}{\partial x}\left(\prod_y^x e^{A(s)ds}\right) = A(x)\prod_y^x e^{A(s)ds}, \quad \frac{\partial}{\partial y}\left(\prod_y^x e^{A(s)ds}\right) = -\prod_y^x e^{A(s)ds}A(y). \tag{12}$$

It is important to keep in mind the relative order of the $A(x)$ and $A(y)$ with respect to the product integral.

The usual elementary way of computing ordinary integrals is by means of the fundamental theorem calculus: $\int_a^b f(x)ds = F(b) - F(a)$, where F is a primitive function of f ($F' = f$). To obtain the corresponding theorem for product integrals, we start by defining the so called L-operation which is a generalization of the logarithmic derivative, for non-singular functions:

Definition 2 *The L-derivative of a non-singular differentiable function $P : [a,b] \to \mathbf{C}_{n\times n}$ is given by:*

$$LP(x) := P'(x)P^{-1}(x). \tag{13}$$

To demonstrate the usefulness of this operation, let us consider the product integral $P(x) = \prod_a^x e^{A(s)ds}P(a)$. From Theorem 2, we have $P'(x) = A(x)\prod_a^x e^{A(s)ds}P(a) = A(x)P(x)$. Then, from the above definition, we get $(LP)(x) = A(x)$ (the derivative of the primitive function is the original function). We are thus led to the analog of the fundamental theorem of calculus for product integrals [9]:

Theorem 3 *For a non-singular and continuously differentiable ($C^1[a,b]$) function $P : [a,b] \to \mathbf{C}_{n\times n}$, we have*

$$\prod_a^x e^{(LP)(s)ds} = P(x)P^{-1}(a). \tag{14}$$

The following elementary properties of the L-operation follow from its definition:

$$(LP^{-1})(x) = (P^{-1})'(P^{-1})^{-1} = (-P^{-1}P'P^{-1})P = -P^{-1}(x)P'(x), \tag{15}$$

and

$$L(PQ)(x) = (P'Q + PQ')Q^{-1}P^{-1} = LP(x) + P(x)(LQ(x))P^{-1}(x). \tag{16}$$

We will rely heavily on the contents of the next three theorems in proving the non-abelian version of Stokes theorem. The proofs are given in reference [9].

Theorem 4 *(*Sum rule*): Given continuous functions $A, B : [a,b] \to \mathbf{C}_{n\times n}$, let $P(x) = \prod_a^x e^{A(s)ds}$. Then*

$$\prod_a^x e^{[A(s)+B(s)]ds} = P(x)\prod_a^x e^{P^{-1}(s)B(s)P(s)ds}. \tag{17}$$

Theorem 5 (Similarity rule): *Given a continuous function $B : [a,b] \to \mathbf{C}_{n\times n}$ and the non-singular function $P : [a,b] \to \mathbf{C}_{n\times n}$, then*

$$P(x)\left(\prod_a^x e^{B(s)ds}\right)P^{-1}(a) = \prod_a^x e^{[LP(s)+P(s)B(s)P^{-1}(s)]ds}. \tag{18}$$

Theorem 6 (Derivative with respect to a parameter): *Given a function $A : [a,b]\times[c,d] \to \mathbf{C}_{n\times n}$ such that $A(s,\lambda)$ is continuous in s for each fixed $\lambda \in [c,d]$. and is differentiable with respect to λ. Then, the product integral$P(x,y;\lambda) = \prod_y^x e^{A(s;\lambda)ds}$ is differentiable with respect to λ, and*

$$\frac{\partial}{\partial\lambda}P(x,y;\lambda) = \int_y^x ds\, P(x,s;\lambda)\frac{\partial A}{\partial\lambda}(s;\lambda)P(s,y;\lambda). \tag{19}$$

To put the above description in its proper perspective, we note that instead of the specific complex Banach space $L^1([a,b]$, product integrals can be defined over more general Banach spaces. Consider, e.g., a set $\mathcal{B}(X)$ of bounded linear operators over a complex Banach space, and let $A : [a,b] \to \mathcal{B}(X)$ be an operator valued function. It is possible to define the product integral of A and establish the analogs of the properties given above in this more general context. Then, the standard topologies (norm, strong, and weak) on the space of bounded linear operators play an important role. Moreover, the notion of Lebesgue integrable functions used on $L^1([a,b])$ space above generalize naturally to Boschner integrable functions [23]. For details we refer again to [9].

3. The representation of Wilson Line and Wilson Loop

As noted in the introduction, Wilson lines and Wilson loops are intimately related to the structure of non-abelian gauge theories. To provide the background for using the product integral formalism of Section 2 to explore their physical properties, we begin with the statement of the problem as it arises in the physics context. Let M be an n-dimensional manifold representing the space-time (target space). Let A be a (connection) 1-form on M. When M is a differentiable manifold, we can choose a local basis dx^μ, $\mu = 1,...,n$, and express A in terms of its components:

$$A(x) = A_\mu(x)\; dx^\mu.$$

We take A to have values in the Lie-algebra, or a representation thereof, of a Lie group. Then, with T_k, $k = 1,..,m$, representing the generators of the Lie group, the components of A can be written as

$$A_\mu(x) = A^k_\mu(x)\; T_k.$$

With these preliminaries, we can express the Wilson line of the non-abelian gauge theories in the form [24]

$$W_{ab}(C) = \mathcal{P}e^{\int_a^b A}.$$

where $\mathcal{P}$ indicates path ordering, and C is a path in M. When the path C is closed, the corresponding loop operator becomes [24]:

$$W(C) = \mathcal{P}e^{\oint A}. \tag{20}$$

The path C in M can be described in terms of an intrinsic parameter σ, so that for points of M which lie on the path C, $x^\mu = x^\mu(\sigma)$. One can then write

$$A_\mu(x(\sigma))dx^\mu = A(\sigma)d\sigma,$$

where

$$A(\sigma) \equiv A^\mu(x(\sigma))\frac{dx^\mu(\sigma)}{d\sigma}.$$

It is the quantity $A(\sigma)$, and the variations thereof, which we will identify with the matrix valued functions of the product integral formalism.

Let us next consider the loop operator. For simplicity, we assume that M has trivial first homology group with integer coefficients, i.e., $H_1(M, \mathbf{Z}) = 0$. This ensures that the loop may be taken to be the boundary of a two dimensional surface Σ in M. More explicitly, we take the 2-surface to be an orientable submanifold of M. It will be convenient to describe the properties of the 2-surface in terms of its intrinsic parameters σ and τ or σ^a, $a = 0, 1$. So, for the points of the manifold M, which lie on Σ, we have $x = x(\sigma, \tau)$. The components of the 1-form A on Σ can be obtained by means of the vielbeins (by the standard pull-back construction):

$$v_a^\mu = \partial_a\, x^\mu(\sigma).$$

Thus, we get

$$A_a = v_a^\mu\, A_\mu.$$

The curvature 2-form F of the connection A is given by

$$F = dA + A \wedge A = \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu.$$

The components of F on Σ can again be obtained by means of the vielbeins:

$$F_{ab} = v_a^\mu\, v_b^\nu\, F_{\mu\nu}.$$

We note at this point that we can construct the pulled-back field strength F_{ab} in another way, as the the field strength of the pulled-back connection A_a. It is easy to check that these two results coincide, ensuring the consistency of the construction.

We want to express the Wilson loop operator in terms of product integrals [25]. To achieve this, we begin with the definition of a Wilson line in terms of a product integral. Consider the continuous map $A : [s_0, s_1] \to \mathbf{C}_{n\times n}$ where $[s_0, s_1]$ is a real interval. Then, we define the Wilson line given above in terms of a product integral as follows:

$$\mathcal{P}e^{\int_{s_0}^{s_1} A(s)ds} \equiv \prod_{s_0}^{s_1} e^{A(s)ds}.$$

Anticipating that we will identify the closed path C over which the Wilson loop is defined with the boundary of a 2-surface, it is convenient to work from the beginning with the matrix valued functions $A(\sigma, \tau)$. This means that our expression for the Wilson line will depend on a parameter. That is, let

$$A : [\sigma_0, \sigma_1] \times [\tau_0, \tau_1] \to \mathbf{C}_{n\times n}, \tag{21}$$

where $[\sigma_0, \sigma_1]$ and $[\tau_0, \tau_1]$ are real intervals on the two surface Σ and hence in M. Then, we define a Wilson line

$$P(\sigma, \sigma_0; \tau) = \prod_{\sigma_0}^{\sigma} e^{A_1(\sigma';\tau)d\sigma'} \equiv \mathcal{P}e^{\int_{\sigma_0}^{\sigma} A_1(\sigma';\tau)d\sigma'}. \tag{22}$$

In this expression, $\mathcal{P}$ indicates path ordering with respect to σ, as defined by the product integral, while τ is a parameter. To be able to describe a Wilson loop, we similarly define the Wilson line

$$Q(\sigma; \tau, \tau_0) = \prod_{\tau_0}^{\tau} e^{A_0(\sigma;\tau')d\tau'} \equiv \mathcal{P}e^{\int_{\tau_0}^{\tau} A_0(\sigma;\tau')d\tau'}. \tag{23}$$

In this case, the path ordering is with respect to τ, and σ is a parameter.

To prove the non-abelian version of the Stokes theorem, we want to make use of product integration techniques to express the Wilson loop operator as an integral over a two dimensional surface bounded by the corresponding loop. In terms of the intrinsic coordinates of such a surface, we can write the Wilson loop operator in the form

$$W(C) = \mathcal{P}e^{\oint A_a d\sigma^a}, \tag{24}$$

where, as mentioned above,

$$\sigma^a = (\tau, \sigma)\,; \quad a = (0, 1). \tag{25}$$

The expression for the Wilson loop depends only on the homotopy class of paths in M to which the closed path C belongs. We can, therefore, parameterize the path C in any convenient manner consistent with its homotopy class. In particular, we can break up the path into piecewise continuous curves along which either σ or τ remains constant. The composition rule for product integrals given by Eq. (11) ensures that this break up of the Wilson loop into a number of Wilson lines does not depend on the intermediate points on the closed path, which are used for this purpose. So, inspired by the typical paths which are used in the actual computations of Wilson loops (see for example [8]), we write

$$W = W_4\, W_3\, W_2\, W_1, \tag{26}$$

In this expression, W_k, $k = 1, .., 4$, are Wilson lines such that $\tau = const.$ along W_1 and W_3, and $\sigma = const.$ along W_2 and W_4.

To see the advantage of parameterizing the closed path in this manner, consider the exponent of Eq. 24 :

$$A_a d\sigma^a = A_0 d\tau + A_1 d\sigma. \tag{27}$$

Along each segment, one or the other of the terms on the right hand side vanishes. For example, along the segment $[\sigma_0, \sigma]$, we have $\tau^{'} = \tau_0 = const.$. As a result, we get for the Wilson lines W_1 and W_2, respectively,

$$W_1 = \prod_{\sigma_0}^{\sigma} e^{A_1(\sigma';\tau_0)d\sigma'} \equiv \mathcal{P}e^{\int_{\sigma_0}^{\sigma} A_1(\sigma';\tau_0)d\sigma'} = P(\sigma, \sigma_0; \tau_0), \tag{28}$$

and

$$W_2 = \prod_{\tau_0}^{\tau} e^{A_0(\sigma;\tau')d\tau'} \equiv \mathcal{P}e^{\int_{\tau_0}^{\tau} A_0(\sigma;\tau')d\tau'} = Q(\sigma; \tau, \tau_0). \tag{29}$$

When the 2-surface Σ requires more than one coordinate patch to cover it, the connections in different coordinate patches must be related to each other in their overlap region by transition functions [6]. Then, the description of Wilson loop in terms of Wilson lines given in Eq. (26) must be suitably augmented to take this complication into account. The product integral representation of the Wilson line and the composition rule for product integrals given by Eq. (11) will still make it possible to describe the corresponding Wilson loop as a composite product integral. For definiteness, we will confine ourselves to the representation given by Eq. (26).

It is convenient for later purposes to define two composite Wilson line operators U and T according to

$$U(\sigma, \tau) = Q(\sigma; \tau, \tau_0)\, P(\sigma, \sigma_0; \tau), \tag{30}$$

$$T(\sigma; \tau) = P(\sigma, \sigma_0; \tau)\, Q(\sigma_0; \tau, \tau_0). \tag{31}$$

Using the first of these, we have

$$W_2\, W_1 = U(\sigma, \tau). \tag{32}$$

Similarly, we have for the two remaining Wilson lines

$$W_3 = P^{-1}(\sigma, \sigma_0; \tau), \tag{33}$$

and

$$W_4 = Q^{-1}(\sigma_0; \tau, \tau_0). \tag{34}$$

From the Eq. (31), it follows that

$$W_4\, W_3 = T^{-1}(\sigma, \tau). \tag{35}$$

Appealing again to Eq. (11) for the composition of product integrals, it is clear that this expression for the Wilson loop operator is independent of the choice of the point (σ, τ). In terms of the quantities T and U, the Wilson loop operator will take the compact form

$$W = T^{-1}(\sigma; \tau) U(\sigma; \tau). \tag{36}$$

4. Non-Abelian Stokes Theorem

As a first step in the proof of the non-abelian Stokes theorem, we obtain the action of the L-derivative operator on W:

$$L_\tau W = L_\tau[T^{-1}(\sigma,\tau)Q(\sigma;\tau,\tau_0)P(\sigma,\sigma_0;\tau_0)]. \tag{37}$$

Using the definition of the L-operation given by Eq. (13), noting that $P(\sigma,\sigma_0;\tau_0)$ is independent of τ, and carrying out the L operations on the right hand side (RHS), we get

$$\begin{aligned} L_\tau W &= L_\tau T^{-1}(\sigma,\tau) + T^{-1}(\sigma,\tau)\,[\,L_\tau Q(\sigma;\tau,\tau_0) + \\ &\quad + Q(\sigma;\tau,\tau_0)(L_\tau P(\sigma,\sigma_0;\tau_0))Q^{-1}(\sigma;\tau,\tau_0)]T(\sigma,\tau). \end{aligned} \tag{38}$$

Simplifying this expression by means of Eqs. (13) and (15), we end up with

$$L_\tau W = T^{-1}(\sigma,\tau)[A_0(\sigma,\tau) - L_\tau T(\sigma,\tau)]T(\sigma,\tau). \tag{39}$$

Next, we prove the analog of Theorem 3, which applies to an elementary Wilson line, for the composite Wilson loop operator defined in Eq. (24) and made explicit in Eq. (26).

Theorem 7 *The Wilson loop operator defined in Eq. (26) can be expressed in the form*

$$W = \prod_{\tau_0}^{\tau} e^{T^{-1}(\sigma,\tau')[A_0(\sigma,\tau') - L_\tau T(\sigma,\tau')]T(\sigma,\tau')d\tau'}. \tag{40}$$

To prove this theorem first we note from the definition of the L operation that the right hand side (RHS) of this equation can be written as

$$RHS = \prod_{\tau_0}^{\tau} e^{[T^{-1}(\sigma;\tau')A_0(\sigma;\tau')T(\sigma;\tau') - T^{-1}(\sigma;\tau')\frac{\partial}{\partial\tau'}T(\sigma;\tau')]d\tau'}. \tag{41}$$

Noting that $-T^{-1}\partial_\tau T = L_\tau T$, we can use Theorem 5 to obtain

$$RHS = T^{-1}(\sigma;\tau)\prod_{\tau_0}^{\tau} e^{A_0(\sigma;\tau')d\tau'}T(\sigma;\tau_0). \tag{42}$$

Moreover, making use of the defining Eq. (23), we get

$$RHS = T^{-1}(\sigma;\tau)\,Q(\sigma;\tau,\tau_0)\,P(\sigma,\sigma_0;\tau_0)\,Q(\sigma;\tau_0,\tau_0) = T^{-1}(\sigma;\tau)\,U(\sigma;\tau). \tag{43}$$

The last line is clearly the expression for W given by Eq. (36).

Finally, we want to express the Wilson loop W in yet another form which we state as:

Theorem 8 *The Wilson loop operator defined in Eq. (26) can be expressed as a surface integral of the field strength:*

$$W = \prod_{\tau_0}^{\tau} e^{\int_{\sigma_0}^{\sigma} T^{-1}(\sigma';\tau')F_{01}(\sigma';\tau')T(\sigma';\tau')d\sigma' d\tau'} \tag{44}$$

where F_{01} is the 0-1 component of the non-Abelian field strength.

To prove this theorem, we note that

$$\frac{\partial}{\partial\sigma}[T^{-1}(\sigma,\tau)A_0(\sigma,\tau)T(\sigma,\tau)] = T^{-1}(\sigma,\tau)\left[\partial_\sigma A_0(\sigma,\tau) + [A_0(\sigma,\tau), A_1(\sigma,\tau)]\right]T(\sigma,\tau). \tag{45}$$

Moreover,

$$\frac{\partial}{\partial\sigma}\{T^{-1}(\sigma,\tau)\left(L_\tau T(\sigma,\tau)\right)T(\sigma,\tau)\} = T^{-1}(\sigma,\tau)\partial_\tau A_1(\sigma,\tau)T(\sigma,\tau). \tag{46}$$

It then follows that

$$\begin{aligned} &\tfrac{\partial}{\partial\sigma}\{T^{-1}(\sigma,\tau)[A_0(\sigma,\tau) - L_\tau T(\sigma,\tau)]\}T(\sigma,\tau)\} \\ &= T^{-1}(\sigma,\tau)[\tfrac{\partial}{\partial\sigma}A_0(\sigma,\tau) - \tfrac{\partial}{\partial\tau}A_1(\sigma,\tau) + [A_0(\sigma,\tau), A_1(\sigma,\tau)]\}T(\sigma,\tau) \\ &= T^{-1}(\sigma,\tau)F_{01}(\sigma,\tau)T(\sigma,\tau). \end{aligned} \tag{47}$$

The last step follows from the definition of the field strength in terms of the connection given above

$$F_{0\,1} := \frac{\partial}{\partial\sigma}A_0(\sigma,\tau) - \frac{\partial}{\partial\tau}A_1(\sigma,\tau) + [A_0(\sigma,\tau), A_1(\sigma,\tau)]. \tag{48}$$

Integrating Eq. (47) with respect to σ, we get

$$\begin{aligned} &T^{-1}(\sigma,\tau)[A_0(\sigma,\tau) - L_\tau T(\sigma,\tau)]\}T(\sigma,\tau) \\ &= \int_{\sigma_0}^{\sigma} T^{-1}(\sigma';\tau')F_{0\,1}(\sigma';\tau')T(\sigma';\tau')d\sigma' d\tau'. \end{aligned} \tag{49}$$

We thus arrive at the surface integral representation of the Wilson loop operator:

$$W = \prod_{\tau_0}^{\tau} e^{\int_{\sigma_0}^{\sigma} T^{-1}(\sigma';\tau')F_{0\,1}(\sigma';\tau')T(\sigma';\tau')d\sigma' d\tau'}. \tag{50}$$

We note that in this expression the ordering of the operators is defined with respect to τ whereas σ is a parameter. Recalling the antisymmetry of the components of the field strength, we can rewrite this expression in terms of path ordered exponentials familiar from the physics literature:

$$W = \mathcal{P}_\tau e^{\frac{1}{2}\int_\Sigma d\sigma^{ab}T^{-1}(\sigma;\tau)F_{ab}(\sigma;\tau)T(\sigma;\tau)}, \tag{51}$$

where $d\sigma^{ab}$ is the area element of the 2-surface. Despite appearances, it must be remembered that σ and τ play very different roles in this expression.

5. A Second Proof

To illustrate the power and the flexibility of the product integral formalism, we give here a variant of the previous proof for the non-Abelian Stokes theorem. This time the proof makes essential use of the non-trivial Theorem 6. We start with the form of W given in Eq. (36) and take its derivatives with respect to τ:

$$\begin{aligned}\frac{\partial W}{\partial \tau} &= \partial_\tau Q^{-1}(\sigma_0;\tau,\tau_0)P^{-1}(\sigma,\sigma_0;\tau)Q(\sigma;\tau,\tau_0)P(\sigma,\sigma_0;\tau_0)+\\ &+Q^{-1}(\sigma_0;\tau,\tau_0)\partial_\tau P^{-1}(\sigma,\sigma_0;\tau)Q(\sigma;\tau,\tau_0)P(\sigma,\sigma_0;\tau_0)+\\ &+Q^{-1}(\sigma_0;\tau,\tau_0)P^{-1}(\sigma,\sigma_0;\tau)\partial_\tau Q(\sigma;\tau,\tau_0)P(\sigma,\sigma_0;\tau_0).\end{aligned} \tag{52}$$

Here, we have made use of the fact that $P(\sigma,\sigma_0;\tau_0)$ is independent of τ. As a preparation for the use of Theorem 3, we start with Eq. (13) for W and use Theorem 1

$$\begin{aligned}L_\tau W = \frac{\partial W}{\partial \tau}W^{-1} = \; & T^{-1}(\sigma;\tau)\left[A_0(\sigma;\tau) - P(\sigma,\sigma_0;\tau)A_0(\sigma_0;\tau)P^{-1}(\sigma,\sigma_0;\tau) - \right.\\ & \left. -\partial_\tau P(\sigma,\sigma_0;\tau)P^{-1}(\sigma,\sigma_0;\tau)\right]T(\sigma;\tau).\end{aligned} \tag{53}$$

Now we can use Theorem 6 to evaluate the derivative of the product integral with respect to the parameter τ:

$$\partial_\tau P(\sigma,\sigma_0;\tau) = \int_{\sigma_0}^{\sigma} d\sigma' P(\sigma,\sigma';\tau)\partial_\tau A_1(\sigma';\tau)P(\sigma',\sigma_0;\tau). \tag{54}$$

Then, after some simple manipulations using the defining equations for the various terms in Eq. (53), we get, with the abbreviation $P(\tau) = P(\sigma,\sigma_0;\tau)$:

$$T^{-1}(\sigma;\tau)\partial_\tau P(\tau)P^{-1}(\tau)T(\sigma;\tau) = \int_{\sigma_0}^{\sigma} d\sigma' T^{-1}(\sigma';\tau)\partial_\tau A_1(\sigma';\tau)T(\sigma';\tau). \tag{55}$$

Using Theorem 2 and the fact that $P(\sigma_0,\sigma_0;\tau)=1$, we can write the rest of Eq. (53) as an integral too:

$$\begin{aligned}&T^{-1}(\sigma;\tau)[A_0(\sigma;\tau) - P(\sigma,\sigma_0;\tau)A_0(\sigma_0;\tau)P^{-1}(\sigma,\sigma_0;\tau)]T(\sigma;\tau) =\\ &= Q^{-1}(\sigma_0;\tau,\tau_0)[P^{-1}(\sigma,\sigma_0;\tau)A_0(\sigma;\tau)P(\sigma,\sigma_0;\tau) - A_0(\sigma_0)]Q(\sigma_0;\tau,\tau_0) =\\ &= \textstyle\int_{\sigma_0}^{\sigma} d\sigma'\, P^{-1}(\sigma',\sigma_0;\tau)(\partial_\tau A_0(\sigma',\tau) + [A_0(\sigma',\tau), A1(\sigma',\tau)])P(\sigma',\sigma_0;\tau).\end{aligned} \tag{56}$$

Combining Eqs. (53), (55), and (56), we obtain:

$$L_\tau W = \frac{\partial W}{\partial \tau}W^{-1} = \int_{\sigma_0}^{\sigma} d\sigma' T^{-1}(\sigma',\tau)F_{01}(\sigma',\tau)T(\sigma',\tau). \tag{57}$$

Using Theorem 3, we are immediately led to Eq. (50) which was obtained by the previous method of proof.

There are two reasons for the relative simplicity of this proof over the one which was given in the previous section. One is due to the use of differentiation with respect to a parameter according the Theorem 6. The other is due to the use of Eq. (13) and Theorem 3 for the composite operator W. In the first proof, the use of this theorem for W was not assumed. Its justification for using it in the second proof lies in the composition law for product integrals given by Eq. (11).

6. Convergence Issues

The definitions of Wilson lines and Wilson loops as currently conceived in the physics literature involve exponentials of operators. The standard method of making sense out of such exponential operators in the physics literature is through their power series expansion:

$$\mathcal{P}e^{\int_a^b A(x)dx} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{P} \left(\int_a^b A(x)dx \right)^n , \tag{58}$$

where a typical path ordered term in the sum has the form

$$\frac{1}{n!}\mathcal{P}(\int_a^b A(x)dx)^n := \int_a^b dx_1 \int_a^{x_1} dx_2 \ldots \int_a^{x_{n-1}} dx_n\, A(x_1)A(x_2)\ldots A(x_n). \tag{59}$$

Without additional specifications, such a power series expansion is purely formal, and it is not clear *á priori* that the series (58) is well defined and convergent. Indeed, in previous attempts [10]- [20] at proving the non-abelian Stokes theorem, the convergence of such series has been taken more or less for granted.

One important advantage of our product integral approach is that, without the need for further input, we can show the convergence of the path ordered exponentials precise, without any further effort by using the established properties of product integrals. [1] They will enable us to prove that the series of partial sums converges uniformly to the product integral. The proof is contained, as a special case, in the following two theorems valid for all product integrals. The detailed proofs of these theorems are given in reference [9].

Theorem 9 *Given the continuous function* $A : [a, b] \to \mathbf{C}_{n\times n}$*, and given* $x, y \in [a, b]$*, let* $L(x, y) = \int_x^y ||A(s)||ds$*. Also let* $J_0(x, y) = I$*, and for* $n \geq 1$ *define iteratively* $J_n(x, y) := \int_x^y A(s)J_{n-1}(s, y)ds$*. Then for any* $n \geq 0$ *the following holds:*

$$||\prod_x^y e^{A(s)ds} - \sum_{k=0}^{n} J_k(x, y)|| \leq \frac{1}{(n+1)!}|L(x, y)|^{n+1}e^{L(a,b)}. \tag{60}$$

This estimate is uniform for all x, y in the interval $[a, b]$. One of the consequences of this estimate is the content of the next theorem.

[1]For example, to estimate a general term in the expansion (58) to show the norm convergence, we don't need to restate the elementary fact that the space of matrices of given rank over the complex numbers form a Banach space.

Theorem 10 *With A and $J_k(x,y)$ as in Theorem 9, we have, in the same notation,*

$$\prod_x^y e^{A(s)ds} = \sum_{k=0}^{\infty} J_k(x,y). \tag{61}$$

The series on the right hand side of this expression converges uniformly for any $x, y \in [a,b]$.

To give a flavor of the proofs involving product integrals we will include the proof of this theorem. According to Theorem 9 we have:

$$|| \prod_x^y e^{A(s)ds} - \sum_{k=0}^{n} J_k(x,y) || \leq M \frac{1}{(n+1)!} |L(b,a)|^{n+1} \longrightarrow 0, \ as\, n \to \infty \tag{62}$$

This follows immediately since, e.g., by Stirling's formula, the asymptotic behavior of the factorial function is roughly $n! \approx n^n$ as $n \to \infty$. Therefore, $\frac{x^n}{n!} \approx \left(\frac{x}{n}\right)^n \to 0$ as $n \to \infty$. q.e.d.

7. Physical Observables

From the Wilson loop operator, one can obtain physical quantities in a variety of ways. The most familiar one is its trace which gives the c-number Wilson loop or the Wilson loop observable:

$$\mathrm{Tr\,W_R(C)} = \mathrm{Tr}\,\mathcal{P}\mathrm{e}^{\frac{1}{2}\int_\Sigma \mathrm{d}\sigma^{\mathrm{ab}}\mathrm{T}^{-1}(\sigma;\tau)\mathrm{F_{ab}}(\sigma;\tau)\mathrm{T}(\sigma;\tau)}. \tag{63}$$

The subscript R in this expression refers to the particular representation of the generators.

Another invariant associated with an operator (a matrix) is its determinant. From its product integral representation, the determinant of the Wilson loop operator is given by

$$\det W = e^{\mathrm{Tr}\,\frac{1}{2}\int_\Sigma \mathrm{d}\sigma^{\mathrm{ab}}\mathrm{T}^{-1}(\sigma;\tau)\mathrm{F_{ab}}(\sigma;\tau)\mathrm{T}(\sigma;\tau)}. \tag{64}$$

After some straight forward manipulations, this can be expressed in the form

$$\det W = e^{\frac{1}{2}\int_\Sigma \mathrm{Tr\,d}\sigma^{\mathrm{ab}}\mathrm{F_{ab}}(\sigma;\tau)}. \tag{65}$$

The generators of simple Lie groups can be represented by traceless matrices so that for these groups $tr\,F_{ab} = 0$, indicating that $\det W = 1$. This is not surprising since the Wilson loop operator is a group element, and for group elements with determinant one this result follows trivially. For non-simple groups such as $U(1)$ and the products thereof the trace reduces to the trace of commuting elements of the algebra with non-zero trace. The corresponding subgroup is commutative, there is no ordering problem, and the surface representation of Wilson loop operator reduces to that of the (abelian) Stokes Theorem.

8. The gauge transforms of Wilson lines and Wilson loops

Under a gauge transformation, the components of the connection, i.e. the gauge potentials, transform according to [24]

$$A_\mu(x) \longrightarrow g(x)A_\mu(x)g^{-1}(x) - g(x)\partial_\mu g(x)^{-1}, \tag{66}$$

The components of the field strength (curvature) transform covariantly:

$$F_{\mu\nu}(x) \longrightarrow g(x)F_{\mu\nu}(x)g^{-1}(x). \tag{67}$$

Using the product integral formalism, we want to derive the effect of these gauge transformations on Wilson lines and Wilson loops.

Let us start with the Wilson line defined by Eq. (22). Under the gauge transformation (66) this quantity transforms as

$$P(\sigma,\sigma_0;\tau) = \prod_{\sigma_0}^{\sigma} e^{A_1(\sigma';\tau)d\sigma'} \longrightarrow \prod_{\sigma_0}^{\sigma} e^{[g(\sigma';\tau)A_1(\sigma';\tau)g^{-1}(\sigma';\tau)-g(\sigma';\tau)\partial_\sigma g^{-1}(\sigma';\tau)]d\sigma'}. \tag{68}$$

By Eq. (13), $g(\sigma;\tau)\partial_\sigma g^{-1}(\sigma;\tau) = -L_\sigma g(\sigma;\tau)$. Thus, we have for the gauge transformed Wilson line

$$\prod_{\sigma_0}^{\sigma} e^{[g(\sigma';\tau)A_1(\sigma';\tau)g^{-1}(\sigma';\tau)+L_\sigma g(\sigma';\tau)]d\sigma'}. \tag{69}$$

Moreover, we use Theorem 4 and recall from Theorem 3 that $\prod_{\sigma_0}^{\sigma} e^{L_\sigma g(\sigma';\tau)d\sigma'} = g(\sigma;\tau)g^{-1}(\sigma_0;\tau)$. Then, the gauge transform of $P(\sigma,\sigma_0;\tau)$ will take the form

$$g(\sigma;\tau)g^{-1}(\sigma_0;\tau)\prod_{\sigma_0}^{\sigma} e^{g(\sigma_0;\tau)A_1(\sigma';\tau)g^{-1}(\sigma_0;\tau)}. \tag{70}$$

Finally, using Theorems 7 and 8 one can readily see that the constant terms in the exponents can be factored out from the product integral so that we get

$$P(\sigma,\sigma_0;\tau) \longrightarrow g(\sigma;\tau)P(\sigma,\sigma_0;\tau)g^{-1}(\sigma_0;\tau). \tag{71}$$

In the physicist's notation, the result can be stated as

$$\mathcal{P}e^{\int_a^b A_\mu(x)dx^\mu} \longrightarrow g(b)\left(\mathcal{P}e^{\int_a^b A_\mu(x)dx^\mu}\right)g^{-1}(a), \tag{72}$$

Thus, we have an unambiguous proof of how the Wilson line transforms under gauge transformations. This is of course consistent with the role of the Wilson line as a parallel transport operator. For a closed path, the points a and b coincide. As a result, the corresponding Wilson loop operator transforms gauge covariantly.

For consistency, we expect that the surface integral representation of the Wilson loop also transforms covariantly under gauge transformations. To show this explicitly, we note from Eq. (50) that in this case we need to know how the operator $T(\sigma,\tau)$ transforms under gauge transformations. To this end, we note that the Wilson line $Q(\sigma;\tau,\tau_0)$ given by Eq. (23) transforms as

$$Q(\sigma;\tau,\tau_0)=\prod_{\tau_0}^{\tau} e^{A_0(\sigma;\tau')d\tau'} \longrightarrow g(\sigma;\tau)Q(\sigma;\tau,\tau_0)g^{-1}(\sigma;\tau_0). \tag{73}$$

The transform of the composite Wilson line $T(\sigma,\tau)$ given by Eq. (31) follows immediately:

$$T(\sigma;\tau)=P(\sigma,\sigma_0;\tau)\,Q(\sigma_0;\tau,\tau_0)\longrightarrow g(\sigma;\tau)T(\sigma;\tau)g^{-1}(\sigma_0;\tau_0). \tag{74}$$

As expected from the composition rule given by Eq. (11), the product of two Wilson lines transforms as a Wilson line.

From the above results, it is straight forward to show that the surface integral representation of Wilson loop transforms as

$$W\longrightarrow\prod_{\tau_0}^{\tau} e^{g(\sigma_0;\tau_0)\left(\int_{\sigma_0}^{\sigma} T^{-1}(\sigma';\tau')F_{10}(\sigma';\tau')T(\sigma';\tau')dt'\right)g^{-1}(\sigma_0;\tau_0)}. \tag{75}$$

As in the case of Wilson line, the constant factors in the exponent factorizes, so that under gauge transformations the Wilson loop transforms covariantly, i.e.,

$$W\longrightarrow g(\sigma_0;\tau_0)\prod_{\tau_0}^{\tau} e^{\int_{\sigma_0}^{\sigma} T^{-1}(\sigma';\tau')F_{10}(\sigma';\tau')T(\sigma';\tau')dt'}g^{-1}(\sigma_0;\tau_0). \tag{76}$$

We view this result as a nontrivial confirmation of our proof of non-abelian theorem. In the familiar physics notation, this transformation law takes the form

$$\mathcal{P}_\tau e^{\frac{1}{2}\int_\Sigma d\sigma^{ab}T^{-1}(\sigma;\tau)F_{ab}(\sigma;\tau)T(\sigma;\tau)}\longrightarrow g(a)\left(\mathcal{P}_\tau e^{\frac{1}{2}\int_\Sigma d\sigma^{ab}T^{-1}(\sigma;\tau)F_{ab}(\sigma;\tau)T(\sigma;\tau)}\right)g^{-1}(a) \tag{77}$$

9. Concluding Remarks

The identification of Wilson lines and Wilson loops of non-Abelian gauge theories with product integrals allows for the possibility of extracting physical consequences from these objects in a consistent and mathematically well defined manner. Although many of the properties of Wilson lines and Wilson loops have been discussed [10]- [20] from various, more intuitive, points of view, there are some issues associated with these operators with respect to which their product integral representations have a decided advantage. One is the existence issue discussed in Section 6, and the other is the supersymmetric generalization of these notions [26].

On the more formal front, the advantage of the product integral formulation of Wilson lines and Wilson loops presented here lies in the possibility of extending these results to cases that are of interest in number theory and algebraic geometry. We conjecture that most of our results hold if we replace the ground field of real numbers with a local field, especially with the field of p-adic numbers. In this respect, we rely on the properties of the norm and differential calculus used in the proofs. Efforts have been made in the last couple of years to formulate quantum field theory, and especially string theory, on local fields. Particular attention was given to the p-adic numbers and their finite algebraic extensions, both in string theory and field theory (see [27]), and in Galois fields with their finite extensions (see [28]). It would be of interest to study the adelic properties of Wilson lines and Wilson loops using these methods. We are optimistic that the present work will help fill the gap in connection with these as well as other issues.

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