Turk J Phy
24 (2000) , 403 – 409.

Spinors and Positive Mass*

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Received 23.03.2000

Abstract

As an example of how spinor methods can be applied to the positive-mass theorem, a simple, spinor-based proof of the theorem is presented for the case of a single black hole. Generalisations of the theorem are also discussed.

1. Introduction

One of the most important applications of spinor methods in general relativity is the proof that Bondi mass is positive for all physically reasonable spacetime [1]. The surprising thing about this proof is its relative simplicity compared with proofs using more traditional methods. But why is this so? Why do spinor methods work so well in this and other contexts? In this paper I shall attempt to answer these questions by highlighting certain properties of spinors and then by showing how they can be applied to the proof of the positive-mass theorem. I shall illustrate this by presenting a simple, spinor-based proof of the theorem in the special case of a spacetime containing a single black hole.

2. Spinors

At the most basic level, a spinor λ^A is simply an element of an abstract, 2-dimensional, complex vector space S. It has a complex conjugate which we denote by $\bar{\lambda}^{A'}$. Elements of the dual of S are denoted by symbols like λ_A, with complex conjugate $\bar{\lambda}_{A'}$. Tensors constructed from S and its complex conjugate are called spin tensors. They differ from ordinary real tensors only in that they can have dashed (conjugate) indices which contract with complex conjugate spinors. A spin-metric on S is defined to be an antisymmetric spin-tensor $\varepsilon_{AB} = -\varepsilon_{BA}$. Since S is 2-dimensional, all spin-metrics are proportional.

*Talk presented in Regional Conference on Mathematical Physics IX held at Feza Gürsey Institute, Istanbul, August 1999.

The connection with spacetime comes when we consider hermitian spin-tensors $x^{AA'} = \bar{x}^{AA'}$. These form a real, 4-dimensional vector space V with a positive cone N^+ described by hermitian tensors of the form $\lambda^A\bar{\lambda}^{A'}$. Thus V has a Minkowski-like structure. At this point it is convenient to elaborate the index notation by replacing the double index AA' with a single lower-case index a (also BB' with b, etc). Thus $x^a = x^{AA'}$ and $\lambda^a = \lambda^A\bar{\lambda}^{A'} \in N^+$.

The existence of a preferred cone in V determines a metric, unique up to a conformal rescaling, by $g_{ab}\lambda^a\lambda^b = 0$ for $\lambda^a \in N^+$. Thus N^+ is a half null-cone with respect to g_{ab}. Note that all such metrics have the form

$$g_{ab} = \varepsilon_{AB}\bar{\varepsilon}_{A'B'} \tag{1}$$

for some spin-metric ε_{AB}. Furthermore, if (λ^A, μ^A) is a basis in S, then

$$(\lambda^a, \alpha^a, \beta^a, \mu^a) = \left(\lambda^A\bar{\lambda}^{A'}, \lambda^A\bar{\mu}^{A'} + \mu^A\bar{\lambda}^{A'}, i(\lambda^A\bar{\mu}^{A'} - \mu^A\bar{\lambda}^{A'}), \mu^A\bar{\mu}^{A'}\right) \tag{2}$$

is a basis in V, and all such bases have the same orientation. Thus V has a preferred orientation.

So far, using only the algebraic structure of S, we have constructed an oriented (both in time and space) Minkowski space with a metric defined up to a conformal factor. A vector λ^a is null and future pointing if it has the form $\lambda^A\bar{\lambda}^{A'}$, and a basis has a positive orientation if has the same orientation as as basis of the form given by equation 2. In order to make the metric unique we select a spin-metric and take g_{ab} to be given by equation 1.

It should be pointed out the the reverse procedure also holds (see [6]): given an orientated Minkowski space (V, g_{ab}), it is possible to construct a spin-space (S, ε_{AB}) from which (V, g_{ab}) can be constructed by the above procedure. Both (V, g_{ab}) and (S, ε_{AB}) are thus essentially equivalent descriptions of the same thing, but in many respects the latter is considerably simpler, especially when applied to problems involving null vectors and where a particular choice of orientation is important (e.g the following proof of the positive-mass theorem).

The traditional approach to general relativity is to start with a 4-dimensional manifold M with a Lorentzian metric, g_{ab}, which gives the tangent space at each point the structure of a Minkowski space. In the spinor approach, on the other hand, the tangent space of each point p of M is identified with a Minkowski space V_p constructed as above from a spin-space S_p. In other words, M is endowed with a spin-structure. The existence of a spin-structure imposes certain topological restrictions on M, for example M must be 4-dimensional[1] and orientable, but apart from this both approaches are equivalent. However, the spinor approach gives a different slant on the structure of spacetime and highlights certain features which tend to be hidden in more traditional methods.

Given a manifold with a spin structure we can now consider spinor fields, which we represent again by symbols like λ^A. The spacetime metric is given by equation 1 where

[1] Spin-structures can be defined for higher dimensional manifolds, but not with respect to a 2-dimensional spin-space.

ε_{AB} is the spin-metric field, and the spacetime connection $\nabla_a = \nabla_{AA'}$ is defined by $\nabla_a \varepsilon_{AB} = 0$, which implies $\nabla_a g_{bc} = 0$.

3. Asymptotic Flatness

In general relativity an isolated gravitating system, such as a star or a black hole, is described by an asymptotically spacetime (M, g_{ab}) satisfying the energy condition $T_{ab} w^a v^a \geq 0$ where v^a and w^a are future pointing (this ensures that energy density is positive). According the Penrose conformal definition of asymptotic flatness [6], there exists an extended spacetime $(\hat{M}, \hat{g}_{ab})$ with boundary $\mathcal{I}$ such that

1. $\hat{M} = M \cup \mathcal{I}$.
2. There exists a smooth function Ω on $\hat{M}$ such that $\Omega > 0$ and $\hat{g}_{ab} = \Omega^2 g_{ab}$ on M, and $\Omega = 0$ and $\nabla_a \Omega \neq 0$ on $\mathcal{I}$.

Though not immediately obvious, this definition does indeed capture the intuitive notion of asymptotic flatness in that it implies that the metric approaches a flat metric near infinity. It is important to note that this conformal definition of asymptotic flatness does not determine a unique conformal factor Ω: if Ω provides a conformal completion then so does $\omega\Omega$ where $\omega > 0$ on $\hat{M}$. Any meaningful geometric equation pertaining to $\hat{M}$ must therefore be conformally invariant under the conformal resaling $\Omega \to \omega\Omega$.

The boundary $\mathcal{I}$ forms a null surface and represents points at null infinity. It consists of two disjoint components: $\mathcal{I}^+$, future null infinity on which $\nabla_a \Omega$ is past pointing, and $\mathcal{I}^-$, past null infinity on which $\nabla_a \Omega$ is future pointing. Both $\mathcal{I}^+$ and $\mathcal{I}^-$ have topology $R \times S^2$. A S^2 cross-section of $\mathcal{I}$ (a cut of $\mathcal{I}$) determines a null surface in M generated by null geodesics which intersect the cut orthogonally.

The conformally rescaled spin-metric is given by $\hat{\varepsilon}_{AB} = \Omega \varepsilon_{AB}$ and the corresponding connection by $\hat{\nabla}_a \hat{\varepsilon}_{AB} = 0$. We employ the convention of lowering indices of hatted spinors by means of $\hat{\varepsilon}_{AB}$ and indices of unhatted spinors by means of ε_{AB}.

According to Bramson's definition [4], a spinor field λ^A is said to be asymptotically constant if there exists a spinor field $\hat{\lambda}^A$ on $\hat{M}$ such that

1. $\hat{\lambda}^A = \lambda^A$ on M,
2. $\hat{\lambda}^A$ is tangent to $\mathcal{I}$ in that $\hat{\lambda}^a \nabla_a \Omega = 0$ on $\mathcal{I}$ where $\hat{\lambda}^a = \hat{\lambda}^A \hat{\lambda}^{A'}$.
3. $\hat{\nabla}_{A'(A} \hat{\lambda}_{B)} = 0$ on $\mathcal{I}$.

Again, though not immediately obvious, this does capture the intuitive notion of an asymptotically constant spinor field in that it implies that $\nabla_a \lambda^A$ tends to zero in a well-defined sense near infinity. Furthermore, the definition can easily be shown to be conformally invariant.

We say that two asymptotically constant spinor fields are equivalent if they coincide on $\mathcal{I}$ and denote the equivalence class of λ^A by $\lambda^{\tilde{A}}$. The space of such equivalence classes can be shown to form a two-dimensional complex symplectic space (asymptotic spin-space) with $\varepsilon_{\tilde{A}\tilde{B}}$ defined by $\varepsilon_{\tilde{A}\tilde{B}}\alpha^{\tilde{A}}\beta^{\tilde{B}} = \varepsilon_{AB}\alpha^A\beta^B$ on $\mathcal{I}$ where α^A and β^A are asymptotically constant.

Physically, asymptotic spin-space should be regarded as spin-space according to asymptotic observers near $\mathcal{I}^+$, and the corresponding Minkowski space as the vector space of translations according to such observers.

4. Bondi Momentum

Observers at a great distance from some isolated system such as a star may be regarded as residing near $\mathcal{I}^+$. Their natural frame of reference is thus asymptotic spin-space, and a measurement of the star's total 4-momentum at some retarded time (i.e. a cut of $\mathcal{I}^+$) will result in a vector $p_{\tilde{a}} = p_{\tilde{A}\tilde{A}'}$ living in asymptotic spin-space. According to an observer with 4-velocity $v^{\tilde{a}}$ the star will have mass/energy $m = p_{\tilde{a}}v^{\tilde{a}}$.

In 1962 Bondi et. al. [5] framed a definition of gravitational mass (now known as Bondi mass) at a retarded time which appeared to capture the properties expected of m. In particular, it was shown that m (according to the definition) reduces to the standard expression for the mass in the case of a stationary spacetime, and, more spectacularly, that m is a decreasing function of retarded time for radiating systems, reflecting 'mass-loss' due to radiation escaping to infinity. However, there remained the possibility that m could eventually become negative due to the flux of radiative energy through $\mathcal{I}^+$. That this can not be the case for any physically-reasonable system was first proved several years later in 1981 using spinor methods [1]. A by-product of this proof is the following elegant 'spinor' definition of Bondi momentum.

Given a spinor field λ^A we construct a complex 2-form

$$F_{ab} = \frac{i}{2}(\bar{\lambda}_{A'}\nabla_b\lambda_A - \lambda_{B'}\nabla_a\lambda_B). \tag{3}$$

It can easily be checked that the imaginary part of F_{ab} has the form $\nabla_{[a}\lambda_{b]}$ (i.e. $F = d\lambda$) and hence $\int_S F$ is real for any closed 2-surface S. If S is a cut of $\mathcal{I}$ and λ^A is asymptotically constant this integral is well-defined and is linear in $\lambda^{\tilde{A}}$ and $\lambda^{\tilde{A}'}$. It thus defines a real vector $p_{\tilde{A}\tilde{A}'}$ in asymptotic spin-space by

$$p_{\tilde{A}\tilde{A}'}\lambda^{\tilde{A}}\lambda^{\tilde{A}'} = \frac{1}{4\pi}\int_S F. \tag{4}$$

This vector turns out to be Bondi momentum at the retarded time defined by the null-surface in M which intersects S - given that S is a cut of $\mathcal{I}^+$. If it is a cut of $\mathcal{I}^-$ it gives total momentum at the corresponding advanced time.

At this stage it is worth pointing out the essential role played by spinor methods in formulating such natural and elegant definition of Bondi momentum. Had we been restricted to traditional tensor methods, whose basic elements are a connection ∇_a and vector fields, the only natural 2-form we could have constructed would be of the form

$\nabla_{[a}\lambda_{b]}$, whose integral is identically zero. On the other hand, using spinor methods, whose basic elements are a connection and spinor fields, the 2-form given by equation 3 is a perfectly natural construction. It can be expressed entirely in terms of tensors, but the resulting expression turns out to be quite complicated.

If S and S' are two cuts of $\mathcal{I}^+$ where S' lies to the future of S, then if can be shown, and indeed follows from the results given below, that $p_{\tilde{A}\tilde{A}'}\lambda^{\tilde{A}}\lambda^{\tilde{A}'} \geq p'_{\tilde{A}\tilde{A}'}\lambda^{\tilde{A}}\lambda^{\tilde{A}'}$. This is the celebrated Bondi mass-loss formula. It represented the first rigourous proof that gravitational radiation carries positive energy, the positive difference $p_{\tilde{A}\tilde{A}'}\lambda^{\tilde{A}}\lambda^{\tilde{A}'} - p'_{\tilde{A}\tilde{A}'}\lambda^{\tilde{A}}\lambda^{\tilde{A}'}$ being interpreted as the flux of radiative energy through the region of $\mathcal{I}^+$ between S and S'.

5. Positivity Equations

Taking the exterior derivative of the 2-form given by equation 3 and using Einstein's equation

$$G_{ab} = -8\pi T_{ab}$$

we get

$$dF = \alpha + \beta$$

where

$$\alpha_{cde} = \frac{2\pi}{3} T_{ab}\lambda^A\bar{\lambda}^{A'} e^b{}_{cde}$$

(e_{abcd} is the Levi-Cevita tensor) and β is the skew part of

$$b_{abc} = -i\nabla_a\bar{\lambda}_{C'}\nabla_b\lambda_C.$$

Consider now two spacelike 2-spheres, S and S', where S lies to the future of S', which are connected by a null 3-surface N. By Stokes we have

$$\int_S F - \int_{S'} F = \int_N \alpha + \int_N \beta.$$

If N is diverging then the positivity condition on T_{ab} implies that

$$\int_N \alpha \geq 0.$$

Similarly, if N is converging, then

$$\int_N \alpha \leq 0.$$

We now show that λ_A can be chosen in a natural way so that this also applies to $\int_N \beta$.

Let $\gamma^a = \gamma^A\bar{\gamma}^{A'}$ be tangent to N. Note that γ^a is automatically null and future-pointing, and is unique up to a complex multiplier. Using γ^A we propagate λ^A along N by means of the equation

$$\bar{\gamma}^{A'}\gamma^B\nabla_{A'A}\lambda_B = 0. \tag{5}$$

This gives a good propagation equation in the sense that λ_A on S' determines λ_A on S. It also imposes a single constraint on the initial and final 2-surface, S' and S, which allows $\gamma^A\lambda_A$ to be chosen freely. Given $\gamma^A\lambda_A$, this constraint determines λ_A uniquely. Most importantly, if λ_A satisfies this propagation equation then $\int_N \beta$ is automatically positive for a diverging N and negative for a converging N.

Another good propagation equation with this property, but which imposes a different constraint on S, is given by

$$\bar{\gamma}^{A'}\gamma_{(B}\nabla_{A)A'}\lambda^B = 0. \tag{6}$$

Another important feature of both of these propagation equations is that they are compatible with λ_A being asymptotically constant. Indeed, in flat spacetime, a spinor which is propagated out to $\mathcal{I}$ by means of equation 5 or equation 6, and which is asymptotically constant, is globally constant on N.

6. Positivity of Bondi Mass

Based on the foregoing results, we now present a simple proof of the positivity of Bondi mass in the case of a spacetime containing a single black hole. In particular, we consider the following situation. We start with a trapped surface, T, and move out from T into the past along a converging null surface N' until we reach a non-trapped surface S. We then move out from S into future along a diverging null surface N to a cut S' of $\mathcal{I}^+$. Our aim is to show that

$$p_{\tilde{A}\tilde{A}'}\lambda^{\tilde{A}}\lambda^{\tilde{A}'} = \frac{1}{4\pi}\int_{S'} F \geq 0. \tag{7}$$

Let $(o_A,\, \iota_A)$ be a spinor diad (i.e. $o_A\iota^A = 1$) such that $l^a = o^A\bar{o}^{A'}$ is tangent to N and $n^a = \iota^A\bar{\iota}^{A'}$ is tangent to N'. Using equation 6 with $\gamma^A = o^A$ to propagate λ^A along N we have

$$\int_{S'} F \geq \int_S F.$$

Asymptotic constancy on S' imposes a restriction on $\lambda_0 = \lambda_A o^A$ when evaluated on S but allows $\lambda_1 = \lambda_A\iota^A$ to be chosen freely on S (this is an important feature of 6, not possessed of 5). We now use this freedom to satisfy the constraint imposed by equation 5 and propagate λ^A along N' using this equation. We now get

$$\int_{S'} F \geq \int_S F \geq \int_T F.$$

The constraint imposed by equation 5 can be shown to imply that

$$\int_T F = \int_T (\rho\lambda_1\bar{\lambda_1} + \rho'\lambda_0\bar{\lambda_0})dT,$$

where the left-hand side is an ordinary surface integral over T and ρ and ρ' are the divergences of N and N' on T. That $\int_T F$ has this form is a particular feature of 5, not possessed of 6. Finally, since the defining feature of a trapped surface is that both divergences are positive, we get

$$\int_{S'} F \geq \int_{S} F \geq \int_{T} F \geq 0.$$

and hence

$$p_{\tilde{A}\tilde{A}'}\lambda^{\tilde{A}}\bar{\lambda}^{\tilde{A}'} \geq 0.$$

7. Stronger Versions of the Positive-Mass Theorem

A drawback of equation 4 is that it gives the component of Bondi momentum along a null vector, and therefore does not immediately provide an integral expression for Bondi mass (i.e. the component of Bondi momentum along a unit, time-like vector). Fortunately, this can be remedied by employing two spinor fields, λ^A and μ^A, to obtain a unit time-like vector $v^a = \lambda^a + \mu^a$. Thus

$$m = p_{\tilde{a}}v^{\tilde{a}} = \frac{1}{4\pi}(\int_S F_\lambda + \int_S F_\mu). \tag{8}$$

This equation is interesting since it remains well-defined even if λ^A and μ^A are not asymptotically constant: all that is required is that v^a be asymptotically constant. This allows the possibility of more general propagation equations and thus stronger versions of the positive-mass theorem, which simply states $m \geq 0$. In particular, in the case of a charged black hole, the Maxwell field can be incorporated into the propagation equations. This leads to the stronger result, $m \geq |e|$, where e is the hole's electric charge [2]. In a similar manner it is also possible to show that

$$m^2 \geq \frac{1}{8\pi}A$$

where A is the hole's area [3].

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