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On the Duality of Quantum Liouville Field Theory*

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Abstract

It has been found empirically that the Virasoro centre and 3-point functions of quantum Liouville field theory with potential $e^{2b\phi(x)}$ and external primary fields $\exp(\alpha\phi(x))$ are invariant with respect to the duality transformations $\hbar\alpha \to q - \alpha$ where $q = b^{-1} + b$. The steps leading to this result (via the Virasoro algebra and 3-point functions) are reviewed in the path-integral formalism. The duality stems from the fact that the quantum relationship between the α and the conformal weights Δ_α is two-to-one. As a result the quantum Liouville potential may actually contain two exponentials (with related parameters). It is shown that in the two-exponential theory the duality appears in a natural way and that an important extrapolation which was previously conjectured can be proved.

1. Introduction

The interest in two-dimensional Liouville theory [1] lies in the fact that it is a non-trivial conformally invariant field theory involving only a single (scalar) field, and that it has recently appeared in many contexts, most notably in string theory [2][3]. However the quantum version of the theory presents some problems. For example, in contrast to the Wess-Zumino-Witten (WZW) four-point function, the Liouville four-point function cannot be expressed in terms of elementary analytic functions (the analogue of the Knitznik-Zamolodchikov equation is second order). Even for the 3-point function, for which the space-time dependence is fixed by conformal invariance, the coefficients are difficult to obtain. On the other hand, what is known about the 3-point functions is quite interesting and exhibits a duality of the theory that is absent at the classical level [4][5]. In this talk I wish to review the quantized Liouville theory from the path-integral point

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of view and to discuss this duality, first describing what it is, then proposing a manner in which it can be incorporated in the functional integral by using a potential with two exponentials.

2. The Liouville Path-Integral

The classical Liouville theory is defined by the Action

$$S(\phi) = \int d^2x \Big[\frac{1}{2}(\partial\phi)^2 + \mu e^{2b\phi}\Big] \tag{2.1}$$

where μ and b are constants and $\phi(x)$ is a scalar field, and in 2 dimensions this Action is conformally invariant. If the theory is embedded in a curved space by generalizing the Action to

$$S(\phi) = \int d^2x \Big[\frac{1}{2}\sqrt{g}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + \mu\sqrt{g}e^{2b\phi} + q\sqrt{g}R\phi\Big] \tag{2.2}$$

where R is the Riemann curvature and q a constant, the conformal invariance converts to Weyl invariance i.e. invariance with respect to the transformation $g_{\mu\nu}(x) \to \lambda(x)g_{\mu\nu}(x)$ for any harmonic scale-factor $\lambda(x)$, provided that the constant q is

$$q = b^{-1} \quad \text{and} \quad q = b^{-1} + \hbar b \tag{2.3}$$

in the classical and quantum theories respectively. The quantum duality referred to above makes its appearance already at this stage in the fact that q is invariant under the transformation $b \leftrightarrow (\hbar b)^{-1}$.

By varying $g^{\mu\nu}$ in (2.2) and then taking the flat space limit one sees that the energy momentum tensor for the flat-space theory is

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}\eta_{\mu\nu}(\partial\phi)^2 + q(\eta_{\mu\nu}\partial^2\phi - \partial_\mu\partial_\nu\phi) \tag{2.4}$$

where η is the Minkowski metric. This energy momentum tensor is traceless on the mass-shell and leads to Virasoro operators of the form

$$T_{++} = (\partial_+\phi)^2 - q\partial_+^2\phi \quad \text{and} \quad T_{--} = (\partial_-\phi)^2 - q\partial_-^2\phi \tag{2.5}$$

For these Virasoros the centres are of the form

$$c = \hbar + kq^2 \tag{2.6}$$

where k is a numerical constant, and the fields $e^{\alpha\phi}$ are primary fields of conformal weights $(\Delta_\alpha, \Delta_\alpha)$ where

$$\Delta_\alpha = \alpha(q - \hbar\alpha) \tag{2.7}$$

the αq part coming from the improvement term and the α^2 part from the normal ordering of the standard energy momentum tensor. The duality makes its second appearance at this stage in the fact that the conformal weights $\hbar\Delta_\alpha$ are invariant with respect to the transformation $\hbar\alpha \leftrightarrow (q - \hbar\alpha)$. This second form of duality is more general than the first, which is actually the special case for which $\Delta_\alpha = 1$. An immediate consequence of the duality is that in quantum conformal field theories there are two primary fields of the form $e^{2\alpha\phi}$ for each conformal weight Δ_α. In other words, there are two weights

$$\alpha_\pm = \frac{q \pm \sqrt{q^2 - 4\hbar\Delta_\alpha}}{2\hbar} \tag{2.8}$$

for every Δ_α. Note that in the limit $\hbar \to 0$ we have $\alpha_+ \to \infty$ and $\alpha_- \to \alpha$ so that only one root remains finite.

From now on we shall set $\hbar = 1$, assume that 2-space is a sphere and use conformal coordinates i.e. coordinates for which

$$g_{\mu\nu}(x) = e^{\sigma(x)}\eta_{\mu\nu} \tag{2.9}$$

In these coordinates the Action reduces to

$$S(\phi) = \int d^2x \Big[\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + \mu e^{2(b\phi+\sigma)} + qe^{2\sigma}R\phi\Big] \tag{2.10}$$

3. The N-Point Functions

In terms of the path integral we write the N-point function as

$$< \prod_a e^{2\alpha_a\tilde{\phi}(x_a)} >= \int d\tilde{\phi}(x)e^{iS(\tilde{\phi}(x))}\prod_a e^{2\alpha_a\tilde{\phi}(x_a)} \tag{3.1}$$

The exponential potential has the property that, if the zero mode part of $\tilde{\phi}$ is seperated from the rest by the decomposition

$$\tilde{\phi}(x) = \phi_o + \phi(x) \qquad \int d^2x\phi(x) = 0, \tag{3.2}$$

the integration over the zero-mode can be seperated from the rest [9] by writing (3.1) in the form

$$< \prod_a e^{2\alpha_a\tilde{\phi}(x_a)} >= \int d\phi e^{iS(\phi)}\prod_a e^{2\alpha_a\phi(x_a)}\Big(\int d\phi_o e^{-2\xi\phi_o + e^{2b\phi_o}V_b(\phi)}\Big) \tag{3.3}$$

where

$$\xi = q - \sum_a \alpha_a \qquad V_b(\phi) = \mu e^{2b\phi} \tag{3.4}$$

and we have used the fact that for the sphere $\int d^2x\sqrt{g}R = \chi = 2$, where χ is the Euler number. Furthermore, modulo some factors, the zero mode integral in (3.3) is just the integral for the Γ-function so it can immediately be evaluated to give

$$\prod_a < e^{2\alpha_a \tilde{\phi}(x_a)} >= \Gamma\Big(-\frac{\xi}{b}\Big) \int d\phi e^{iS(\phi)} \prod_a e^{2\alpha_a \phi(x_a)} \Big(V_b(\phi)\Big)^{\frac{\xi}{b}} \tag{3.5}$$

Here we run into the major difficulty with Liouville theory, which is that the path-integral is calculable only when ξ/b takes positive integer values, and these are precisely the points at which the coefficient $\Gamma(-\xi/b)$ has poles. So in practice all we can do is calculate the residues at the poles and try to extrapolate to other points. There is some consolation in the fact that the residues

$$R_N(x_a, \alpha_a, m) = \frac{(-1)^m}{m!} \int d\phi e^{iS(\phi)} \prod_a e^{2\alpha_a \phi(x_a)} \Big(V_b(\phi)\Big)^m \tag{3.6}$$

turn out to be free-field integrals which are already available in the literature [10]. More specifically, if we extract the standard Polyakov factor $\exp(1 + kq^2) \int R\Delta^{-2}R$, which includes the conformal anomaly, renormalize by extracting the singularities of the Green's function in the usual way, and take the flat space limit, we obtain for the residues

$$R_N(x_a, \alpha_a, m) = \frac{(-\mu)^m}{m!} \prod_{a \neq b} |x_{ab}|^{-2\alpha_a \alpha_b} \prod_{i \neq j}^{i=m} \int d^2 x_i |x_{ij}|^{-2b^2} |x_{ia}|^{-4b\alpha_a} \tag{3.7}$$

where $x_{ab} \equiv x_a - x_b$. Conformal invariance implies that the kernel of the integral in the second factor is $SL(2, C)$ invariant. Since the group $SL(2, C)$ has 3-parameters this means that the partition function and the 1-point function do not exist while the two point function has a factor $\delta(\Delta_1 - \Delta_2)$, where the Δ's are the conformal weights of the two external fields. The most interesting case is for $N = 3$, when the residue takes the form

$$R_3(x_a, \alpha_a, m) = C_3(\alpha_a, m) \prod_{a \neq b}^{3} |x_{ab}|^{-2\Delta_{ab}} \tag{3.8}$$

where C_3 is a numerical coefficient and

$$\Delta_{ab} = \Delta_a + \Delta_b - \Delta_c \qquad \Delta_a = \alpha_a (q - \alpha_a) \tag{3.9}$$

For $N > 3$ the residues take the form

$$R_N(x_a, \alpha_a, m) = |\frac{x_{23}}{x_{12} x_{31}}|^{2\xi(\xi+q)} \prod_{a \geq 2} |x_{1a}|^{4\xi\alpha_a} K(\alpha_a, r_c) \tag{3.10}$$

where the $r_c(x)$ are conformally invariant ratios of the form $r_c = x_{c2} x_{31}/x_{c1} x_{23}$ and

$$K(\alpha_a, r_c) = \prod_{c \geq 4} \int d^2 x_i |x_{ij}|^{-2b^2} |x_i|^{-4b\alpha_2} |x_i - 1|^{-4b\alpha_3} |x_i - r_c|^{-4b\alpha_c} \tag{3.11}$$

4. Three-Point Coefficient

The values of the coefficient $C_3(\alpha_a, -\xi/b)$ of the 3-point function at the poles $\xi = -mb$ of the Γ-function can be computed from the free-field integral

$$C_3(\alpha_a, m) = \frac{(-\mu)^m}{m!} \prod_{i \neq j}^{m} \int d^2x_i |x_{ij}|^{-2b^2} |x_{ia}|^{-4b\alpha_a} \tag{4.1}$$

and this can be integrated [10] to give

$$C_3(\alpha_a, m) = (\mu_r)^{mb} \frac{k'(0)}{k'(-mb)} \Big[\frac{k(2\alpha_1)k(2\alpha_2)k(2\alpha_3)}{k(\alpha_1+\alpha_2-\alpha_3)k(\alpha_2+\alpha_3-\alpha_1)k(\alpha_3+\alpha_1-\alpha_2)} \Big] \tag{4.2}$$

where $\mu_r = \left[-\mu\gamma(b^2)\right]^{1/b}$ is a renormalized version of μ and the function $k(\alpha)$, which is implicitly a function of b, is defined by the following, rather remarkable, properties

$$k(\alpha) = k(q-\alpha) \tag{4.3}$$

$$k(\alpha+b) = \gamma(b\alpha) b^{1-2b\alpha} k(\alpha) \tag{4.4}$$

and

$$k(\alpha+c) = \gamma(c\alpha) c^{1-2c\alpha} k(\alpha) \tag{4.5}$$

where

$$q = b + c \qquad c = b^{-1} \quad \text{and} \quad \gamma(x) = \Gamma(x)/\Gamma(1-x) \tag{4.6}$$

Thus the function exhibits the symmetry $\alpha \leftrightarrow q - \alpha$ explicitly and is doubly periodic modulo some simple multiplicative factors. It is actually an entire function and its only zeros are simple ones at the points

$$(m+1)b + (n+1)c \quad \text{and} \quad -(mb+nc) \qquad m, n = 0, 1, 2.... \tag{4.7}$$

Thus it has a double series of zeros. For the full function $C_3(\alpha_a, -\xi/b)$ we then have

$$C_3(\alpha_a, -\xi/b) \to \frac{(\mu_r)^{mb}}{(\xi+mb)} \frac{k'(0)}{k'(-mb)} \Big[\frac{k(2\alpha_1)k(2\alpha_2)k(2\alpha_3)}{k(\alpha_1+\alpha_2-\alpha_3)k(\alpha_2+\alpha_3-\alpha_1)k(\alpha_3+\alpha_1-\alpha_2)} \Big] \tag{4.8}$$

for $\xi \to -mb$ and the question is : what is the correct extrapolation $C_3(\alpha_a, -\xi/b)$ that satisfies this equation?

5. The DOZZ Conjecture

Recently there has been a suggestion by Dorn and Otto [4] and the Zamilodchikovs [5] that the simple extrapolation

$$C_3(\alpha_a, -\xi/b) = (\mu_r)^{\xi} \frac{k'(0)}{k(-\xi)} \Big[\frac{k(2\alpha_1)k(2\alpha_2)k(2\alpha_3)}{k(\alpha_1+\alpha_2-\alpha_3)k(\alpha_2+\alpha_3-\alpha_1)k(\alpha_3+\alpha_1-\alpha_2)} \Big] \tag{5.1}$$

is the correct one. We shall call this the DOZZ conjecture and, apart from simplicity, there are three strong arguments in its favour:

I: Maximal Analyticity: The function $1/k(-\xi)$ has simple poles at the points given in (4.7). The replacement of $1/k'(-\xi)$ with $1/k(-\xi)$ does not introduce any new poles but its replacement by any other function would do so. Thus the DOZZ minimizes the number of new poles that are introduced.

II: Reflection Invariance: The *numerator* of (5.1) is covariant with respect to reflection symmetry in the sense that under the reflection $\alpha_a \to q - \alpha_a$ we have

$$k(\alpha_1)k(\alpha_2)k(\alpha_3) \to R(\alpha_a)\Big(k(\alpha_1)k(\alpha_2)k(\alpha_3)\Big) \tag{5.2}$$

where

$$R(\alpha_a) = \frac{k(2\alpha_a)}{k(2\alpha_a - q)} \tag{5.3}$$

for any choice of $a = 1, 2, 3$. In the *denominator* of (5.1), however, two of the factors in large bracket in (5.1) interchange, and the third one changes to $k(-\xi)$. Hence, with the DOZZ Ansatz the numerator in (5.1) is reflection *invariant* but otherwise it has no particular reflection properties.

III: Unitarity: If it is assumed that the $e^{\alpha\phi(x)}$ form a complete set of states $|\alpha>$ one can express the 4-point functions in terms of the 3-point functions by inserting a complete set of states according to

$$\prod_{a=1}^{4} < |e^{\alpha_a\phi(x_a)}| >= \int d\alpha \prod_{i=1}^{2} < |e^{\alpha_i\phi(x_i)}\Big(|\alpha >< \alpha|\Big) \prod_{j=1}^{2} < e^{\alpha_j\phi(x_j)}| > \tag{5.4}$$

Although the 4-point functions are unknown for most values of the parameters α there are some special values for which they are known, and Teschner [11] has shown that for these values (5.4) is true only if C_3 satisfies the DOZZ Ansatz.

6. 6. Duality

Once the DOZZ Ansatz is accepted one sees that the three point coefficient has the very simple reflection property just mentioned, namely

$$G_N(x_1, x_a, q - \alpha_1, \alpha_a) = R(\alpha_1)G_N(x_1, x_a, \alpha_1, \alpha_a) \tag{6.1}$$

and similarly for $\alpha_{2,3}$. This shows that the symmetry $\alpha \to q - \alpha$ found previously for the Virasoro centre and the primary fields extends to the three-point functions and hence, by the Teschner argument, to the N-point functions.

The most important result, however, is that, although we started off from the residues at the poles $\xi = -mb$ of the zero-mode integral, the fluctuating part of the integral yields a dual set of poles, namely those at the points $\xi = mb + nc$. The invariance of this lattice of poles under the transformation $\xi \to q - \xi$ extends the duality found earlier for the Virasoro centre and primary fields to the N-point functions and hence to the entire theory. This is the ultimate expression of the quantum duality mentioned in the title.

A minor problem is that the dual property is not shared by the zero-mode part of the path-integral, whose factor $\Gamma(-\xi/b)$ has poles only at $\xi = mb$. This means that there is a discrepancy between the zero-mode part of the path-integral and the rest. The DOZZ conjecture removes this discrepancy, but in a rather ad hoc way, by replacing Γ-function by a function with a double set of poles. More precisely, the conjecture replaces

$$\Gamma(-\xi/b)\Gamma(\xi/b+1) \equiv \frac{1}{\sin(\xi/b)} \quad \text{with poles at} \quad \xi = mb \qquad m \in Z \tag{6.2}$$

by

$$k^{-1}(-\xi) \quad \text{with poles at} \quad \xi = mb \quad \text{and} \quad \xi = nc \qquad m,n \in Z \tag{6.3}$$

modulo $m = 0, n \in Z_+$ and $n = 0, m \in Z_+$. This is rather unsatisfactory since it is ad hoc and in section (8) we shall propose a path integral for which the conjecture is automatic and thus the discrepancy is removed.

7. Two-Point Function

The fact that there are two fields $e^{\alpha\phi(x)}$ and $e^{(q-\alpha)\phi(x)}$ for each conformal weight $\Delta_\alpha = \alpha(q-\alpha)$ suggests that there might be two states for each conformal weight also, which would be an undesirable feature of the theory. We show that this is not the case by computing the two point function and interpreting it in the usual way as the inner-product of two states. As mentioned above the two point function can not be computed directly because of the SL(2,C) invariance of the integral but it can be obtained as the limit

$$< e^{\alpha_1\phi(x_1)} e^{\alpha_2\phi(x_2)} >= \mathrm{Lt}_{\alpha_3=i0} < e^{\alpha_1\phi(x_1)} e^{\alpha_1\phi(x_1)} e^{\alpha_1\phi(x_1)} > \tag{7.1}$$

of the 3-point function, in which case it is

$$G_2(x_1,x_2,\alpha_1,\alpha_2) = |x_1-x_2|^{2(\Delta_1+\Delta_2)}\Big(\frac{k(\xi)}{k(-\xi)}\xi + \eta\Big)\delta(\Delta_1-\Delta_2) \qquad \begin{matrix} \xi = q-\alpha_1-\alpha_2 \\ \eta = \alpha_1-\alpha_2 \end{matrix} \tag{7.2}$$

This has all the correct properties of a two-point function and if we interpret it as the inner product between states of weight $|x,\alpha>$ we find that

$$\begin{pmatrix} < x,\alpha|y,\alpha > & < x,\alpha|y,q-\alpha > \\ < x,q-\alpha|y,\alpha > & < x,q-\alpha|y,q-\alpha > \end{pmatrix} = \begin{pmatrix} 1 & R(\alpha) \\ R^{-1}(\alpha) & 1 \end{pmatrix} |x-y|^{4\Delta} \tag{7.3}$$

where $\Delta = \Delta_1 = \Delta_2$ and $R(\alpha)$ is the reflection coefficient defined in section (5.3). The matrix in (7.3) is diagonalizable and has a zero eigenvalue. Accordingly, there is a state which has zero norm and decouples, namely $|x,\alpha> - R^{-1}(\alpha)|x,q-\alpha>$. Thus effectively there is only one state for each conformal weight. At first sight this may seem surprising but actually it is only the two-point manifestation of the reflection invariance

$$< x, \alpha | x_a, \alpha_a \chi >= R(\alpha) < x, q-\alpha | x_a, \alpha_a \chi > \qquad a = 2...N \tag{7.4}$$

of all N-point functions implied by (6.1).

8. Generalized Proposal

A way out of this zero-mode difficulty is to incorporate the double pole prescription into the theory from the beginning [12] as follows: Instead of the Liouville theory being defined as a theory of a single scalar field with an exponential potential it can be defined as a conformal invariant theory of a single scalar field, without any a priori prejudice as to the form of the potential. In the classical theory the fact that the potential should have conformal weight $(1,1)$ limits the potential to being the usual exponential one, but in the quantum theory the fact that there are two primary fields for each conformal weight permits the potential to be a sum of these two primary fields with related coefficients i.e. permits the path-integral to take the form

$$\int d\tilde{\phi} e^{i \int d^2x \left(\frac{1}{2}(\partial\tilde{\phi})^2 + \mu_b e^{2b\tilde{\phi}(x)} + \mu_c e^{2c\tilde{\phi}(x)} + qR\tilde{\phi} \right)} \tag{8.1}$$

where it can be verified that the necessary and sufficient condition for weyl invariance is

$$q = b + c \qquad bc = 1 \tag{8.2}$$

So a theory with path-integral (8.1) is undoubtedly a conformally invariant theory of a single scalar field. In fact, such a potential was already suggested some years ago [13]. With such a potential everything concerning the double-pole structure falls into place. In particular

(I) The fluctuating part of the path integral gives exactly the same result as for the one-exponential theory (for $bc = 1$). This is quite remarkable because the form of the integral is much more complicated.

(II) The zero-mode contribution to the path integral exhibits the required double-pole structure. More precisely it is of the form

$$\frac{1}{\sin(\pi\xi/b)} = b \int du \frac{\delta(\xi - bu)}{\sin(\pi u)} \quad \rightarrow \quad \int du dv \frac{\delta(\xi - bu - cv)}{\sin(\pi u)\sin(\pi v)} = l(\xi) \tag{8.3}$$

where $l(\xi)$ is the principal part of the Barnes di-logarithmic function. In fact $l(\xi)$ is a natural partner of the function $k'(-\xi)/k(-\xi)$ in the sense that it has simple poles at the same points, with the same residues 1 for positive ξ but the opposite residues -1 for negative ξ. In other words it is even with respect to the symmetry $\alpha \rightarrow q - \alpha$ where $k'(\xi)/k(\xi)$ is odd.

(III) The DOZZ conjecture is automatically satisfied. This happens as follows: From the previous considerations it is clear that the coefficient of the 3-point function can in any case be written as

$$h(\xi)\Big[\frac{k(2\alpha_1)k(2\alpha_2)k(2\alpha_2)}{k(-\xi)k(\alpha_1+\alpha_2-\alpha_3)k(\alpha_1+\alpha_3-\alpha_2)k(\alpha_2+\alpha_3-\alpha_1)}\Big] \tag{8.4}$$

where the quantity in brackets is just that conjectured by DOZZ and $h(\xi)$ is a function which is unity at the residues, $h(mb+nc)=1$, and whose extrapolation to other points is to be determined. The point is that, from the translational invariance of the measure there is a sum rule of the form

$$\int d^2x_N\Big(b\mu_b\lambda^b G_{N+1}(x_I,x_N,\alpha_I,b)+c\mu_c\lambda^c G_{N+1}(x_I,x_N,\alpha_I,c)\Big)=\xi G_N(x_I,\alpha_I) \tag{8.5}$$

relating N and $N+1$ point functions, and by applying this relation to the 2- and 3-point functions, using also the Weyl condition and renormalization group equation, this sum rule implies that the function $h(\xi)$ is periodic with respect to b and c. Since (for incommensurate real b and c) the only such function is a constant, this implies $h(\xi)$ must be constant, in which case its value at the poles shows that it is unity. In the one-exponential case the same argument would imply only that $h(\xi)$ is periodic in b.

9. Conclusion

The manner in which the quantum Liouville theory exhibits a dual symmetry which is absent in the classical theory has been described within the path integral formalism. The duality is linked to the fact that the primary fields $e^{\alpha\phi(x)}$ and $e^{(q-\alpha)\phi(x)}$ both have the same conformal weight $\Delta_\alpha=\alpha(q-\alpha)$ and appears in the Virasoro algebra and the N-point functions. The duality can be incorporated naturally in the theory from the beginning by using a potential of the form $\mu_b e^{b\phi(x)}+\mu_c e^{c\phi(x)}$ where $bc=1$ in the path integral.

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