

# Parameterized Absolute Parallelism: A Geometry for Physical Applications\*

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## Abstract

Absolute parallelism (AP) geometry is frequently used for physical applications. Although it is wider than the Riemannian geometry, it has two main defects. The first is that its path equation does not represent physical trajectories of any test particle. The second is the identical vanishing of its curvature tensor. The present work shows that parameterizing this geometry would solve the two problems. Furthermore, the resulting parameterized (AP)-structure is more general than both the conventional (AP)-structure and the Riemannian structure. Also, it is shown that it can be reduced to one or the other, of these two geometric structures, in some special cases. The structure obtained is more appropriate for physical applications, especially in constructing field theories gauging gravity.

## 1. Introduction

Geometrization of physics is an important philosophy introduced by Albert Einstein at the beginning of the twentieth century. It represents, together with the quantization philosophy, the main two ideas of fundamental physics of the century. It proved to be very successful in describing and interpreting gravitational interaction. During the first half of the century, and until the end of his life, Einstein tried to include other interactions in his scheme, especially the electromagnetic one, in a series of attempts known as *Unified Field Theories*. Unfortunately, all these attempts were not successful or incomplete.

On the other hand, in the second half of the century, several attempts have been done to unify the four interactions known to physicists, so far, using the quantization philosophy. A partial success is achieved in the trial known as *The Standard Model* of Glashow,

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Weinberg and Salam (cf. [1]), which unifies weak and electromagnetic interactions. The success of this trial encouraged authors to rush in this direction. Another trial, of less success, is done to include the strong interaction together with weak and electromagnetic interactions, known as *Grand Unified Theories* (cf. [1]). Extensive efforts have been done to include gravity in this scheme. The problem is that, it seems very difficult or probably impossible to unify gravity with the other interactions using the conventional quantization scheme. It is not convincing that the continuation of the attempts in this direction will bring us more nearer to the solution of the problem.

The situation now is either to suggest a third philosophy or to return back and reexamine carefully the geometrization scheme used to construct general relativity (GR). In the present work, I am going to follow the easier way, i.e. to reexamine the geometrization scheme. Consequently, I am going to suggest a modified version of *Absolute Parallelism* (AP) geometry more appropriate for physical applications than the conventional version.

## 2. Scheme of Geometrization

Geometrization philosophy can be summarized in the following statement: *"to understand nature one should start with geometry and end with physics"*. The theory of GR is a successful application of this philosophy. The application of this philosophy implies the following scheme. It contains the following criteria:

- 1- A certain geometry is chosen to represent a model for nature. It should be wide enough to accommodate all required physical quantities.
- 2- Building blocks in this geometry should be of one type.
- 3- Spaces of the chosen geometry should be affinely connected in order to guarantee general covariance of mathematical expressions.
- 4- A one-to-one correspondence is set between geometric objects and physical quantities.
- 5- Identities in the chosen geometry represent laws of nature.
- 6- Curves or paths in this geometry represent trajectories of test particles.
- 7- A theory in which matter is not represented by a phenomenological material-energy tensor is much preferable [2].
- 8- A theory in which different interactions do not enter as logically distinct structures would be much preferable [3].

To illustrate the above criteria, we use the most successful application of the geometrization scheme, GR in free space. For the first criterion, the geometry chosen for this application is the Riemannian geometry. It is just sufficient for the description of gravitational interaction. The building block of the geometry is the metric tensor only. All other geometric objects are constructed using this tensor. The affine connection defined is Christoffel symbol, and consequently Riemannian spaces are affinely connected. The Bianchi identity is used to represent conservation, as a law of nature. Geodesic and null-geodesic paths are used to represent, successfully, trajectories of test particles and of photons respectively. There is no phenomenological material-energy tensor imposed on the theory, and the sources of the gravitational field arise as constants of integration of

the field equations. The eighth criterion is not applicable in this case since the theory is for gravitational interaction only. One can classify theories constructed using the above scheme, in which the eight criteria are satisfied, as *"Pure Geometric Field Theories"*.

Einstein spent the last thirty years of his life trying to construct geometric field theories, in two main series of attempts to unify gravity with electromagnetism. These attempts are known in the literature as *"Unified Field Theories"*. In the first series of these attempts, that we call *"Einstein's Absolute Parallelism Theory"* (EAP), he used the geometry of absolute parallelism [4] to construct a theory unifying gravity and electromagnetism. For the same reason he started the second series [3], known as *"Einstein's Non-Symmetric Theory"* (ENS), using another type of non-symmetric geometry in which he dropped the symmetry conditions from the metric tensor and from the affine connection. Table 1 summarizes the application of the geometrization scheme, mentioned above, in the two series of attempts compared to GR.

**Table 1.** Are the criteria of the geometrization scheme satisfied?

Criterion	GR	EAP	ENS
1-Geometry	Yes, Riemannian	Yes, AP	Yes, Non-symmetric
2-Building Blocks	Yes, $g_{(\mu\nu)}$	Yes, $\lambda_i^\mu$	(?), $g_{\mu\nu}$
3-Affine Connection	Yes, $g_{\mu\nu;\alpha} = 0$	Yes, $\lambda_{i+ \nu}^\mu = 0$	Yes, $g_{+ - \alpha}^{\mu\nu} = 0$
4-Correspondence	Yes	Yes	Yes
5-Identities	Bianchi	(?)	Generalized Bianchi
6- Paths and Motion	Yes, Geodesic	No	No, in general.
7-Exclusion of Phenomenological Objects	Yes, Free space	Yes	Yes
8-Unique Structure	Yes	Yes	No

From the above table it is clear that all the criteria of the geometrization scheme are satisfied in the case of GR in free space.

The EAP-theory [4] has been found to be unsatisfactory since it did not produce the Schwarzschild field, in the case of spherical symmetry, when the electromagnetic field vanished. Also, it has been found, as shown in Table 1, that autoparallel paths do not represent trajectories of any known particles. Furthermore, some authors (e.g. [5], [6]) claim that the AP-geometry is a flat one, since a curvature tensor, defined in a certain way, vanishes. The last two objections indicate, in my opinion, the incompleteness of the version of the AP-geometry used in constructing the theory. I am going to stress on these objections, and to give them more attention in the next Section.

For the ENS-theory [3], most of the criteria are satisfied. one objection is that the metric tensor is non-symmetric. It is made of two parts: one is symmetric and the other is skew. The skew part has no contribution to the quadratic expression giving the metric of space. The term containing the skew part, in this expression, vanishes identically. So the geometry may be considered as a symmetric one to which one can add any second order skew tensor. Then if the symmetric part is related to the gravitational field, as usually done, and the skew part to the electromagnetic field, so the two fields enter as logically two distinct structures. This violates the 8th criteria. Another objection, of less importance, is that the non-symmetric connection in this geometry is defined using a metricity condition:

$$g_{+|\alpha}^{\mu \nu} = 0, \tag{1}$$

whose solution, in this case, gives a complicated expression which makes subsequent calculations tedious. A third objection, against this geometry, is that paths of this geometry do not, in general, represent trajectories of test particles except for some restricted cases (cf. [7]).

Several authors tried to construct field theories using the above mentioned geometries. Although their attempts have satisfied many of the criteria mentioned above, yet these attempts violate others. Some of these attempts are listed in Table 2, in an ascending order of the year of publication in the first column. The second column of this table gives the aim of the trial. The headings of the last eight columns give the criterion number, in the scheme of geometrization given above. The table shows how many of these criteria are satisfied by the listed attempts.

### 3. Why Absolute Parallelism?

Before giving an answer to this question, I will review briefly the fundamental bases of the conventional AP-geometry. An AP- space is an n-dimensional space each point of which is labeled by a set of variables  $x^\nu$  ( $\nu = 1, 2, 3, \dots, n$ ). At each point we define n-contravariant vectors  $\lambda_i^\mu$  ( $\mu = 1, 2, 3, \dots, n$  stands for the coordinate components and  $i = 1, 2, 3, \dots, n$  stands for the vector number). We are going to use Latin indices for vector numbers and

**Table 2.** Confrontation between Geometrization attempts and Criteria

Trial (year)	Aim	1	2	3	4	5	6	7	8
Levi-Civita(1950) [8]	Simplification of EAP	✓	✓	✓	✓	x	x	x	✓
Mikhail(1952) [9]	Creation of Matter	✓	✓	✓	x	x	x	x	✓
Møller (1961) [10]	Energy Localization	✓	✓	✓	x	✓	x	x	✓
Mikhail(1964) [11]	Unification	✓	✓	✓	x	x	x	✓	✓
Mikhail&Wanas(1977) [12]	Unification	✓	✓	✓	✓	✓	x	✓	✓
Møller (1978) [13]	Singularity Free	✓	✓	✓	x	✓	x	x	✓
Hayashi&Shirafuji(1979) [5]	Microscopic Gauge Gravity	✓	✓	✓	?	?	?	x	✓
Hammond(1990) [14]	Dynamical Torsion	✓	x	✓	✓	?	x	x	x
Hammond(1995) [15]	Dirac Field & GR	✓	x	✓	✓	x	?	x	x
Moffat(1995) [16]	Non-Symmetric Gravity	✓	✓	✓	?	✓	?	x	✓
Hammond(1998) [17]	Geometrization of Strings	✓	x	✓	✓	x	?	x	x

✓ = Criterion satisfied, x = Criterion not satisfied, ? = Not clear

Greek indices for the coordinate components. These vectors are subject to the condition:

$$\lambda_{i+|\nu}^{\mu} = 0, \tag{2}$$

which is the absolute parallelism condition. One can define a set of n- covariant vectors, conjugate to  $\lambda_i^{\mu}$  such that,

$$\lambda_i^\mu \lambda_j^\mu = \delta_{ij}, \quad (3)$$

$$\lambda_i^\mu \lambda_\nu^\mu = \delta_{\nu}^i. \quad (4)$$

Summation convention will be applied to both types of indices. Using these vectors we can define the following 2nd order symmetric tensor:

$$g_{\nu\mu} \stackrel{\text{def}}{=} \lambda_\nu^\mu \lambda_\mu^\nu, \quad (5)$$

$$g^{\nu\mu} \stackrel{\text{def}}{=} \lambda_i^\nu \lambda_i^\mu. \quad (6)$$

One can then use  $g_{\mu\nu}$  to play the role of the metric tensor of the Riemannian geometry, and so it can be used with its conjugate to lower and raise Greek indices respectively.

Affine Connections: In this geometry we have more than one affine connection. A non-symmetric connection is immediately obtained as a solution of the AP-condition given by (2),

$$\Gamma_{\cdot\mu\nu}^\alpha \stackrel{\text{def}}{=} \lambda_i^\alpha \lambda_{\mu,\nu}^i. \quad (7)$$

Also its dual behaves as an affine connection under the group of general coordinate transformations. This connection is written in the form:

$$\Gamma_{\cdot\mu\nu}^\alpha \stackrel{\text{def}}{=} \Gamma_{\cdot\nu\mu}^\alpha. \quad (8)$$

Since (7) is non-symmetric, its symmetric part,

$$\Gamma_{\cdot(\mu\nu)}^\alpha \stackrel{\text{def}}{=} \frac{1}{2}(\Gamma_{\cdot\mu\nu}^\alpha + \Gamma_{\cdot\nu\mu}^\alpha), \quad (9)$$

behaves as an affine connection. Now since (5) is defined as a metric tensor, then as a consequence of a metricity condition,

$$g_{\mu\nu}{}_{;\sigma} = 0, \quad (10)$$

we can define Christoffel symbol in the usual manner,

$$\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} \stackrel{\text{def}}{=} \frac{1}{2}g^{\alpha\sigma}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}). \quad (11)$$

It is worth of mention that the symmetric part of the non-symmetric connection (9) is different from Christoffel symbol. We can define using this connection a third order skew tensor, known as the torsion tensor,

$$\Lambda^{\alpha}_{\cdot\mu\nu} \stackrel{\text{def}}{=} \Gamma^{\alpha}_{\cdot\mu\nu} - \Gamma^{\alpha}_{\cdot\nu\mu} = -\Lambda^{\alpha}_{\cdot\nu\mu}. \quad (12)$$

Consequently, one can define another non-symmetric connection by adding (11) to (12),

$$\Omega^{\alpha}_{\cdot\mu\nu} \stackrel{\text{def}}{=} \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + \Lambda^{\alpha}_{\cdot\mu\nu}, \quad (13)$$

from which it is clear that its symmetric part is Christoffel symbol. The dual of (13) is also an affine connection, given by

$$\hat{\Omega}^{\alpha}_{\cdot\mu\nu} \stackrel{\text{def}}{=} \Omega^{\alpha}_{\cdot\nu\mu}, \quad (14)$$

So, in the AP-geometry one can define at least six quantities which behave as affine connections under the effect of the group of general coordinate transformations. Some of these quantities are symmetric (9) and (11), while others are non-symmetric. (7), (8), (13) and (14).

**Absolute Derivatives:** Using the above defined affine connections, one can define the following absolute derivatives:

$$A^{\mu}_{+|\nu} = A^{\mu}_{\cdot,\nu} + A^{\alpha} \Gamma^{\mu}_{\cdot\alpha\nu}, \quad (15)$$

$$A^{\mu}_{-|\nu} = A^{\mu}_{\cdot,\nu} + A^{\alpha} \Gamma^{\mu}_{\cdot\alpha\nu}, \quad (16)$$

$$A^{\mu}_{0|\nu} = A^{\mu}_{\cdot,\nu} + A^{\alpha} \Gamma^{\mu}_{\cdot(\alpha\nu)}, \quad (17)$$

$$A^{\mu}_{+||\nu} = A^{\mu}_{\cdot,\nu} + A^{\alpha} \Omega^{\mu}_{\cdot\alpha\nu}, \quad (18)$$

$$A^{\mu}_{-||\nu} = A^{\mu}_{\cdot,\nu} + A^{\alpha} \hat{\Omega}^{\mu}_{\cdot\alpha\nu}, \quad (19)$$

$$A^{\mu}_{0||\nu} = A^{\mu}_{\cdot,\nu} + A^{\alpha} \Omega^{\mu}_{\cdot(\alpha\nu)}, \quad (20)$$

where  $A^{\mu}$  is any contravariant vector. The last derivative is equivalent to the conventional covariant derivative since the symmetric part of the connection  $\Omega^{\mu}_{\cdot\alpha\nu}$  is Christoffel symbol, as stated above.

**Basic Vector:** We can define a third order tensor, called the contortion, as

$$\gamma^{\alpha}_{\cdot\mu\nu} \stackrel{\text{def}}{=} \lambda^{\alpha}_{\cdot i} \lambda^{\mu}_{\cdot i\nu}. \quad (21)$$

It can be easily shown that

$$\Gamma^{\alpha}_{\cdot\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + \gamma^{\alpha}_{\cdot\mu\nu}, \quad (22)$$

$$\Lambda^{\alpha}_{\cdot\mu\nu} \stackrel{\text{def}}{=} \gamma^{\alpha}_{\cdot\mu\nu} - \gamma^{\alpha}_{\cdot\nu\mu} = -\Lambda^{\alpha}_{\cdot\nu\mu}, \quad (23)$$

$$\gamma_{,\mu\nu}^{\alpha} \stackrel{\text{def}}{=} \frac{1}{2}(\Lambda_{\cdot\mu\nu}^{\alpha} - \Lambda_{\mu,\nu}^{\alpha} - \Lambda_{\nu,\mu}^{\alpha}). \quad (24)$$

Contracting we can define a covariant basic vector,

$$C_{\mu} \stackrel{\text{def}}{=} \Lambda_{\cdot\mu\alpha}^{\alpha} = \gamma_{\cdot\mu\alpha}^{\alpha}. \quad (25)$$

Path Equation: Historically, it is known that the only path defined in the AP-geometry is the autoparallel path given by:

$$\frac{dA^{\mu}}{dS} + \Gamma_{\cdot\alpha\beta}^{\mu} A^{\alpha} A^{\beta} = 0. \quad (26)$$

One of the reasons for which EAP-theory and the AP-geometry were neglected for about twenty years (from Robertson's paper [18] in 1932 to Mikhail's Ph.D. [9] in 1952) is that this path does not represent trajectories of any known particles. However, recently [19] it is shown that the AP-geometry admits other paths, whose equations can be written in the form:

$$\frac{dV^{\mu}}{dS^{+}} + \{\mu_{\alpha\beta}\} V^{\alpha} V^{\beta} = -\Lambda_{(\alpha\beta)\cdot}^{\mu} V^{\alpha} V^{\beta}, \quad (27)$$

$$\frac{dW^{\mu}}{dS^0} + \{\mu_{\alpha\beta}\} W^{\alpha} W^{\beta} = -\frac{1}{2}\Lambda_{(\alpha\beta)\cdot}^{\mu} W^{\alpha} W^{\beta}, \quad (28)$$

$$\frac{dU^{\mu}}{dS^{-}} + \{\mu_{\alpha\beta}\} U^{\alpha} U^{\beta} = 0, \quad (29)$$

where  $S^{+}$ ,  $S^0$  and  $S^{-}$  are the evolution parameters characterizing the three paths respectively ; and  $V^{\alpha}$ ,  $W^{\alpha}$  and  $U^{\alpha}$  are the tangents to the corresponding paths . We will discuss this set of equations at the end of this Section and in the next one.

Curvature Tensors: In general, there are at least two different methods to define curvature tensors in any affinely connected geometry.

*The First Method:* In this method we simply replace Christoffel symbol, in the definition of Riemann-Christoffel tensor, by the non-symmetric connection (7), then we get:

$$B_{\cdot\mu\nu\sigma}^{\alpha} = \Gamma_{\cdot\mu\sigma,\nu}^{\alpha} - \Gamma_{\cdot\mu\nu,\sigma}^{\alpha} + \Gamma_{\epsilon\nu}^{\alpha}\Gamma_{\cdot\mu\sigma}^{\epsilon} - \Gamma_{\cdot\epsilon\sigma}^{\alpha}\Gamma_{\cdot\mu\nu}^{\epsilon}. \quad (30)$$

Unfortunately the curvature tensor, defined in this way, vanishes identically because of the AP-condition (2). Some authors believe that Ap-space is a flat space because of the vanishing of this curvature. There is no convincing reason for which authors choosing a certain connection, (7), while neglecting others, then claiming that the space is flat! I will show, in the next Section, that the space is not flat and (30) can be considered as one of the advantages of the AP-geometry.

*The Second Method:* An alternative method, for defining curvature tensors, is to consider it as a measure of non-commutation of the absolute derivatives given above. To make calculations easier, and the geometry self consistent, it is better to use the



contravariant components of the tetrad vectors in the definition of the following curvature tensors [12], [20],

$$\lambda_i^+|\nu\sigma - \lambda_i^+|\sigma\nu = \lambda_i^\alpha B_{.\alpha\nu\sigma}^\mu, \quad (31)$$

$$\lambda_i^-|\nu\sigma - \lambda_i^-|\sigma\nu = \lambda_i^\alpha L_{.\alpha\nu\sigma}^\mu, \quad (32)$$

$$\lambda_i^0|\nu\sigma - \lambda_i^0|\sigma\nu = \lambda_i^\alpha N_{.\alpha\nu\sigma}^\mu, \quad (33)$$

$$\lambda_i^+||\nu\sigma - \lambda_i^+||\sigma\nu = \lambda_i^\alpha M_{.\alpha\nu\sigma}^\mu, \quad (34)$$

$$\lambda_i^-||\nu\sigma - \lambda_i^-||\sigma\nu = \lambda_i^\alpha K_{.\alpha\nu\sigma}^\mu, \quad (35)$$

$$\lambda_i^0||\nu\sigma - \lambda_i^0||\sigma\nu = \lambda_i^\alpha R_{.\alpha\nu\sigma}^\mu. \quad (36)$$

Here again the curvature defined by (31) vanishes identically because of the AP- condition. The tensor defined by (36) is the Riemann-Christoffel curvature tensor of the associated Riemannian space, since the symmetric part of the connection  $\Omega_{.\alpha\nu}^\mu$  is Christoffel symbol as stated above.

It is worth of mention that the two methods, in the case of Riemannian geometry, give identical results, since the affine connection in this case is unique in this geometry.

Now, we answer the question given in the title of the present Section: Why absolute parallelism? Recalling that the structure of any AP-space is defined completely, in four dimensions, by a tetrad vector field subject to the condition (2), we can summarize the advantages of using the AP-geometry, in the geometrization scheme, in the following points:

- 1- Calculations using this geometry are more easier than those using Einstein's non-symmetric geometry. All tensors of different orders, and affine connections, are explicitly defined by relatively simple expressions.
- 2- The AP-geometry is more wider than the Riemannian one. It admits tensors of third orders (21)&(23), a basic vector (25) and a number of second order skew and symmetric tensors. It also admits more than one affine connection, some of which are symmetric and others are non- symmetric.
- 3- This type of geometry admits non-vanishing torsion (12). Recently, it is shown that torsion is necessary to couple Dirac field to gravity [15]. Also, it is suggested that torsion is necessary to geometrize strings [17].Furthermore, it appears that gauge formulation of gravity needs non-vanishing torsion [15].
- 4- Tetrads, defining the structure of the AP-space, are used as fundamental variables in attempts to quantize gravity [15].
- 5- A metric tensor is defined, in the AP-spaces, whenever needed. In other words, there is always a Riemannian space associated with any AP-space. This facilitates comparison between any field theory written in the AP-space and GR.

- 6- The use of the AP-geometry helped in solving some of the problems of GR, e.g. the problem of localization of gravitational energy [10].
- 7- AP-geometry admits a number of path equations (27), (28), (29) [19], in which the effect of the torsion appears explicitly.
- 8- Quantum features are recently discovered in such type of geometry [22].
- 9- The AP-condition (2) is necessary to describe the dynamics of spinning particles [23].
- 10- Using the tetrad vectors, one can always associate a set of scalars with each tensor defined in the AP-geometry (cf. [24]).

#### 4. Physical Needs for Parameterizing AP-Geometry

We have at least two convincing physical reasons for parameterizing this type of geometry. These two reasons are:

*The First Reason:* Let us examine the structure of the curvature tensor given by (30). As stated before, this tensor vanishes identically because of the AP-condition (2). This tensor can be written in the form,

$$B^{\alpha}_{\cdot\mu\nu\sigma} \stackrel{\text{def}}{=} R^{\alpha}_{\cdot\mu\nu\sigma} + Q^{\alpha}_{\cdot\mu\nu\sigma}, \tag{37}$$

where  $R^{\alpha}_{\cdot\mu\nu\sigma}$  is the Riemann-Christoffel curvature tensor, of the associated Riemannian space, given by,

$$R^{\alpha}_{\cdot\mu\nu\sigma} \stackrel{\text{def}}{=} \left\{ \begin{matrix} \alpha \\ \mu\sigma \end{matrix} \right\}_{\cdot\nu} - \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}_{\cdot\sigma} + \left\{ \begin{matrix} \beta \\ \mu\sigma \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \beta\nu \end{matrix} \right\} - \left\{ \begin{matrix} \beta \\ \mu\nu \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ \beta\sigma \end{matrix} \right\}, \tag{38}$$

and

$$Q^{\alpha}_{\cdot\mu\nu\sigma} \stackrel{\text{def}}{=} \gamma^{\alpha}_{\cdot\mu\sigma}{}^{+}{}_{\cdot+-|\nu} - \gamma^{\alpha}_{\cdot\mu\nu}{}^{+}{}_{\cdot++|\sigma} + \gamma^{\beta}_{\cdot\mu\sigma} \gamma^{\alpha}_{\cdot\beta\nu} - \gamma^{\beta}_{\cdot\mu\nu} \gamma^{\alpha}_{\cdot\beta\sigma}. \tag{39}$$

It is clear from (38) that  $R^{\alpha}_{\cdot\mu\nu\sigma}$  is made from Christoffel symbols only. Also from (39) we can see that  $Q^{\alpha}_{\cdot\mu\nu\sigma}$  is made from the contortion (or the torsion via (24)) only. Some authors believe that  $R^{\alpha}_{\cdot\mu\nu\sigma}$  and  $Q^{\alpha}_{\cdot\mu\nu\sigma}$  are equivalent (cf.[6]). Others consider  $Q^{\alpha}_{\cdot\mu\nu\sigma}$  as giving an alternative definition of  $R^{\alpha}_{\cdot\mu\nu\sigma}$ . Let us examine these two tensors from a different point of view. It is well known that Christoffel symbol is related, in applications, to the gravitational field. So, its existence in (38) indicates that gravity is responsible for the curvature of space-time. The identical vanishing of the curvature  $B^{\alpha}_{\cdot\mu\nu\sigma}$  may indicate that there is another physical interaction (anti-gravity, say) which is related to the contortion (or the torsion) and is represented by the tensor  $Q^{\alpha}_{\cdot\mu\nu\sigma}$ . This interaction balances the effect of gravity in such a way that the total effect vanishes. If so, it is better to call the tensor  $Q^{\alpha}_{\cdot\mu\nu\sigma}$  "The Curvature Inverse of Riemann-Christoffel Tensor". But since gravity dominates our observable universe, which indicates that  $R^{\alpha}_{\cdot\mu\nu\sigma}$  is more effective than  $Q^{\alpha}_{\cdot\mu\nu\sigma}$ , thus one has to parameterize torsion terms in AP- expressions.

*The Second Reason:* The set of equations (27), (28), (29) represents paths, admitted by the AP-geometry, that are different from those of the Riemannian geometry. Although the first equation of this set is similar to the geodesic (or null-geodesic) equation, the other two equations cannot be reduced to the geodesic one, unless the torsion vanishes. It has been shown that the vanishing of the torsion of the AP-space will reduce the space to a flat one [20]. This is because all curvature tensors, defined in the previous Section, can be written in terms of the torsion (or contortion) tensor. Consequently, all of these tensors vanish as a result of the vanishing of the torsion. So, what are the trajectories of particles that can be represented by these two equations? Clearly there are no particles that move along these paths. The reason is that the effect of the Christoffel symbol term, in these equations, is of the same order of magnitude as the effect of the torsion term. So, for these equations to represent physical trajectories, the torsion term in the equations should be parameterized, in order to reduce its effect [25].

### 5. Parameterized AP-Geometry

As it is shown in the previous Section, the two reasons for which we parameterize the geometry are the vanishing of the curvature tensor (30) and the problem of the physical meaning of the set of path equations (27), (28), (29) . As stated in the previous Section, the common factors between these two features are the affine connections. So, it is necessary to start parameterizing these connections first.

Parameterized Connection: One way to parameterize the AP-geometry is to define a general affine connection by linearly combining the affine connections defined in the geometry. In doing so, we get after some manipulations [25]:

$$\nabla_{\alpha\beta}^{\mu} = a_1 \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} + (a_2 - a_3)\Gamma_{\alpha\beta}^{\mu} - (a_3 + a_4)\Lambda_{\alpha\beta}^{\mu} \quad , \quad (40)$$

where  $a_1, a_2, a_3$  and  $a_4$  are parameters. It can be easily shown that  $\nabla_{\alpha\beta}^{\mu}$  transforms as an affine connection under the group of general coordinate transformations. It is clear that this parameterized connection is non-symmetric.

Parameterized Absolute derivatives: If we characterize absolute derivatives, using the connection (40), by a treble stroke, then we can define the following derivatives:

$$A_{+||\nu}^{\mu} \stackrel{\text{def}}{=} A^{\mu}_{,\nu} + A^{\alpha}\nabla_{\alpha\nu}^{\mu}, \quad (41)$$

$$A_{-||\nu}^{\mu} \stackrel{\text{def}}{=} A^{\mu}_{,\nu} + A^{\alpha}\nabla_{\nu\alpha}^{\mu}, \quad (42)$$

$$A_{0||\nu}^{\mu} \stackrel{\text{def}}{=} A^{\mu}_{,\nu} + A^{\alpha}\nabla_{(\alpha\nu)}^{\mu}, \quad (43)$$

where  $A^{\mu}$  is any arbitrary vector. If we need metricity, using the parameterized connection, to be preserved i.e.,

$$g_{+||\sigma}^{\mu\nu} = 0, \quad (44)$$

then one should take,

$$a + b = 1, \tag{45}$$

where  $a = a_1$ ,  $b = a_2 + a_4$ , ( $a_3 = -a_4$ ) are two parameters. Now, due to (45), we have only one parameter. In this case the general affine connection (40) can be written in the form:

$$\nabla_{\cdot\mu\nu}^\alpha = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + b \gamma_{\cdot\mu\nu}^\alpha. \tag{46}$$

It is clear from this equation that we have parameterized the contortion (or equivalently the torsion) term in a general connection of the AP-geometry. Now we will explore the consequences of this parameterization.

Parameterized Path Equation: Using the parameterized connection (46) and following the same approach followed before to get the set (27), (28), (29), we can get the following parameterized path equation admitted by the geometry [25],

$$\frac{dZ^\mu}{d\tau} + \left\{ \begin{matrix} \mu \\ \nu\sigma \end{matrix} \right\} Z^\nu Z^\sigma = -b \Lambda_{(\nu\sigma)\cdot}^\mu Z^\nu Z^\sigma, \tag{47}$$

where  $Z^\mu$  ( $\stackrel{\text{def}}{=} \frac{dx^\mu}{d\tau}$ ) is the tangent to the path and  $\tau$  is the evolution parameter along it. This equation replaces the set given by (27), (28), (29). Discussion of this result is given in the next Section.

Parameterized Curvature Tensors: Using the first method, given in Section 3, for defining curvature tensors we can define the following tensor,

$$\hat{B}_{\mu\nu\sigma}^\alpha \stackrel{\text{def}}{=} \nabla_{\cdot\mu\sigma,\nu}^\alpha - \nabla_{\cdot\mu\nu,\sigma}^\alpha + \nabla_{\cdot\mu\sigma}^\beta \nabla_{\cdot\beta\nu}^\alpha - \nabla_{\cdot\mu\nu}^\beta \nabla_{\cdot\beta\sigma}^\alpha. \tag{48}$$

Using the definition of  $\nabla_{\cdot\mu\nu}^\beta$  given by (40) and applying the metricity condition (45), then we can write,

$$\hat{B}_{\cdot\mu\nu\sigma}^\alpha = R_{\cdot\mu\nu\sigma}^\alpha + b \hat{Q}_{\cdot\mu\nu\sigma}^\alpha, \tag{49}$$

where

$$\hat{Q}_{\cdot\mu\nu\sigma}^\alpha \stackrel{\text{def}}{=} \gamma_{\cdot\mu\sigma}^+{}^\alpha - \gamma_{\cdot\mu\nu}^+{}^\alpha + b (\gamma_{\cdot\mu\sigma}^\beta \gamma_{\cdot\beta\nu}^\alpha - \gamma_{\cdot\mu\nu}^\beta \gamma_{\cdot\beta\sigma}^\alpha). \tag{50}$$

It is clear that the tensor  $\hat{B}_{\cdot\mu\nu\sigma}^\alpha$  is a parameterized replacement of the tensor  $B_{\cdot\mu\nu\sigma}^\alpha$  given in Section 3. But here  $\hat{B}_{\cdot\mu\nu\sigma}^\alpha$  is, in general, non-vanishing.

Using the second method, given in Section 3, for defining curvature tensors we get the following tensors,

$$\lambda_i^\mu |||\nu\sigma - \lambda_i^\dagger |||\sigma\nu = \lambda_i^\alpha W_{\cdot\alpha\nu\sigma}^\mu, \tag{51}$$

$$\lambda_i^\mu |||\nu\sigma - \lambda_i^- |||\sigma\nu = \lambda_i^\alpha L_{\cdot\alpha\nu\sigma}^\mu, \tag{52}$$

$$\lambda_i^\mu |||\nu\sigma - \lambda_i^0 |||\sigma\nu = \lambda_i^\alpha N_{\cdot\alpha\nu\sigma}^\mu, \tag{53}$$

Note that every tensor with a hat is the parameterized replacement of that without a hat. We can show that the tensors given by the second method are more general than those obtained using the first method. For example we can write,

$$W_{\cdot\mu\nu\sigma}^{\alpha} = \hat{B}_{\cdot\mu\nu\sigma}^{\alpha} - b(b-1)\gamma_{\cdot\mu\beta}^{\alpha}\Lambda_{\cdot\nu\sigma}^{\beta}. \tag{54}$$

We are going to discuss this result in the next Section.

## 6. Discussion and Conclusion

Two main problems, concerning gravitational interactions, are now well defined. The first is that quantization of gravity, using conventional quantization schemes, is still a difficult task if not impossible. The second, which may be a consequence of the first, is that unification of gravity with other known interactions is still beyond the reach of investigators. One way to overcome these problems, may be in reexamining carefully the geometrization scheme suggested and applied by Einstein in constructing his theory of GR in free space. We are convinced that this scheme is successful in constructing this theory, but subsequent application of it, is the subject that needs a careful examination.

It as well known that GR in free space is the most satisfactory application of the geometrization scheme as shown in Section 2. Recalling that the first criteria in this scheme is the choice of an appropriate geometry for application; so in order to generalize or modify this theory, one should first look for a geometric structure more wider than the Riemannian one. For this reason Einstein started two series of attempts, to construct unified field theories, using two different geometries. We believe that these attempts are incomplete rather than unsatisfactory. The reasons are probably rooted in the geometric structures used. As shown in Section 2, the geometric structures used are the AP-geometry and the non-symmetric (NS) geometry. Table 1 shows how many of the geometrization criteria, given in Section 2, are satisfied by Einstein's two attempts, compared with GR in free space as a standard theory of gravity. It is to be considered that the version of GR, used for comparison in the present work, is that for free space. It is the theory with less problems than the version written for a material distribution. As stated before, Einstein's two attempts are not successful. The reason may probably be the incompleteness of the geometries used. We mean by a complete geometry, a geometry that satisfies the following requirements:

- i- All tensors, of different orders, should be well defined in terms of the building blocks of the geometry, especially those measuring curvature. Different relations between these objects should be clarified.
- ii- All affine connections, admitted by the geometry, should be known and well studied, with subsequent covariant derivatives defined using these connections.
- iii- Identities, especially those of the differential type, are obtained for the geometry considered.
- iv- Different paths, admitted by the geometry, are obtained.

It is obvious that Riemannian geometry satisfies all these requirements. So, it could be classified as a complete geometry. But it is not wide enough to modify or generalize GR. It is found to be just sufficient to describe gravitational interactions, in free space, with some limits concerning the distance from the source of the field. The present work is a step towards making the AP-geometry as complete as possible. This step is necessary before using this geometry to overcome the problems mentioned above.

Table 2 displays geometrization attempts of some authors. The AP-geometry is used, as a basic structure, in some of these attempts [5], [8], [9], [10], [11], [12] and [13]. The NS-geometry is used in another type of attempt [16]. In the rest of the attempts, listed in the table, [14], [15] and [16] while a tetrad defined in spaces with torsion is used, it is not clear whether the AP- condition (2) is imposed on this tetrad. So, one cannot decide whether these attempts were carried out using a version of the AP-geometry or not.

In Section 3, ten reasons, for preferring AP-geometry for physical applications, are given. However, the conventional version of this geometry suffers from two main problems. The first is the vanishing of its curvature tensor (30), which leads to the conclusion that the geometry is flat. The second is that the path, defined in this version, (26) does not represent physical trajectories of any known particles. In the present work the two problems are solved.

*For The First Problem:* This problem is solved on two levels. It is shown that the AP-space is not flat. On the first level we have shown that the AP-space admits non-vanishing curvature tensors (31)-(36), defined by the second method given in Section 3. On the second level, by parameterizing AP- connection (40), we were able to define a non-vanishing curvature tensor (48) even by using the first method given in Section 3. The curvature tensors, defined using the second method, (51), (52) and (53) are also non-vanishing and more general than those given in Section 3. Thus, the AP-geometry is no longer flat. Even before defining the non-vanishing curvature tensors admitted by the geometry, there were some evidences indicating that this the AP-space is not flat. One of these evidences is that GR can be written in this space with satisfactory solutions (cf. [26]).

*For The Second Problem:* Also, this problem is solved on two levels. On the first, we discovered that the AP-geometry admits other path equations different from (26). These equations are given by the set (27), (28), (29) . This does not solve the problem completely, since none of these equations represent physical trajectories. But this was a necessary step towards the solution, and to make the geometry more complete. On the second level, we parameterized a general path equation admitted by the geometry (47), which is used now to represent physical trajectories in gravitational fields [25], [27]. This step makes the AP- geometry more complete and solves the second problem.

In modifying or generalizing GR in spaces with torsion (e.g. [28]), the resulting theories are required to reduce to GR when torsion vanishes. Such theories would encounter a problem in conventional AP-geometry. The vanishing of the torsion in this case would lead to the vanishing of all curvature tensors of the structure [20]. So, the resulting GR would be a trivial one, and gravity would no longer represented in such structures. These theories are rendered unviable in the conventional version of the AP-geometry. The situation now is different if such theories are written in the parameterized AP-geometry. There

is no need for the torsion to vanish. It is sufficient, in order to reduce the theory to GR, to switch the parameter off, which would correspond to some physical condition. So, the problem of such theories is solved using the parameterized version of the AP-geometry.

We can summarize the conclusion of the present work in the following points.

- 1- It is shown that the AP-geometry is not flat since it admits a number of non-vanishing curvature tensors.
- 2- Paths admitted by the parameterized version of the geometry can be used to study physical trajectories of test particles.
- 3- The parameterization carried out for the AP-geometry make it more complete and more appropriate for physical applications.
- 4- The parameterized version of this geometry is more wider than the Riemannian geometry and the conventional AP-geometry. It can be reduced to the first upon taking  $b = 0$ ,  $a = 1$  in (45), and to the second if we take  $b = 1$ ,  $a = 0$ .
- 5- The parameterized connection represents simultaneously non-vanishing curvature and non-vanishing torsion. This result is the contrary to what obtained by some authors [6].
- 6- The parameterized version of the AP-geometry is more suitable for constructing field theories in spaces with torsion, especially theories gauging gravity.

Finally, it is worth of mention that the geometrization criteria, given in Section 2, are necessary but not sufficient to construct geometric field theories.

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