

Canonical Treatment of Regular Lagrangians with Holonomic Constraints as Singular Systems

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Abstract

Regular Lagrangians with holonomic constraints are treated as singular systems using the canonical method. The Lagrange multipliers are introduced as generalized coordinates. The regular Lagrangians are extended to be singular and the Hamiltonian formulation is obtained. The equations of motion are written as total differential equations in terms of the time t and the Lagrange multipliers. It is also shown that Lagrange multipliers can be determined from the integrability conditions.

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Introduction

The investigation of constrained dynamic systems may be discussed under two basic headings: the investigation of regular Lagrangians with given constraints and the investigation of systems with singular Lagrangians.

The study of regular Lagrangians with holonomic constraint equations, $f_\alpha(q_i, t) = 0, i = 1, 2, \dots, n$ and $\alpha = n + 1, n + 2, \dots, n + m$ are discussed in standard texts [1,2]. Hamilton-Jacobi differential equation of systems with regular Lagrangian and constraint equations has been constructed by Guler [3]. Lagrange multipliers are introduced as generalized velocities and the constraint functions are the generalized momenta conjugate to the Lagrange multipliers.

Singular Lagrangians, however, need considerably more care and somewhat specialized techniques, Since the study of singular Lagrangians, having a degenerate Hessian matrix does not allow all the velocities to be expressed in terms of the coordinates and momenta. The formalism for treating singular systems was initiated by Dirac [4]. He showed that, in the presence of constraints, the number of degrees of freedom of the dynamic system can be reduced. His approach was extended to continuous systems [5]. Other researchers

[6-10] followed Dirac and showed interest in singular field theories.

A powerful approach, the canonical method, has been developed for investigating singular systems [11-13]. In this approach the equations of motion are written as total differential equations and the formulation leads to a set of Hamilton-Jacobi partial differential canonical equations which are familiar in regular systems.

The purpose of the present work is to treat the regular Lagrangians with given holonomic constraints as singular systems. The Lagrange multipliers λ_α are treated as generalized coordinates and the singular Lagrangian is constructed.

In section 2, the Hamiltonian formulation is proposed and three simple examples are given in section 3.

Hamiltonization

The standard method for incorporating the constraint functions to the equations of motion is the use of the so-called Lagrange multipliers. The motion of a holonomic systems could in principle be determined by making use of the n Euler Lagrange Equations,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} \quad , i = 1, 2, \dots, n \tag{1}$$

together with the m constraints

$$f_\alpha(q_i, t) = 0, \quad \alpha = n + 1, n + 2, \dots, n + m \tag{2}$$

where L is regular which is a function of n generalized coordinates q_i and n generalized velocities \dot{q}_i as well as the time t .

Now, let us construct the new Lagrangian by adding the holonomic constraints multiplied by the Lagrange multipliers to the regular Lagrangian i.e.

$$L'(q_i, \lambda_\alpha, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \lambda_\alpha f_\alpha \tag{3}$$

where a repeated suffix assumes a summation over values of that suffix. We introduce the Lagrange multipliers as generalized coordinates. Thus, the new Lagrangian is singular. The Hesse determinant of L' is $(n+m) \times (n+m)$ determinant formed by the partial derivatives of this extended Lagrangian with respect to \dot{q}_i and $\dot{\lambda}_\alpha$. In other words, the following determinant vanishes

$$\begin{vmatrix} \frac{\partial^2 L'}{\partial \dot{q}_i \partial \dot{q}_j} & \frac{\partial^2 L'}{\partial \dot{q}_i \partial \dot{\lambda}_\alpha} \\ \frac{\partial^2 L'}{\partial \dot{\lambda}_\beta \partial \dot{q}_j} & \frac{\partial^2 L'}{\partial \dot{\lambda}_\beta \partial \dot{\lambda}_\alpha} \end{vmatrix} \tag{4}$$

The Euler-Lagrange equations for the extended Lagrangian will be increased by m equations

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}_i} \right) - \frac{\partial L'}{\partial q_i} = 0 \tag{5}$$

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{\lambda}_\alpha} \right) - \frac{\partial L'}{\partial \lambda_\alpha} = 0 \tag{6}$$

Equation (5) leads to Equation (1) while Equation (6) gives the holonomic constraints (2).

The canonical Hamiltonian

$$\mathbf{H}_0 = \dot{q}_i p_i + \dot{\lambda}_\alpha p_\alpha - L' \quad (7)$$

can be calculated using the definition of the canonical momenta

$$P_i = \frac{\partial L'}{\partial \dot{q}_i} \quad (8)$$

$$P_\alpha = \frac{\partial L'}{\partial \dot{\lambda}_\alpha} = 0 \quad (9)$$

Making use of Eqs (8) the generalized velocities can be obtained in terms of generalized coordinates q_i and generalized momenta P_i , i.e.

$$\dot{q}_i = w_i(p_i, q_i, t)$$

Thus, the canonical Hamiltonian H_0 can be written as

$$H_0 = w_i(P_i, q_i, t)P_i - L(q_i, \dot{q}_i \equiv w_i(P_i, q_i, t), t) - \lambda_\alpha f_\alpha(q_i, t) \quad (10)$$

following to the canonical method the set of the Hamilton-Jacobi partial differential equations read as

$$H'_0 = P_0 + H_0 \quad (11)$$

$$H'_\alpha = P_\alpha \quad (12)$$

which can be written in a compact form as

$$\frac{\partial s}{\partial t} + H_0 = 0 \quad (13)$$

$$\frac{\partial s}{\partial \lambda_\alpha} = 0 \quad (14)$$

According to previous publications [11-13], these calculations lead to the following equations of motion which are total differential equations

$$dq_i = \frac{\partial H_0}{\partial P_i} dt + \frac{\partial H'_\alpha}{\partial P_i} d\lambda_\alpha \quad (15)$$

$$dP_i = -\frac{\partial H_0}{\partial q_i} dt - \frac{\partial H'_\alpha}{\partial q_i} d\lambda_\alpha \quad (16)$$

$$dP_\alpha = -\frac{\partial H_0}{\partial \lambda_\alpha} dt - \frac{\partial H'_\alpha}{\partial \lambda_\alpha} d\lambda_\alpha \quad (17)$$

The last equation leads to the holonomic constraints which are total time derivative of the generalized momenta P_α

$$\dot{P}_\alpha = f_\alpha(q_i, t) \quad (18)$$

Since the equations of motion are total differential equations, integrability conditions should be checked. As mentioned in previous publications, equations of motion are integrable if the variations of H'_α vanish identically. If they do not vanish identically we consider them as new constraints. This procedure continues until the Lagrange multipliers are determined. According to these conditions the variations of the holonomic constraints should be equal to zero.

$$df_\alpha(q_i, t) = 0 \quad (19)$$

Making use of equations (15) and (16), equations (19) determine the Lagrange multipliers or lead to new relations between coordinates and momenta. Thus, taking the variations of the new relations one can determine the Lagrange multipliers.

Sample Illustrations

The procedure described in the previous sections will be made clear by discussing the following examples:

1. A disk rolling down an inclined plane

As a first example let us discuss the motion of a disk of radius R that is rolling down an inclined plane without slipping. The Lagrangian of this system reads as:

$$L = \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mgy \sin \Psi \quad (20)$$

where M is the mass of the disk, g is the acceleration of gravity and Ψ is the angle of inclination. The equation of the holonomic constraint giving the relation between the coordinates y and θ is

$$f(y, \theta) = y - R\theta = 0 \quad (21)$$

Thus, the equivalent Lagrangian L' constructed as,

$$L' = L + \lambda(y - R\theta) \quad (22)$$

The canonical momenta are

$$P_y = M\dot{y} \quad (23)$$

$$P_\theta = \frac{1}{2}MR^2\dot{\theta} \quad (24)$$

$$P_\lambda = 0 \quad (25)$$

The Hamiltonian H_0 is calculated as

$$H_0 = \frac{P_y^2}{2M} + \frac{P_\theta^2}{MR^2} - Mgy \sin \psi - \lambda(y - R\theta) \quad (26)$$

Making use of Eqs (15-17) the equations of motion take the form

$$dy = \frac{P_y}{M} dt \quad (27)$$

$$d\theta = \frac{2P_\theta}{MR^2} dt \quad (28)$$

$$dP_y = (Mg \sin \psi + \lambda) dt \quad (29)$$

$$dP_\theta = -\lambda R dt \quad (30)$$

$$dP_\lambda = (y - R\theta) dt \quad (31)$$

The last equation (31) leads to the holonomic constraint (21). According to the integrability conditions the total differential of the holonomic constraints should be equal to zero.

$$df = dy - R d\theta = 0 \quad (32)$$

using equation (27) and equation (28) one gets a new relation between momenta

$$RP_y - 2P_\theta = 0 \quad (33)$$

Taking the total differential of this new relation, and making use of equations (29) and (30), the Lagrange multiplier can be determined as

$$\lambda = -\frac{1}{3} Mg \sin \psi \quad (34)$$

This gives the magnitude of force of constraint resulting from a frictional force, and there are no further constraints.

Using equations (27-30) with the aid of (34) one gets

$$\ddot{y} = \frac{2}{3} g \sin \psi \quad (35)$$

$$\ddot{\theta} = \frac{2}{3} \frac{g}{R} \sin \psi \quad (36)$$

These results are in exact agreement with those obtained using the Euler-Lagrange equations with the aid of Lagrange multipliers.

2. A bead placed on top of a vertical hoop.

As a second example, let us consider a bead of mass M placed at the top of a vertical hoop. The Lagrangian of this system reads as

$$L = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) - Mgr \cos \theta \quad (37)$$

and the holonomic constraint is

$$f(r) = r - R \quad (38)$$

where R is the radius of the hoop. Thus the extended Lagrangian is

$$L' = L + \lambda(r - R) \quad (39)$$

Using the definition (10) the Hamiltonian is calculated as

$$H_0 = \frac{P_r^2}{2M} + \frac{P_\theta^2}{2Mr^2} + Mgr \cos \theta - \lambda(r - R) \quad (40)$$

Thus, the equations of motion are

$$dr = \frac{P_r}{M} dt \quad (41)$$

$$d\theta = \frac{P_\theta}{Mr^2} dt \quad (42)$$

$$dP_r = -(Mg \cos \theta - \frac{P_\theta^2}{Mr^3} - \lambda) dt \quad (43)$$

$$dP_\theta = Mgr \sin \theta dt \quad (44)$$

$$dP_\lambda = (r - R) dt = 0 \quad (45)$$

Eq.(45) gives the holonomic constraints (38). Taking the total differential of $f(r)$ gives

$$dr = 0 \quad (46)$$

The Lagrange multiplier is determined using this condition with the above equations

$$\lambda = Mg \cos \theta - MR\dot{\theta}^2 \quad (47)$$

$$\ddot{\theta} = \frac{g}{R} \sin \theta \quad (48)$$

These equations are the Euler-Lagrange equations and they can be solved to obtain the reaction of the hoop on the bead.

$$\lambda = Mg(3 \cos \theta - 2) \quad (49)$$

3. A bead slides on a helix

Finally, let us discuss the motion of a bead that is free to slide on a smooth wire bent so as to form a helix, the equations of which, in cylindrical coordinates, are $Z = k\theta$ and $r = R$, where k and R are constants. The origin is the center of an attractive force that varies directly with distance. The Lagrangian of this system is

$$L = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2}\alpha(r^2 + z^2) \quad (50)$$

where α is a constant. The holonomic constraints are

$$f_1 = r - R \quad (51)$$

$$f_2 = z - k\theta \quad (52)$$

Thus, the extended Lagrangian is

$$L' = L + \lambda_1(r - R) + \lambda_2(z - k\theta) \quad (53)$$

The canonical Hamiltonian is calculated as

$$H_0 = \frac{P_r^2}{2M} + \frac{P_\theta^2}{2Mr^2} + \frac{P_z^2}{2M} + \frac{1}{2}\alpha(r^2 + z^2) - \lambda_1(r - R) - \lambda_2(z - k\theta) \quad (54)$$

and the equations of motion are

$$dr = \frac{P_r}{M}dt \quad (55)$$

$$d\theta = \frac{P_\theta}{Mr^2}dt \quad (56)$$

$$dz = \frac{P_z}{M}dt \quad (57)$$

$$dP_r = \left(\frac{P_\theta^2}{Mr^3} - \alpha r + \lambda_1\right)dt \quad (58)$$

$$dP_\theta = -\lambda_2 k dt \quad (59)$$

$$dP_z = (-\alpha z + \lambda_2)dt \quad (60)$$

$$dP_1 = (r - R)dt \quad (61)$$

$$dP_2 = (z - k\theta)dt \quad (62)$$

Noting that P_1 and P_2 are the generalized momenta which conjugate to λ_1 and λ_2 . Equations (61) and (62) give the holonomic constraints (51) and (52).

According to the integrability conditions the total differential of these constraints should be equal to zero thus,

$$dr = 0 \quad (63)$$

$$dz - k d\theta = 0 \quad (64)$$

The first condition determines the Lagrange multiplier λ_1 , i.e.

$$\lambda_1 = \alpha R - \frac{P_\theta^2}{MR^3} \quad (65)$$

and the second leads to new relation

$$kP_\theta - R^2P_z = 0 \quad (66)$$

Again, calculating the total differential of this equation we determine the Lagrange multiplier λ_2

$$\lambda_2 = \frac{R^2\alpha}{k^2 + R^2}z \quad (67)$$

Then, the equations of motion (55-60) can be written as,

$$\dot{\theta} = \frac{P_\theta}{MR^2} \quad (68)$$

$$\dot{z} = \frac{P_z}{M} \quad (69)$$

$$\dot{P}_\theta = -\frac{kR^2\alpha}{k^2 + R^2}z \quad (70)$$

$$\dot{P}_z = -\alpha z + \frac{R^2\alpha}{k^2 + R^2}z \quad (71)$$

$$(72)$$

Obviously, solutions of these equations determine the Lagrange multipliers.

Conclusion

Regular Lagrangian with holonomic constraints are investigated by the canonical method. The Lagrange multipliers are treated as generalized coordinates. It seems that our formulation predicts $n+m$ generalized coordinates corresponding to $n+m$ generalized momenta. In other words the phase-space is extended to obtain singular Lagrangian, in $(2n+2m)$ -dimensional phase space.

As a conclusion it is observed that the holonomic constraints are equal to the total time derivatives of the momenta conjugates to the Lagrange multipliers. In addition, the solutions of the three given examples are in complete agreement with the Euler-Lagrange equations.

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