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Elementary charges in classical electrodynamics

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Abstract

In the framework of classical electrodynamics elementary particles are treated as capacitors. The electrostatic potentials satisfy equations of the Schrödinger type. An interesting "quantization condition" for elementary charges is derived.

Introduction

The problem of charges carried by elementary particles is one of the most fundamental problems of physics. More than a half of the century it was not attacked in the framework of classical electrodynamics because it is commonly believed that such kind of problems may be solved only in the framework of quantum electrodynamics. As a result many interesting statements derived in classical physics are completely unknown to the physics community.

Recently, one of us has shown [1] that taking into account the maximal experimental value of the electric field intensity in air

$$E_{max} = 3 \times 10^6 \frac{N}{C} \tag{1}$$

and using the obvious formula for the minimum size of a body with charge Q

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$$R_{crit} = \sqrt{\frac{k|Q|}{E_{max}}}, \tag{2}$$

where

$$k = \frac{1}{4\pi\epsilon_0} \simeq 8.99 \times 10^9 \frac{N.m^2}{C^2}, \tag{3}$$

we get for a body with one electron charge,

$$Q = -1.6021917 \times 10^{-19} C, \tag{4}$$

the unexpected result

$$R_{crit} = 2 \times 10^{-8} m. \tag{5}$$

Clearly, the obtained value of R_{crit} is much larger than the characteristic size of elementary particles which are of the order of 10^{-15} m. This fact shows that classical electrodynamics must be modified at short distances before some quantum effects start to play their role.

Elementary charges as elementary capacitors

In a previous paper [2] we proposed a new model of elementary charges in which these objects are treated as capacitors. We start with the standard equations of electrostatics [3]

$$rot\vec{E}(\vec{x}) = 0 \tag{6}$$

and

$$div\vec{D}(\vec{x}) = \rho(\vec{x}) \tag{7}$$

where $\vec{E}(\vec{x})$ and $\vec{D}(\vec{x})$ are the electric and displacement fields, respectively, while $\rho(\vec{x})$ is the charge density which is the source of the field.

To solve equations (6) and (7) the electrostatic potential $\psi(\vec{x})$ is introduced by the standard formula

$$\vec{E}(\vec{x}) = -\vec{\nabla}\psi(\vec{x}) \tag{8}$$

in conjunction with some kind of constitutive relation specified by the properties of the medium. For homogeneous media we put

$$\vec{D}(\vec{x}) = \epsilon\vec{E}(\vec{x}), \tag{9}$$

where ϵ is the permittivity of the medium. For macroscopic bodies ϵ is usually taken as a known parameter. For charge distributions inside elementary particles we should treat

ϵ as a parameter of the solution because there are no experimental data concerning the electromagnetic properties of the material from which the elementary particles are done.

In terms of the electrostatic potential $\psi(\vec{x})$ the basic equation of electrostatics is [3]

$$\epsilon \nabla^2 \psi(\vec{x}) = -\rho(\vec{x}). \tag{10}$$

In the present paper we shall consider charge distributions which do not vanish only in finite space domains where they are functions of the electrostatic potential. We take therefore the basic equations for the electrostatic potential in the form

$$\epsilon_0 \Delta \psi(\vec{x}) = 0 \quad for \quad [\vec{x}| \geq R \tag{11}$$

and

$$\epsilon \Delta \psi(\vec{x}) = -\rho(\psi(\vec{x})) \quad for \quad [\vec{x}| \leq R, \tag{12}$$

where R is the radius of the charge distribution (in our approach we treat R as an experimentally fixed parameter). It is clear that equation (12) may have different mathematical properties than the Poisson equation (10).

Electrostatic fields generated by sources of the type present in equation (12) such that

$$\rho(\psi(\vec{x}) = 0) = 0 \tag{13}$$

shall be called here **self-induced electrostatic fields.**

In the linear approximation (with respect to ψ, not with respect to $\vec{x}$) the charge densities for self-induced fields are of the form

$$\rho(\vec{x}) = \Omega(\vec{x})\psi(\vec{x}), \tag{14}$$

where $\Omega(\vec{x})$ is some **structure function** specific to the particular distribution of charge in space. For simplicity we shall consider only spherically symmetric charge distributions for which we have

$$\Omega(\vec{x}) \equiv \Omega(|\vec{x}|). \tag{15}$$

Under condition (14) equations (11) and (12) are linear and this is the only justification for that condition because otherwise we shall have non-solvable models. We may apply therefore the superposition principle to the solutions of these equations. For $|\vec{x}| \leq R$ we may look for solutions of equations (11) and (12) in the following more general form

$$\vec{E}(\vec{x}) = -\sum_{n=1}^{N} \vec{\nabla}\psi_n(\vec{x}) \tag{16}$$

and

$$\vec{D}(\vec{x}) = -\sum_{n=0}^{N} \sim \epsilon_n \vec{\nabla}\psi_n(\vec{x}), \tag{17}$$

where ψ_n is the solution of (12) corresponding to $\epsilon = \widetilde{\epsilon}_n$.

Here we have taken into account that the medium in which the charge is distributed may exist in different quantized states. Coefficients $\widetilde{\epsilon}_n$ are the permittivities of the medium in the corresponding states described by potentials ψ_n. The "pure" states, with one selected value of permittivity, are distinguished by the requirement that for such states, like for free space, the displacement vector $\vec{D}$ is parallel to the electric field $\vec{E}$.

For $|\vec{x}| \geq R$ we assume the standard $\vec{E}$ and $\vec{D}$ fields generated by the standard potential

$$\psi(\vec{x}) = \frac{Q}{4\pi\epsilon_0|\vec{x}|}, \tag{18}$$

where Q is the total charge of the system and ϵ_0 is the permittivity of the free space. At the boundary of the charge distribution the solutions outside and inside are glued together by the usual continuity conditions. This means that at distance R the inner solution must satisfy the condition.

$$R\psi'(R) + \psi(R) = 0. \tag{19}$$

In the case when the solution is represented in the form

$$\psi(r) = \frac{u(r)}{r} \tag{20}$$

condition (19) reads

$$u'(R) = 0. \tag{21}$$

Conditions (19) and (21) may be satisfied only for some special values of parameters present in the functional form of $\psi(r)$. These conditions play therefore the role of **quantization conditions.**

The electrostatic energy of the self-induced fields is equal to

$$W = \frac{1}{2}\int_{|\vec{x}|\leq R} \Omega(\vec{x})\psi^2(\vec{x})d^3x. \tag{22}$$

We shall equate this energy to the rest energy of the charged body with mass M (fixed by the experimental data). This leads to the following normalization condition for the self-induced fields:

$$\int_{|\vec{x}|\leq R} \Omega(\vec{x})\psi^2(\vec{x})d^3x = 2Mc^2. \tag{23}$$

Obviously, this condition may be satisfied only if the function $\Omega(\vec{x})$ satisfies some positivity condition. The strongest form of such a condition is

$$\Omega(\vec{x}) \geq 0. \tag{24}$$

The total charge which generates the self-induced field is given by the expression

$$Q = \int_{|\vec{x}|\le R} \Omega(\vec{x})\psi(\vec{x})d^3x. \tag{25}$$

On the other hand, the charge contained in the sphere of radius r is given by

$$Q_r = 4\pi r^2 D_r(r) = \begin{cases} Q & for \quad r \ge R \\ -4\pi r^2 \sum_{n=1}^{N} \overset{\sim}{\epsilon}_n \, \psi_n'(r) & for \quad r \le R \end{cases} \tag{26}$$

In this paper we shall discuss self-induced fields for which the structure function $\Omega(\vec{x})$ is of the form

$$\Omega(\vec{x}) = A - B(\vec{x}), \tag{27}$$

where A is some constant and $B(\vec{x})$ is some "structure potential". Under this assumption the basic equation of electrostatics of self-induced fields (12) takes the form of the **Schroedinger-type equation**

$$-\epsilon\Delta\psi(\vec{x}) + B(\vec{x})\psi(\vec{x}) = A\psi(\vec{x}). \tag{28}$$

Simple example

It is customary in physics to test all new ideas on simple examples. In our case we choose the simplest "structure functions" for which we may expect that the positivity condition for energy is satisfied and for which the corresponding Schroedinger equation for the electrostatic potential may be solved explicitly.

In the present paper we shall consider only-solutions of equation (28) because we want to exclude charge distributions with higher moments such as dipole or quadruple. This means that in spherical coordinates r, θ, φ we choose solutions which depend only on the variable r. As a result for $r \le R$ we obtain the radial equation

$$-\epsilon \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\psi(r)\right) + B(r)\psi(r) = A\psi(r). \tag{29}$$

As a first example we shall consider the almost trivial case for which

$$B(\vec{x}) = 0 \tag{30}$$

$$A > 0. \tag{31}$$

The solution to equation (29), bounded and normalized according to (23), is given by

$$\psi(k;r) = \sqrt{\frac{Mc^2}{\pi AR}}\frac{\sin kr}{r}, \tag{32}$$

where k is a real number such that

$$\epsilon = \frac{A}{k^2}. \tag{33}$$

This solution will continuously match the outer solution (18) provided the total charge contained in the internal domain is equal to

$$Q = 4c\epsilon_0\sqrt{\frac{\pi M}{AR}}\sin kR. \tag{34}$$

The continuity of the radial component of electric field intensity then requires condition (21) which in the present case takes the form

$$\cos kR = 0. \tag{35}$$

This is a quantization condition for our solution. It will be satisfied only for discrete values of the constant k given by

$$k_n = \pm\frac{(2n+1)\pi}{2R}, \tag{36}$$

where n is an arbitrary integer. Substituting these values into (31) and (32) we get the quantized values of permittivity and charge:

$$\tilde{\epsilon}_n = \frac{4AR^2}{(2n+1)^2\pi^2} \tag{37}$$

$$Q = \pm 4c\epsilon_0\sqrt{\frac{\pi M}{AR}}. \tag{38}$$

This result shows that in all quantized states of the medium there are only two possible values of the quantized charge. It is in excellent agreement with the situation observed in Nature where all elementary particles carry either positive or negative elementary charge e.

The experimental value of elementary charge e is independent of the mass of the particle however. This can be achieved provided the constant A is chosen to be equal to

$$A = \frac{16\pi\epsilon_0^2 Mc^2}{e^2 R}. \tag{39}$$

With this choice the potential at short distances is

$$\psi_n(r) = \frac{e}{4\pi\epsilon_0}\frac{\sin k_n r}{r}. \tag{40}$$

The quantized permittivity then takes the values

$$\tilde{\epsilon}_n = \frac{64RMc^2}{(2n+1)^2\pi e^2}\epsilon_0^2. \tag{41}$$

Introducing the standard electromagnetic radius R_{em} by the formula

$$R_{em} = \frac{e^2}{4\pi\epsilon_0 M c^2}, \tag{42}$$

we get from (40) the relation

$$\frac{\widetilde{\epsilon}_n}{\epsilon_0} = \frac{16}{(2n+1)^2\pi^2}\frac{R}{R_{em}}. \tag{43}$$

This relation provides for all the information on the electromagnetic properties of the medium inside the charge distribution.

Solutions (39) with $n \geq 1$ corresponds to alternating shells with opposite charge inside the same elementary charge. The recent LEP experiments show that such shells of opposite charge distribution do not exist for the electron. This means that only the solution with n=0 may have some physical application. In this way we arrive at the final form of the electromagnetic potential,

$$\psi(r) = \pm\frac{e}{4\pi\epsilon_0}\frac{\sin(\pi r/2R)}{r}, \tag{44}$$

and the permittivity of the medium is

$$\frac{\widetilde{\epsilon}_0}{\epsilon_0} = \frac{16}{\pi^2}\frac{R}{R_{em}}. \tag{45}$$

From the argument in [1] it follows that, in the framework of classical electromagnetism,

$$\frac{R}{R_{em}} \geq 10^7 \tag{46}$$

and this implies that

$$\frac{\widetilde{\epsilon}_0}{\epsilon_0} \geq \frac{16.10^7}{\pi^2}. \tag{47}$$

Such a result means that in the framework of classical electromagnetism the medium inside the charge distribution is highly polarized.

On the other hand, from the LEP experiments we get

$$\frac{R}{R_{em}} \leq 10^{-6} \tag{48}$$

and this implies that inside the electron we have

$$\frac{\widetilde{\epsilon}_0}{\epsilon_0} \leq \frac{16.10^{-6}}{\pi^2}. \tag{49}$$

Clearly this means that inside the electron the electromagnetic medium is highly non-classical because for all classical media this ratio should be greater than one [3].

The example considered above may seem naive and oversimplified. We suspect however that this may be the only analytically solvable example.

Some possible generalizations

In this Section we shall indicate some possible generalizations of the model. It seems that the most obvious generalization consists in taking nonzero functions $B(\vec{x})$ in (28), e.g., the oscillator type structure function

$$B(\vec{x}) = B|\vec{x}|^2 \tag{50}$$

which also leads to a soluble equation for the potential. The solutions are given by hypergeometric functions. The trouble however is in the quantization condition (19) which cannot be solved analytically.

Another generalization consists in considering two charge centers which create potentials $\varphi_1(\vec{x})$ and $\varphi_2(\vec{x})$, each of which satisfies equation of the type (12) with its own source term. Assuming that the charges behave like small capacitors we arrive at the following system of differential equations:

$$-\epsilon\Delta\varphi_1 = A_{11}\varphi_1 + A_{12}\varphi_2 \tag{51}$$

$$-\epsilon\Delta\varphi_2 = A_{21}\varphi_1 + A_{22}\varphi_2 \tag{52}$$

where the coefficients A_{jk} are some unknown parameters of the model which must be determined.

Assuming the solutions of (50) and (51) are of the form

$$\varphi_j(\vec{x}) = N_j \frac{\sin kr}{r}, \tag{53}$$

we get the algebraic equations for the amplitudes N_j in the form

$$\epsilon k^2 N_1 = A_{11}N_1 + A_{12}N_2, \tag{54}$$

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Nontrivial solutions exits provided the condition

$$(\epsilon k^2 - A_{11})(\epsilon k^2 - A_{22} = A_{12}A_{21} \tag{56}$$

is satisfied. As a result we get two possible values of k:

$$k_{\pm}^2 = \frac{A_{11} + A_{22} \pm \sqrt{(A_{11} - A_{22})^2 + 4A_{12}A_{21}}}{2\epsilon} \tag{57}$$

To simplify the calculations we may treat the case of symmetric charge centers. For such a case we have

$$A_{11} = A_{22} \equiv A, \qquad A_{12} = A_{21} \equiv C \tag{58}$$

and consequently

$$k_{\pm}^2 = \frac{A \pm C}{\epsilon}, \tag{59}$$

The total potential given by the superposition

$$\varphi = \varphi_1 + \varphi_2 \tag{60}$$

satisfies then exactly the same equation as (29) with the constant A replaced by A+C. The profit from this generalization is however in obtaining less "trivial" solutions in the form of superpositions of simple solutions (52). However, again the quantization condition (19) does not lead in general to a simple equation. Only for some particular choice of the parameters may we have soluble equations. This is the reason for our rather pessimistic conclusion made at the end of the previous Section.

Conclusion

The hypothesis that at small distances the basic equation of electrostatics is of the Schroedinger type has been shown to lead to interesting results. Among them the possibility of obtaining a simple quantization procedure for the electric charge is worth pointing out. The obtained connection (38) between the mass of bodies and the electromagnetic structure function of charged bodies indicates that the realistic theory of charge quantization may be achieved only in the unified theory of gravity and electromagnetism.

References

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