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The Wigner problem in electrodynamics

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Abstract

The relation between canonical commutation rules and a shape of acting force is analysed. It is shown that canonical commutation relations and Newtonian equations of motion for a single particle dynamics do imply the force to be of the Lorentz type but the inverse is not true. The example of a single particle motion in a constant magnetic field shows that equations of motion allow an alternative to canonical commutation relations to exist and it is a Lie algebra. The algebra found is of the type of algebras found in studies leading towards noncommutative geometry approaches to physical problems.

1. Introduction

Almost 10 years ago F. Dyson reminded the physical community of "Feynman's proof of the Maxwell equations"- a construction which R. P. Feynman had found and showed to him 40 years earlier [1,2]. In the framework of Feynman's scheme, under canonical commutation relations and Newtonian equations of motion for a single particle assumed to be valid, one is able to predict the Lorentzian shape of any force. Moreover, its "electric" and "magnetic" parts are to be constructed in an usual way from fields which satisfy the first pair of the Maxwell equations, *i.e.* come from a generalized potential.

In [2] Dyson writes about Feynman's motivation for research:

"In 1948 Feynman was still doubting all the accepted dogmas of quantum mechanics. He was exploring possible alternatives to the standard theory. His motivation was to discover a new theory, not to reinvent the old one. He was well aware that, if he assumed the existence of a momentum p_k satisfying the commutation rule $[x_j, p_k] = i\hbar\delta_{jk}$ in addition to $m[x_j, \dot{x}_k] = i\hbar\delta_{jk}$, he would only recover the standard formalism of electrodynamics. That was not his purpose. His purpose was to explore as widely as possible the universe

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of particle dynamics. He wanted to make as few assumptions as he could. In particular he wanted to avoid assuming the existence of momentum and Lagrangian related by $p_k = \partial L/\partial \dot{x}_k$ and $\dot{p}_k = \partial L/\partial x_k$. ... He hoped that by going along this road he might be led to new physics. He hoped to find physical models that would not be describable in terms of ordinary Lagrangians and Hamiltonians."

And further, he explains why Feynman never had published his result:

"His proof of the Maxwell equations was a demonstration that his program has failed and that his assumptions were not leading to new physics. The road that he had been exploring was a dead end. From Feynman's point of view, the proof was a failure, not a success. That is why he was not interested in publishing it."

In the following it is our aim to show that Feynman's assumptions still contain restrictions. Although Feynman did not explore canonical formalism explicitly he went along a road very close to it. The result disappointing him was a consequence of canonical commutation relations left as a basis of all considerations because this assumption implicitly leads to complete agreement with traditional formulation emerging from canonical formalism [3]. New and nonstandard physics, if it is really reachable within similar study, may appear as a result of analysis in the framework of which one explores only equations of motion as fundamental information on a problem given. To follow such an idea means that we are going to look for an answer to the question "Do the equations of motion determine the canonical commutation relations?" which E.P. Wigner asked almost fifty years ago and gave a negative answer to it presenting an example of noncanonical commutation relations which do agree with all dynamical laws of a harmonic oscillator [4].

2. The Feynman's proof

We are going to present Feynman's statement and to show its proof in a way possibly similar to Dyson's presentation. Nevertheless what follows contains necessary modification which defines uniquely the order in operator products [5].

Feynman's statement was:

Let us consider a single particle motion characterized by its position $\vec{x}$ and velocity $\dot{\vec{x}}$ and let us assume

i.) the newtonian form of dynamical equations

$$m\ddot{x}_j = F_j \tag{1}$$

with F_j denoting components of the acceleration independent force acting on particle, and

ii.) the usual canonical form of x_j and $\dot{x}_k$ commutators

$$\begin{aligned} [x_j, x_k] &= 0, \\ m\,[x_j, \dot{x}_k] &= i\hbar \end{aligned} \tag{2}$$

Then the assumptions above determine the shape of the force as

$$F_j = E_j + \varepsilon_{jkl}\langle \dot{x}_k B_l \rangle, \tag{3}$$

where $\langle\cdots\rangle$ denotes completely symmetrized operator product called Weyl ordering of operators. Moreover, the fields $\vec{E}$ and $\vec{B}$ depend on $\vec{x}$ and t only and they satisfy equations

$$\begin{aligned} \operatorname{div} \vec{B} &= 0, \\ \frac{\partial \vec{B}}{\partial t} + \operatorname{curl} \vec{E} &= 0, \end{aligned} \tag{4}$$

i.e. the first pair of the Maxwell equations (homogenous) which imply the representation of the fields $\vec{E}$ and $\vec{B}$ through derivatives of the generalized potential.

Here it must be said explicitly that we will be wrong if we will consider the second pair of Maxwell equations (nonhomogenous) as definitions of a charge density and an electric current which may be added to the statements above as a supplement independent from the other assumptions. Such a statement, present in Dyson's paper, leads to the paradoxal conclusion that one derives Lorentz covariant equations starting from purely Galilean covariant suppositions. It has been criticized by many authors [6] who have explained that homogenous Maxwell equations are both Lorentz and Galilean covariant while the nonhomogenous Maxwell equations are Lorentz covariant only and we must not propose them in a selfconsistent way within a Galilean covariant scheme. The Galilean covariant version of the Maxwell electromagnetism has been known for many years [7] and it is shown there that correct equations of Galilean electromagnetism differ from the relativistic ones.

We shall give the proof of Feynman's statement in a way which enables us to avoid the explicit use of differentiation as a consequence of realization of canonical commutation relations in terms of ordinary multiplication and differentiation. We shall formulate all statements within an abstract algebraic scheme demanded to obey a Lie algebra structure. In order to achieve it we assume the time evolution to be unitary and define it through the Heisenberg equations

$$\begin{aligned} [x_k, H] &= i\hbar \dot{x}_k, \\ [\dot{x}_k, H] &= i\hbar \ddot{x}_k = \frac{i\hbar}{m} F_k \end{aligned} \tag{5}$$

with no reference to a particular form of the operator H assumed only to be selfadjoint and an element of the algebra.

Within a Lie algebra structure the Jacobi identities must be satisfied. Applying it to the triplet $\{H, x_j, \dot{x}_k\}$ and taking into account (5), (1) and (2) we obtain

$$m\,[\dot{x}_j, \dot{x}_k] + [x_j, F_k] = 0. \tag{6}$$

The latter, when put into the Jacobi identity for $\{x_i, \dot{x}_j, \dot{x}_k\}$, results in

$$[x_i, [x_j, F_k]] = 0. \tag{7}$$

The relation (6) also implies

$$[x_j, F_k] = -\,[x_k, F_j]\,, \tag{8}$$

which, by definition, allows one to introduce a vector $\vec{B}$

$$[x_j, F_k] = -\frac{i\hbar}{m}\varepsilon_{jkl}B_l. \tag{9}$$

The components of $\vec{B}$ may equivalently be written as

$$B_i = -i\frac{m^2}{2\hbar}\varepsilon_{ijk}\left[\dot{x}_j, \dot{x}_k\right] \tag{10}$$

because of (6). Now the equality

$$[x_j, B_k] = 0 \tag{11}$$

holds as simple reformulation of (7), and

$$[\dot{x}_k, B_k] = 0 \tag{12}$$

comes from a straightforward consequence of the Jacobi identities:

$$\varepsilon_{ijk}\left[\dot{x}_i, \left[\dot{x}_j, \dot{x}_k\right]\right] = 0. \tag{13}$$

Evolution of the vector $\vec{B}$ has the form

$$[B_i, H] = m\varepsilon_{ijk}\left[\dot{x}_j, F_k\right] \tag{14}$$

and, without loss of generality, may be written as

$$[B_i, H] = m\varepsilon_{ijk}\left[\dot{x}_j, E_k\right] + m\varepsilon_{ijk}\left[\dot{x}_j, \varepsilon_{klm}\langle\dot{x}_l B_m\rangle\right] \tag{15}$$

with the force $\vec{F}$ of (1) replaced by

$$F_j = E_j - \varepsilon_{jkl}\langle\dot{x}_k B_l\rangle, \tag{16}$$

which may be treated as a definition of $\vec{E}$ with properties to be derived from the algebra structure. The consequence of (9), (10) and (11) is

$$[x_j, E_k] = 0. \tag{17}$$

Expansion of the second commutator in (15), with use of (12) and symmetry of the Weyl product, gives

$$[B_i, H] = m\varepsilon_{ijk}\left[E_j, \dot{x}_k\right] + m\langle\dot{x}_j\left[B_i, \dot{x}_j\right]\rangle. \tag{18}$$

It is possible to calculate the remaining commutators but they are not needed to get Feynman's conclusion. Assuming B_i and E_i to be elements of the envelopping algebra of (2) one can see that (11) and (17) imply that B_i and E_i depend on x_j and t only. Further, in the standard representation of canonical commutation relations, we are able to write down commutators in terms of differentiations. Relation (12) gives

$$\text{div}\vec{B} \quad = \quad 0 \tag{19}$$

while (18) leads to

$$i\hbar\left(\frac{\partial B_i}{\partial t} + \langle\frac{\partial B_i}{\partial \dot{x}_j}\dot{x}_j\rangle\right) \quad = \quad i\hbar\left(-\varepsilon_{ijk}\frac{\partial E_k}{\partial \dot{x}_j} + \langle\frac{\partial B_i}{\partial \dot{x}_j}\dot{x}_j\rangle\right), \tag{20}$$

which althogether form the first pair of the Maxwell equations. Such a conclusion should end the Feynman's proof with the only remark that the canonical commutation rules of the Heisenberg algebra (2) have been really important to achieve this goal.

3. The Wigner's approach

Now the question is: can we go backwards in the Feynman's construction, *i.e.* start from (19), (20) and the force law (3) in order to arrive at (2)? Hughes gives [3] the positive answer to this question, and in fact to the Wigner's question quoted in Introduction, but it is a consequence of the principles of Lagrangian formulation of mechanics and the standard form of Poisson bracket, assumed and extensively used through out the proof. Together with the minimal coupling rule, derivable in Lagrangean approach from the Lorentz force law, it leaves no freedom for the shape of the Poisson brackets of coordinates and velocities. In the context of [3], Feynman's arrival at the Lorentz force and the first pair of Maxwell equations is only a rederivation of the Helmholtz conditions for forces independent on acceleration, which allow the Lagrangian to exist but do not need to be restricted to electromagnetic ones. It, however, must be pointed out that the time evolution, if expressed through the standard form of the Poisson brackets, is compatible with a Lie algebra structure only if Hamilton equations of motion are satisfied [5]. Using Lagrangians and Hamiltonians means that we remain in the framework of the canonical formalism and it would be wrong if the results will not agree with fundamental assumptions of it.

In spite of that we consider the Wigner question as still open in the case of the Lorentz force and we shall look for its answer (for the Lorentz force) using only the principles of a Lie algebra-based approach. In order to find an explicit solution we will consider the simple case of motion in a constant external magnetic field. The algebra which describes motion of a particle with mass m and charge e put in a constant classical magnetic field chosen to be parallel to the z-axis, $\vec{B} = B\hat{e}_3$, is of the form

$$\begin{array}{llll} [x_1,\ H] = i\hbar\dot{x}_1, & [\dot{x}_1,\ H] = i\hbar\frac{e}{mc}B\dot{x}_2, & [x_1,\ S] = -i\hbar x_2, & [\dot{x}_1,\ S] = -i\hbar\dot{x}_2, \\ [x_2,\ H] = i\hbar\dot{x}_2, & [\dot{x}_2,\ H] = -i\hbar\frac{e}{mc}B\dot{x}_1, & [x_2,\ S] = i\hbar x_1, & [\dot{x}_2,\ S] = i\hbar\dot{x}_1, \\ [x_3,\ H] = i\hbar\dot{x}_3, & [\dot{x}_3,\ H] = 0, & [x_3,\ S] = 0, & [\dot{x}_3,\ S] = 0 \end{array} \tag{21}$$

and

$$[H,\ S] = 0, \qquad [H,\ B] = 0, \qquad [S,\ B] = 0, \tag{22}$$

where S denotes a generator of rotation around $\hat{e}_3$ axis while H is a generator of the time evolution.

It is obvious that c-number field $\vec{B}$ and vanishing $\vec{E}$ satisfy the commutation conditions (11) and (17) as well as (12) and (18), which have replaced the Maxwell equations in an algebraic formulation. Going backwards with respect to Feynman's proof we may ask for the existence of extensions of (21) to a Lie algebra assumed to be invariant under space reflection and time inversion according to the rules given in [8]. Inspection of the Jacobi identities leads to a conclusion that the completion of (21) in agreement to the Heisenberg algebra (2) (ϵ_{ik} means antisymmetric symbol):

$$\begin{array}{ccc} [x_1, x_2] = 0, & [x_1,\ x_3] = 0, & [x_2,\ x_3] = 0, \\ m\,[x_1,\ \dot{x}_1] = i\hbar, & m\,[x_2,\ \dot{x}_2] = i\hbar, & m\,[x_3,\ \dot{x}_3] = i\hbar, \\ [\dot{x}_1, \dot{x}_2] = i\hbar\epsilon_{12}\frac{e}{mc}B, & [\dot{x}_1, \dot{x}_3] = 0, & [\dot{x}_2, \dot{x}_3] = 0 \end{array} \tag{23}$$

is a particular choice characterized by the fact that it fixes all unknown commutators in the center of the algebra. Besides it, there exists another minimal extension:

$$\begin{array}{ccc} [x_1,\ x_2] = i\hbar\epsilon_{12}\ S, & [x_1,\ x_3] = 0, & [x_2,\ x_3] = 0, \\ [\dot{x}_1,\ \dot{x}_2] = i\hbar\epsilon_{12}\frac{e}{mc}\ B\ H, & [\dot{x}_1,\ \dot{x}_3] = 0, & [\dot{x}_2,\ \dot{x}_3] = 0, \\ [x_1,\ \dot{x}_1] = i\hbar H, & [x_2,\ \dot{x}_2] = i\hbar H, & [x_3,\ \dot{x}_3] = \frac{i\hbar}{m} \end{array} \tag{24}$$

and $[x_j,\ \dot{x}_k] = 0, \mathrm{for} j \neq k$, which has a structure of a quotient Lie algebra $\mathcal{A}/\mathcal{H}$, with a canonical ideal $\mathcal{H}$ sponned by generators $x_3, \dot{x}_3$. The challenger satisfies all properties demanded and essentially differs from the canonical algebra because in (24):

i.) there are components of the position operator $\vec{x}$ which do not commute and it occurs even in the limit $B \to 0$,

ii.) some commutators $[x_i,\ \dot{x}_i]$ can not be assumed to belong to the center of the algebra,

iii.) the algebra is more general than the standard one and it contracts to the canonical algebra. In order to see that one can construct central extension of (21) and (22) replacing anywhere H by $H \to \tilde{H} = H + \mathcal{C}$ where $\mathcal{C}$ is a central element, *i.e.* commutes with all generators. $H + \mathcal{C}$ appears in (24) instead of H and the contraction procedure $H \to 0$, $S \to 0$ results in canonical algebra within the first nonvanishing term accuracy which keeps nontrivial time evolution.

The above result means that we cannot treat the relation between canonical commutation relations and the Lorentzian shape of electromagnetic force as equivalent. Although the algebra (24) unavoidably contains noncommutativity of to the components of coordinate operator $\vec{x}$ the price for the new mathematics is not so high as it might seem at first glance. It is well understood that 50 years ago such an idea looked to be crazy and perspectiveless (nevertheless it appeared!, [9]) but we must not think in such a manner now. New mathematics - noncommutative geometry [10] - allows one to consider it as an alternative to standard approach.

4.Conclusion

Elementary and exactly solvable example presented above shows that the Wigner problem, related to the problem of solution of Heisenberg equations for quantum mechanical operators, admits solutions different from the canonical form and we do not see any arguments in favour of the canonical case to be chosen. The canonical choice contains an identification between generalized coordinates and momenta of the canonical formalism and mechanical coordinates and velocities given as solutions of Newtonian equations of motion. Such an identification is known to work perfectly in standard classical mechanics but it leads to problems even in simple cases going out of this theory. The example may be the old problem of *Zitterbewegung* [11], where algebraic analysis, performed in Wigner's manner, gives unexpected results [12] and moreover known to be shared by any dynamics which coincides with the dynamics of the harmonic oscillator [13]. In our study of a textbook example we have found an "equal opportunity" alternative to the canonical commutation rules which characteristic properties remain very close to those which have appeared since new methods have entered quantum physics. It should be mentioned among them:

- Deformation theory of Lie algebras [14] which opens ways how to construct the "inverse procedure" to the İnönü - Wigner contraction [15] in order to obtain relativistic kinematical algebras [16,17].

- Connection to q-deformations which appear because any solution of the Heisenberg equations remains their solution when multiplied by a function of integrals of motion and it defines a nonlinear noncanonical transformation [18,19]. In our example it has been enough to restrict considerations to a Lie algebraic structure, but in general the Hopf algebra based approach seems to be necessary.

- Compact quantum systems and quantum mechanics in discrete space time [9],[20] - the solution of the Wigner problem may form an algebra of a compact group (*Zitterbewegung* and harmonic oscillator serve as examples).

- Noncommutative geometry of physical space - time [10]. We have obtained that x_i do not commute even in the limit of vanishing forces. When one treats the Heisenberg evolution equations as a fundamental principle such a property is consistent with rotations, including Lie algebra structure, only for noncanonical form of $[x_i,\ \dot{x}_k]$. The algebra found may be interpreted as a "shadow of noncommutativity of space time" [21].

References

[1] F. Dyson, *Physics Today* **42** (1989) 32.

[2] F. Dyson, *American Journal of Physics* **58** (1990) 209.

[3] R. Hughes, *American Journal of Physics* **60** (1992) 301.

[4] E.P. Wigner, *Physical Review* **77** (1950) 711.

[5] S. Tanimura, *Annals of Physics NY* **220** (1992) 229.

[6] N. Dombey, *American Journal of Physics* **59** (1991) 85, R. W. Brehme, *American Journal of Physics* **59** (1991) 85,J. L. Anderson, *American Journal of Physics* **59** (1991) 86, I. E. Farquhar, *American Journal of Physics* **59** (1991) 87.

[7] M. LeBellac, J.-M. Levy - Leblond, *Il Nuovo Cimento***14**B (1973) 217.

[8] J. D. Jackson, *Classical Electrodynamics*, (John Wiley and Sons Inc., NY, 1962), Ch.6.

[9] H. Snyder, *Physical Review* **71** (1947) 38, *ibid.* **72** (1947) 68.

[10] A. Connes, *Noncommutative Geometry*, (Academic Press, 1994).

[11] E. Schrödinger, *Sitzungsb. Preuss. Akad. Wiss. Phys. - Math. Kl* **24** (1930) 418, *ibid.* **3** (1931) 1.

[12] A.O. Barut, A.J. Bracken, *Physical Review* **D23** (1981) 2453.

[13] E. Kapuścik, A. Horzela, Ch. A. Uzes, *Noncommutative geometry or noncommutative dynamics*, in *Proc. of 21-st Int. Colloquium on Group Theoretical Methods in Physics, Goslar 1996*, H.-D. Doebner and V. Dobrev Eds., (Heron Press (1997)).

[14] F. Bayen, M. Flato, C. Fronsdal, C. Lichnerowicz, D. Sternheimer, *Annals of Physics, NY* **111** (1978) 61, 111.

[15] E. Inonü, E.P. Wigner, *Proc. Nat. Acad. Sci. USA***39** (1956) 510.

[16] V. Aldaya, J. Bisquert, J. Navarro - Salas, *Physics Letters* **156** (1991) 381.

[17] R. Vilela - Mendes, *J. Phys* **A27** (1994) 8091, *Physics Letters* **A210** (1996) 232.

[18] A. O. Barut, W. Thacker, *Physical Review* **D31** (1984) 1386.

[19] V. I. Man'ko, G. Marmo, E.C.G. Sudarshan, F. Zaccaria, *Physica Scripta* **55** (1997) 528.

[20] R. López - Pena, V. I. Man'ko, G. Marmo, *Physical Review* **A56** (1997) 1126.

[21] A.O. Barut, A.J. Bracken, *Journal of Math. Phys.* **26** (1985) 2515.

[22] M. Dubois-Violette, J. Madore, R. Kerner, *Journal of Math. Phys* **39** (1998) 730