

## On the concept of spin in electrodynamics

Ivangoe PESTOV\*

*Joint Institute for Nuclear Research  
Dubna, 141 980 RUSSIA*

Received 02.02.1999

### Abstract

It is substantiated that spin is a notion associated with the group of internal symmetry that is tightly connected with the geometrical structure of spacetime. The wave equation for the description of a particle with spin one half is proposed. On this ground it is shown that the spin of electron is exhibited through the quantum number and accordingly the Dirac equation describes properties of particles with the projection of spin  $\pm\hbar/2$ . On the contrary, we put forward the conjecture that the spin of the quark cannot be considered as a quantum number, but only as an origin of a non-abelian gauge field. The reason is that the quark and electron from physical, geometrical and group-theoretical points of view differ from each other. It is from such deep reason understanding quark-lepton symmetry and such important phenomena as quark confinement is important.

### 1. Introduction

This report is devoted to the new approach to the problem of theoretical description of the particle with spin one half and, especially, in the context of the theory of leptons and quarks. As is well known, spin is a fundamental notion of modern physics and its meaning has tendency to grow with time. But what stands out is the absence of spin one half operators  $S_1, S_2, S_3$  with the properties

$$[S_i, S_j] = ie_{ijk}S_k \quad (1)$$

$$S_1^2 + S_2^2 + S_3^2 = S^2 = \frac{3}{4} = s(s+1) \quad (2)$$

$$[H, S_i] = 0, \quad (3)$$

in the Dirac theory of electron, where  $H$  is the Hamiltonian proposed by the Dirac. As a consequence of nonvalidity of the last relation in the framework of the Dirac theory the

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\*e-mail :pestov@thsun1.jimr.ru

corresponding tensor of spin angular momentum is not conserved. Description of the spin in the framework of the Poincare group is not satisfactory because by this method we can derive important information on the state of a field that corresponds to a particle in question but not on its internal property such as spin. In view of this, for the description of a particle with spin one half, a new wave equation is proposed below which verifies that equations (1),(2),(3) are fulfilled.

## 2. On the wave equation for a particle with spin one half

Our proposal is the following system of equations for the description of particles with spin 1/2:

$$P^i \psi_i = \frac{mc}{\hbar} \psi, \quad (4)$$

$$P_i \psi_j - P_j \psi_i - ie_{ijkl} P^k \psi^l = \frac{mc}{\hbar} \psi_{ji}, \quad (5)$$

$$P^i \psi_{ij} - P_j \psi = \frac{mc}{\hbar} \psi_j. \quad (6)$$

with

$$P_i = \nabla_i - (ie/\hbar c) A_i,$$

where  $A_i$  is the vector potential of an electromagnetic field,  $\nabla_i$  is the covariant derivative with respect to the Levi- Civita connection

$$\Gamma_{jk}^i = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}),$$

$\psi$  is a scalar field,  $\psi_i$  is a vector field and  $\psi_{ij}$  is a self-dual bivector

$$i\psi_{ij} = \frac{1}{2} e_{ijkl} \psi^{kl} = \tilde{\psi}_{ij}.$$

Completely antisymmetric Levi-Civita tensor  $e_{ijkl}$  is normalized as follows:  $e_{0123} = \sqrt{-g}$ , where  $g$  is the determinant of the metric tensor. Thus, the wave function  $\Psi = (\psi, \psi_i, \psi_{ij})$  has eight components. We consider the general case of curved spacetime with metric  $ds^2 = g_{ij} dx^i dx^j$  since in our approach its geometrical properties are tightly connected with the concept of spin as a fundamental quantum number.

Spin one half group is the symmetry group of the wave equation and it is defined as follows:

$$\begin{aligned} \psi &\Rightarrow \psi' = -\frac{1}{4} \Sigma_{ij} \psi^{ij} \\ \psi_{ij} &\Rightarrow \psi'_{ij} = \frac{1}{2} \Sigma_{ik} \psi_j^k - \frac{1}{2} \Sigma_{jk} \psi_i^k + \Sigma_{ij} \psi, \\ \psi_i &\Rightarrow \psi'_i = \Sigma_{ik} \psi^k, \end{aligned}$$

where  $\Sigma_{ij}$  is a self-dual bivector,  $i\Sigma_{ij} = (1/2) e_{ijkl} \Sigma^{kl} = \tilde{\Sigma}_{ij}$ . It is easy to recognize the origin of this group of transformations when equations of second order are derived from

(4),(5),(6). These eqs. are of the form

$$\begin{aligned} (P^i P_i + \frac{m^2 c^2}{\hbar^2})\psi_j &= \frac{ie}{\hbar c} H_{jk} \psi^k, \\ (P^i P_i + \frac{m^2 c^2}{\hbar^2})\psi &= \frac{ie}{4\hbar c} H_{jk} \psi^{jk}, \\ (P^i P_i + \frac{m^2 c^2}{\hbar^2})\psi_{jk} &= \frac{ie}{2\hbar c} (H_{jl} \psi_k^l - H_{kl} \psi_j^l) - \frac{ie}{\hbar c} H_{jk} \psi, \end{aligned}$$

where  $H_{jk} = F_{jk} - i\tilde{F}_{jk}$ ,  $F_{ij} = \partial_i A_j - \partial_j A_i$  is the bivector of the electromagnetic field.

However, the transformations of the spin one half group act in the space of solutions of the wave equation only under the condition

$$\nabla_i \Sigma_{jk} = 0. \tag{7}$$

This equation is very important because it connects the spin with the geometrical structure of spacetime. In the flat spacetime with Minkowski metric equation (7) has three linear independent solutions which define the operators  $S_1, S_2, S_3$  with the necessary properties

$$[S_i, S_j] = ie_{ijk} S_k, \quad S_1^2 + S_2^2 + S_3^2 = \frac{3}{4}, \quad [H, S_i] = 0,$$

where  $H$  is the Hamiltonian of the wave equation in question. Thus, in the Minkowski spacetime, the spin one half symmetry can be considered as global and we have no nonformal reason to introduce the gauge field corresponding to this symmetry in spite of the fact that it is internal.

Let  $\Psi_{\pm}$  be the wave functions that satisfy the equations

$$S_3 \Psi_{\pm} = \frac{1}{2} \Psi_{\pm}.$$

It can be shown that  $\Psi_+$  ( or  $\Psi_-$  ) is equivalent to the Dirac wave function in the sense that wave equation for  $\Psi_+$  is equivalent to the Dirac equation. So, the Dirac equation describes properties of particles with the projection of spin  $\pm\frac{1}{2}$ . Now it is clear why the tensor of spin angular momentum is not conserved in the Dirac theory.

### 3. On the wave equation for quarks

Equation (7) has no solutions in the spacetime of constant curvature. This follows from the integrability conditions of this equation and the main property of spacetime manifolds of that kind. The conditions of integrability of equation (7) follow from Ricci identities and are of the form  $R_{ijk}^m \Sigma_{ml} + R_{ijl}^m \Sigma_{km} = 0$ . When  $R_{ijl}^m = K(g_{il} \delta_j^m - g_{jl} \delta_i^m)$  equation (7) has only the trivial solution  $\Sigma_{km} = 0$ . Thus, in the spacetime of constant curvature the spin one half group can be realized only as a group of local symmetry. It means that in the case in question spin is the origin of the Yang-Mills fields and the

Planck constant characterizes the strength of interactions. We should not introduce a special constant of interaction because in the case of non-abelian gauge group there is no gauge-invariant conserving quantity like the electric charge in electrodynamics.

Now it is natural to put forward the idea that in the realm of strong interactions spacetime geometrically can be represented as a one-sheeted hyperboloid,

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = -a^2,$$

in the 5d Minkowski spacetime. Here,  $a$  is a constant which can be interpreted geometrically as the radius of a 3-dimensional sphere, considered here as a space section, and physically as the size of region of quark confinement. We here treat the quark as a point-like particle in the space of constant positive curvature  $S^3$ . For comparison of leptons and quarks we note that free motion of the electron is represented as a straight line in the usual Euclidean space and free motion of the quark is a circumference on the 3d sphere. This correlation between leptons and quarks will be continued with an example of the Coulomb law for these objects. As is well known, the electron Coulomb potential

$$\phi_e(r) = \frac{e}{r}$$

is the fundamental solution of the Laplace equation  $\Delta\phi = \text{div grad } \phi = 0$ . In accordance with our conjecture the Coulomb potential for quarks can be derived as follows. Consider the stereographic projection  $S^3$  from point  $(0,0,0,-a)$  on the sphere  $x^2 + y^2 + z^2 \leq a^2$ :

$$x^1 = fx, \quad x^2 = fy, \quad x^3 = fz, \quad x^4 = a(1 - f),$$

where  $f = 2a^2/(a^2 + r^2)$ . Then it follows that the element of length on the 3d sphere can be represented in the form  $ds^2 = f^2(dx^2 + dy^2 + dz^2)$  and hence the Laplace equation on  $S^3$  can be written as  $\Delta\phi = f^{-3}\text{div}(f\text{grad } \phi) = 0$ . We look for the solution to this equation that is invariant under the transformation group  $SO(3)$  with generators

$$x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad \text{etc.}$$

This subgroup of the  $SO(4)$  group is determined by fixing the point  $(0, 0, 0, -a)$ . Let

$$\psi = f \frac{1}{r} \frac{d\phi}{dr}.$$

Since

$$\Delta\phi = f^{-3} \left( r \frac{d\psi}{dr} + 3\psi \right) = r^{-2} f^{-3} \frac{d}{dr} (r^3 \psi),$$

then  $r^3 \psi = c_1 = \text{constant}$ . Thus,

$$\frac{d\phi}{dr} = c_1 \frac{a^2 + r^2}{2a^2 r^2} = c_1 \left( \frac{1}{2r^2} + \frac{1}{2a^2} \right)$$

and hence

$$\phi_q = c_1\left(-\frac{1}{2r} + \frac{r}{2a^2}\right) + c_2. \quad (8)$$

Generally speaking, expression (8), as we have derived it, coincides with the well-known Cornell potential [1],[2]. If we demand that  $\phi_e(a) = \phi_q(a)$ , then  $c_2 = e/a$  and the Coulomb law for quarks has the form

$$\phi_q(r) = q\left(\frac{1}{2r} - \frac{r}{2a^2}\right) + \frac{e}{a}, \quad (9)$$

where  $q$  is the quark charge. From this consideration it follows that equations (4)-(6) can be considered as basic equations for quarks in the new geometrical framework described.

#### 4. Conclusion

Let us summarize the results obtained and the problem to be solved. Now we can really avoid the introduction of the mysterious and artificial concept of "isotopic space" in the realm of high energy physics if the spin is really a fundamental concept. It is shown that the interquark potential expresses the Coulomb law for quarks and, in fact, coincides with the well-known Cornell potential that was first very successfully used by the Cornell group. At large  $a$ , when  $a \rightarrow \infty$ , from the theory of quarks one can deduce the theory of electrons but with electrons evidently deconfined, because in this case the region of confinement is 3d Euclidean space. Thus, the symmetry between quarks and leptons has a natural explanation. Equation (9) can also be considered as a modification of the Coulomb law on short distances. Wave equations (4)-(6) describe not only the spin but the fine structure of the hydrogen atom. This is not surprising because of the connection with the Dirac equation. It is of some interest that solutions may be written explicitly by using only spherical vector harmonics. In the context of quark hypothesis it is very important to investigate equations (4)-(6) for the case of 3d sphere in more detail. Much has to be done, but the problem is worth efforts as many interesting applications become possible.

#### References

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