Turk J Phy
25 (2001) , 159 – 173.

Path Integral Treatment for Spinless Relativistic Equation in the Two Component Theory

M. MERAD, T. BOUDJEDAA and L. CHETOUANI
Département de Physique, Faculté des Sciences,
Université Mentouri, 25000 Constantine-ALGÉRIE

Received 11.04.2000

Abstract

In this paper we have set up a path integral formalism for Feshbach-Villars equation by using the fermionic Schwinger model for Pauli matrices which describe an isocharge symmetry. This choice is made in analogy with spin model and the coherent state representation is then used. We have also given a general method of treating the problem of vanishing scalar potential by reducing it the to non-relativistic case and then, via Foldy-Wouthuysen canonical transformation, an explicit solution is constructed. The free case and constant magnetic field interaction are explicitly exposed. In each cases the propagators are evaluated and the energy spectrum and the corresponding wave functions are deduced.

P.A.C.S.03.65 Ca.Formalism
P.A.C.S.03.65 Db.Functional analytical methods
P.A.C.S.03.65 Pm.Relativistic wave equations
P.A.C.S.03.65 Ge.Solutions of wave equations: bound states.

1. Introduction

Relativistic wave mechanics could be considered the first of physical theories in which there is an attempt to the principles of relativity and those of quantum theory. The scheme has not yet been accomplished because of the meaninglessness of the single particle in terms of its wave function. This ambiguity arries in the domain of large velocities namely near the speed of light where the creation and annihilation of particles pairs are inherent and can not, in principle, be ignored. Moreover, to correctly describe physical phenomenons, relativistic wave mechanics should be completed by a relativistic quantum field theory where the dynamics of these pairs is taken into account. But in spite of these limitations, there exist many physical situations where relativistic wave mechanics could be an acceptable approximation. That is, for example, the case of Dirac equation which describes particles of spin 1/2 and has, for A long time, been successfully applied to concrete physical problems. Unfortunately, the same approach has been omitted for Klein-Gordon equation which describes the spinless particles because of the presence of the second time derivatives which generate difficulties as essentially the negative density of probability. The Feshbach-Villars (F.V.) equation [1]-[3] is an attempt to avoid these obstacles and at the same time give a solid probabilistic interpretation to spinless single particle. The procedure consists mainly in reducing the second order derivatives by the use of a two component wave function. Consequently, the equation would exhibit charge symmetry which is, in principle, a requirement of the relativistic theory and thus giving rise to negative energy solutions and their interpretation as “antiparticles ”.

In this paper, our purpose is to use this equation to describe spinless particles in the path integral framework. This latter formalism has been, during the last decade, a powerful tool in treating problems in non-relativistic quantum theory [4]. However, its extension to relativistic problems remains contestable because of the use of fifth parameter: Schwinger proper time, or a “time” of evolution[5]-[8]. In F.V. formalism, this default is overcome thanks to its Hamiltonian form by which the physical meaning is found again. In section II, after giving some review on the F.V. equation, we formulate a path integral approach for this equation. The Pauli matrices describing the charge symmetry are replaced by a fermionic Schwinger model [9] and an enlarged dynamics space is then used[10]. It is obvious that this representation has naively been chosen in analogy with spin system and at our knowledge there is no equivalent representation for the isospin case. In section III, we expose the general formal method in treating the problems where the scalar potential vanishes. The Foldy-Wouthuysen (F.W.) transformation[2] is used to reduce the Hamiltonian to its diagonal and Hermitian form by which we could easily perform the calculations. At last, the application of this method is explicitly constructed in the case of free particle and magnetic interaction. Section IV is devoted to concluding remarks.

2. Path integral formulation for relativistic spinless particle

A. Hamiltonian form for Klein-Gordon equation

As has previously mentioned, the Klein-Gordon equation has been rejected as a particle equation because it involues second time derivatives and that the density is not positive definite. To rise above these connected difficulties, Feshbach and Villars[1] brought it into an Hamiltonian form by using a two component wave function instead a scalar one. According to their method, we should write (for a spinless particle), the Schrodinger type equation as

$$i\frac{\partial \Psi}{\partial t} = \widehat{H}\Psi, \tag{1}$$

with Ψ as a two component wave function defined by

$$\Psi \equiv \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi + \frac{i}{m}\left(\frac{\partial}{\partial t} + ieV\right)\Phi \\ \Phi - \frac{i}{m}\left(\frac{\partial}{\partial t} + ieV\right)\Phi \end{pmatrix} \tag{2}$$

where Φ is a solution of Klein-Gordon equation, and $\widehat{H}$ is the Hamiltonian operator given by

$$\widehat{H} = -\frac{1}{2m}\left(\nabla - ie\mathbf{A}\right)^2\left(\tau_3 + i\tau_2\right) + m\tau_3 + eV. \tag{3}$$

Here, $(V, \mathbf{A})$ is the electromagnetic potential field and (τ_1, τ_2, τ_3) are the Pauli matrices describing the two degrees of freedom related to the charge and defined as

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

Equation (1) is known as the Feshbach-Villars (F.V.) equation. This lets the fundamental charge symmetry required by relativity to be apparent in the formalism and has various advantages from which one can cite the uniqueness of two component wave function determined from its initial value. Thus, the perturbation theory could be developed similar to Schrodinger theory. The negative energy solutions still exist and are interpreted as antiparticles. In effect, the density and current become

$$\rho = \overline{\Psi}\Psi, \tag{5}$$

$$\mathbf{j} = \frac{1}{2im}\left[\overline{\Psi}\left(\tau_3 + i\tau_2\right)\nabla\Psi - \nabla\overline{\Psi}\left(\tau_3 + i\tau_2\right)\Psi - \frac{e}{m}\mathbf{A}.\overline{\Psi}\left(\tau_3 + i\tau_2\right)\Psi\right], \tag{6}$$

where

$$\overline{\Psi} = \Psi^{\dagger}\tau_3. \tag{7}$$

ρ is positive for positive energy and negative for negative energy and is interpreted, respectively, as the charge density of the particle and the antiparticle. Let us also remark that the positive solution and negative solution are connected one to each other by a charge conjugaison transformation defined by

$$\Psi \rightarrow \Psi_c = \tau_1\Psi^*. \tag{8}$$

Namely, if Ψ is a solution of the equation

$$i\frac{\partial\Psi}{\partial t} = \widehat{H}(e)\Psi, \tag{9}$$

Ψ_c will be a solution of the conjugated equation

$$i\frac{\partial\Psi_c}{\partial t} = \widehat{H}(-e)\Psi_c. \tag{10}$$

where we have replaced charge (e) with ($-e$). Accordingly, the density-current vector transforms through this charge conjugation like

$$\rho \to -\rho_c \quad and \quad \mathbf{j} \to \mathbf{j}_c. \tag{11}$$

Following this interpretation, we could say that if Ψ describes a particle (π^+ meson) Ψ_c should then describe an antiparticle (π^- meson). In the case where $\Psi = \Psi_c$, we are describing a neutral particle (π^0 meson).

In what follows, our aim is to set up a path integral formalism for this F.V. equation. We shall use an extended space to describe the evolution, namely in addition to the dynamics of the exterior motion, we shall give a "dynamics" to the charge symmetry. A simple and a direct way to do this is to replace the Pauli matrices describing this symmetry by a fermionic Schwinger model. Finally, before elaborating the path integral propagator, let us introduce some definitions and notations useful for further developments relative to the fermionic operators and their corresponding coherent state representation.

Let (a, a^+) be a pair of two fermion operators verifying the usual anticommutation relations

$$\left[a, a^+\right]_+ = 1, [a, a]_+ = \left[a^+, a^+\right]_+ = 0. \tag{12}$$

The coherent state of fermionic oscillator algebra is defined as an eigenvector of the annihilation operator

$$a \mid \eta >= \eta \mid \eta >, \tag{13}$$

where η is a Grassmann variable.

These states can be generated from a vacuum state $\mid 0 >$ by the relation

$$\mid \eta >= e^{-\eta A^+} \mid 0 >, \tag{14}$$

and have the following two main properties:

non orthogonality

$$< \eta|\eta' >= \exp[\overline{\eta}\eta'], \tag{15}$$

and resolution of unity

$$\int d\overline{\eta} d\eta \exp[-\overline{\eta}\eta] \mid \eta >< \eta \mid = 1. \tag{16}$$

Finally, if one furthers the calculations, one often encounters the integral of exponential of quadratic forms. Then, one may find useful the following identity

$$\int \prod_{j=1}^{n} d\overline{\eta}_j d\eta_j \exp[-\overline{\eta} M \eta + \overline{\eta} J + \overline{J}\eta] = \det M \exp[\overline{J} M^{-1} J]. \tag{17}$$

B. Path integral for F.V. equation

Let us consider a spinless particle of mass m interacting with an electromagnetic field. The Hamiltonian governing the evolution of the system is given by the formula (3). By using the Schwinger model, this Hamiltonian will convert from matrix form to the fermionic one as

$$\widehat{H} = -\frac{1}{2m} \left(\nabla - ie\mathbf{A}\right)^2 C^\dagger \left(\tau_3 + i\tau_2\right) C + mC^\dagger \tau_3 C + eV, \tag{18}$$

with

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad and \quad C^\dagger = \left(c_1^\dagger, c_2^\dagger\right), \tag{19}$$

and $\left(c_1, c_1^\dagger\right)$ and $\left(c_2, c_2^\dagger\right)$ are pairs of fermionic operators.

In order to describe the dynamics of this system, we choose the following extended state $\mid \mathbf{x}; \eta\rangle$, where $\mathbf{x}$ is an exterior coordinate and $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ is a pair of Grassmann variables related to the symmetry of charge involved in the problem.

The propagator related to this system and governed by the Hamiltonian (18) is written in the representation $\mid \mathbf{x}; \eta\rangle$ as

$$\mathcal{K}(f, i; T) = \langle \mathbf{x}_f; \eta_f \mid U(T) \mid \mathbf{x}_i; \eta_i \rangle, \tag{20}$$

where

$$U(T) = \mathbb{T}_D \exp\left(-i \int_0^T H dt\right), \tag{21}$$

$\mathbb{T}_D$ is Dyson time ordering symbol.

In order to get a path integral form for the propagator $\mathcal{K}(f, i; T)$ we, divide, as usual, the time interval T into $(N+1)$ equal parts $\epsilon = \frac{T}{N+1}$ and take the limit $N \to \infty$. Hence, we can easily get the following expression:

$$\mathcal{K}(f,i;T)=\lim_{N\to\infty}\langle \mathbf{x}_f;\eta_f \mid \left(e^{-i\epsilon H}\right)^{N+1} \mid \mathbf{x}_i;\eta_i\rangle. \tag{22}$$

Next, we introduce between each pair of the infinitesimal evolution operators the completeness relation (16) to obtain a discretized form of the propagator (22):

$$\mathcal{K}(f,i;T)=\lim_{N\to\infty}\int\prod_{j=1}^{N}d\mathbf{x}_j\prod_{j=1}^{N+1}\frac{d\mathbf{p}_j}{(2\pi)^3}\prod_{j=1}^{N}d\overline{\eta}_j d\eta_j\exp[-\overline{\eta}_j\eta_j]$$

$$\prod_{j=1}^{N+1}\exp\left[\overline{\eta}_j\eta_{j-1}++i\mathbf{p}_j\Delta\mathbf{x}_j-i\epsilon\overline{\eta}_jQ(j)\eta_{j-1}-i\epsilon eV(\mathbf{x}_j)\right], \tag{23}$$

where $\Delta\mathbf{x}_j=\mathbf{x}_j-\mathbf{x}_{j-1},\mathbf{x}_j=\mathbf{x}(t_j)$ with $\mathbf{x}_0=\mathbf{x}_a$ and $\mathbf{x}_{N+1}=\mathbf{x}_b$, $\mathbf{p}_j=\mathbf{p}(t_j),\eta_j=\eta(t_j),\overline{\eta}_j=\overline{\eta}(t_j)$ with $\eta_0=\eta_a$ and $\overline{\eta}_{N+1}=\overline{\eta}_b$, and $Q(j)$ is the Hamiltonian matrix defined by

$$Q(j)=\frac{1}{2m}\left(\mathbf{p}_j-e\mathbf{A}_j\right)^2(\tau_3+i\tau_2)+m\tau_3. \tag{24}$$

In continuous form the discretized expression (23) of propagator would be

$$\begin{aligned}\mathcal{K}(f,i;T) &= \exp(\frac{\overline{\eta}_b\eta_b+\overline{\eta}_a\eta_a}{2})\int\mathcal{D}^3x\mathcal{D}^3p\mathcal{D}\overline{\eta}\mathcal{D}\eta\\ &\exp\left\{i\int_0^T dt\left[\frac{i}{2}\left(\overline{\eta}\dot{\eta}-\dot{\overline{\eta}}\,\eta\right)+\mathbf{p}\dot{\mathbf{x}}-\overline{\eta}Q\eta-eV(\mathbf{x})\right]\right\}.\end{aligned} \tag{25}$$

In next section, we are concerned by calculations of the propagators relative to the free case and a constant magnetic interaction using this formalism. To this aim, we shall at first expose the general method in treating the problems of vanishing scalar potential $(V=0)$ via a canonical transformations analogous to the F.W. transformation.

3. Method and applications

A. General method via Foldy-Wouthuysen transformation

It is well known that F.W. transformation has been introduced to transform the free Dirac equation into a form in which it is easy to associate operators with classical dynamical variables. The F.V. equation displays the same peculiar properties and it is also possible in this case to find an analogous canonical transformation by which the difficulties related to the interpretation of the theory disappear[2]. Indeed, in this representation F.V. Hamiltonian becomes Hermitian and the positive and negative energy solutions are completely decoupled. So, the conventional probability interpretation is restored. In the following, we want to use this canonical transformation in path integral calculations of the F.V. propagator where $V=0$. Furthermore, to get a solution of this problem, we should only suppose that the corresponding non-relativistic problem might be solvable.

Thus, replacing $V=0$ and arranging the path integral integrations in the expression (25) , one will get

$$\begin{aligned}\mathcal{K}(f,i;T) &= \exp(\frac{\overline{\eta}_b\eta_b+\overline{\eta}_a\eta_a}{2})\int\mathcal{D}\overline{\eta}\mathcal{D}\eta\exp\left\{i\int_0^T dt\left[\frac{i}{2}\left(\overline{\eta}\dot{\eta}-\dot{\overline{\eta}}\eta\right)-m\overline{\eta}\tau_3\eta\right]\right\}\\ &\quad\widetilde{\mathcal{K}}_\eta(f,i;T)\end{aligned}\tag{26}$$

where

$$\widetilde{\mathcal{K}}_\eta(f,i;T)=\int\mathcal{D}^3x\mathcal{D}^3p\exp\left\{i\int_0^T dt\left[\mathbf{p}\dot{\mathbf{x}}-\frac{1}{2m}(\mathbf{p}-e\mathbf{A})^2\overline{\eta}\left(\tau_3+i\tau_2\right)\eta\right]\right\}\tag{27}$$

As previously mentioned if we suppose that this latter propagator, which is nothing but the corresponding non-relativistic case, is solvable we could, in principle, write the following result:

$$\widetilde{\mathcal{K}}_\eta(f,i;T)=\sum_{n=0}^{\infty}\Psi_n(\mathbf{x}_f)\Psi_n^*(\mathbf{x}_i)e^{-iE_n\int_0^T dt\overline{\eta}(\tau_3+i\tau_2)e^{i\lambda}\eta}\tag{28}$$

where E_n is the spectrum related to the problem and $\Psi_n(\mathbf{x})$ is the corresponding wave functions. We note that in the presence of the scattering states the method remains valid.

Now, at this level, we introduce the canonical transformation defined by

$$\eta\rightarrow e^{-iS(E_n)}\xi;\qquad\overline{\eta}\rightarrow\overline{\xi}e^{iS(E_n)},\tag{29}$$

such that

$$e^{iS(E_n)}(E_n(\tau_3+i\tau_2)+m\tau_3)e^{-iS(E_n)}=\tau_3\mathcal{E}_n\equiv\mathcal{H},\tag{30}$$

where

$$\mathcal{E}_n=\sqrt{m^2+2mE_n}.\tag{31}$$

This equation represents the spectrum of the relativistic problem which is closely related to the non-relativistic one.

Now, it is easy to show that

$$S(E_n)=\tau_1\theta(E_n)\tag{32}$$

where

$$\theta(p) = -\frac{i}{2}\tanh^{-1}\left(\frac{E_n}{E_n + m}\right). \tag{33}$$

Inserting all these modifications in the expression (26) we get for the propagator the following result

$$\begin{aligned}\mathcal{K}(f,i;T) &= \sum_{n=0}^{\infty}\Psi_n(\mathbf{x}_f)\Psi_n^*(\mathbf{x}_i)\exp\left(\frac{\overline{\xi}_b\xi_b + \overline{\xi}_a\xi_a}{2}\right)\\ &\int \mathcal{D}\overline{\xi}\mathcal{D}\xi\exp\left\{i\int_0^T dt\left[\frac{i}{2}\left(\overline{\xi}\,\dot{\xi} - \dot{\overline{\xi}}\,\xi\right) - \mathcal{E}_n\overline{\xi}\tau_3\xi\right]\right\}.\end{aligned} \tag{34}$$

It clear that the path integral present in Eq.(34) could be easily done due to the its diagonal form. Hence, by projecting the result of integrations on the charge space, one will obtain

$$\mathcal{K}(f,i;T) = \sum_{n=0}^{\infty}\Psi_n(\mathbf{x}_f)\Psi_n^*(\mathbf{x}_i)\left[e^{-i\mathcal{E}_nT}u(\mathcal{E}_n)\overline{u}(\mathcal{E}_n) - e^{i\mathcal{E}_nT}v(\mathcal{E}_n)\bar{v}(\mathcal{E}_n)\right] \tag{35}$$

where

$$u(\mathcal{E}_n) = \tfrac{1}{2(m\mathcal{E}_n)^{1/2}}\begin{pmatrix} m+\mathcal{E}_n \\ m-\mathcal{E}_n\end{pmatrix}, \tag{36a}$$

and

$$v(p) = \tfrac{1}{2(m\mathcal{E}_n)^{1/2}}\begin{pmatrix} m-\mathcal{E}_n \\ m+\mathcal{E}_n\end{pmatrix}. \tag{36b}$$

$\bar{u} = u^\dagger\tau_3, \qquad \bar{v} = v^\dagger\tau_3.$

The expression (35) is known as a spectral decomposition of F.V. propagator from which we easily deduce a formal expression of the spectrum and corresponding wave functions.

B. Applications

1. *The free case*

In this case we put $V = 0$ and $\mathbf{A} = \mathbf{0}$ in the propagator given by Eq.(25)

$$\mathcal{K}_0(f,i;T) = \exp\left(\frac{\overline{\eta}_b\eta_b + \overline{\eta}_a\eta_a}{2}\right)\int \mathcal{D}^3x\mathcal{D}^3p\mathcal{D}\overline{\eta}\mathcal{D}\eta$$

$$\exp\left\{i\int_0^T dt\left[\frac{i}{2}\left(\overline{\eta}\dot{\eta} - \dot{\overline{\eta}}\,\eta\right) + \mathbf{p}\dot{\mathbf{x}} - \overline{\eta}\left(\frac{\mathbf{p}^2}{2m}(\tau_3 + i\tau_2) + m\tau_3\right)\eta\right]\right\}. \tag{37}$$

The path integration over $\mathcal{D}^3x$ is straightforward and gives the impulse $\mathbf{p}$ as a constant of motion. Then, the propagator reduces to

$$\mathcal{K}_0(f,i;T) = \exp(\frac{\overline{\eta}_b\eta_b + \overline{\eta}_a\eta_a}{2}) \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}_f - \mathbf{x}_i)} \int \mathcal{D}\overline{\eta}\mathcal{D}\eta$$

$$\exp\left\{ i\int_0^T dt \left[\frac{i}{2}\left(\overline{\eta}\dot{\eta} - \dot{\overline{\eta}}\,\eta\right) - \overline{\eta}\left(\frac{\mathbf{p}^2}{2m}(\tau_3 + i\tau_2) + m\tau_3\right)\right]\eta\right\}. \tag{38}$$

At this stage, we introduce the canonical transformation defined by

$$\eta \to e^{-iS(p)}\xi \qquad \overline{\eta} \to \overline{\xi}e^{iS(p)},$$

such that

$$e^{iS(p)}Qe^{-iS(p)} = \tau_3 E_p \equiv \mathcal{H} \tag{39}$$

where

$$Q(p) = (p^2/2m)(\tau_3 + i\tau_2) + m\tau_3 \text{and} E_p = \sqrt{p^2 + m^2}. \tag{40}$$

It is easy to show that

$$S(p) = \tau_1\theta(p), \tag{41}$$

where

$$\theta(p) = -\frac{i}{2}\tanh^{-1}\left(\frac{p^2/2m}{p^2/2m + m}\right). \tag{42}$$

Inserting this canonical transformation in Eq.(38), it is seen that the integration over $\left(\overline{\xi},\xi\right)$ becomes easy since the Hermitian Hamiltonian $\mathcal{H}$ is diagonal. Thus, the result will be

$$\widetilde{\mathcal{K}}_0(f,i;T) = \int \tfrac{d^3p}{(2\pi)^3}.e^{i\mathbf{p}(\mathbf{x}_f - \mathbf{x}_i)} \exp\left(\overline{\xi}_f e^{-i\tau_3 E_p T}\xi_i\right). \tag{43}$$

By returning to the old representation, we shall get

$$\mathcal{K}_0(f,i;T) = \int \tfrac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}_f - \mathbf{x}_i)} \exp(\frac{\overline{\eta}_b\eta_b + \overline{\eta}_a\eta_a}{2}) \exp\left(\overline{\eta}_f e^{-iS(p)}e^{-i\tau_3E_pT}e^{iS(p)}\eta_i\right). \tag{44}$$

Projecting this result on the isocharge space, we obtain the matrix representation of the free F.V. propagator:

$$\mathbb{K}_0(\mathbf{x}_f,\mathbf{x}_i;T) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}_f - \mathbf{x}_i)} \left[e^{-iE_pT}u(p)\bar{u}(p) - e^{iE_pT}v(p)\bar{v}(p)\right], \tag{45}$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2}$ and $\bar{u} = u^\dagger \tau_3, \qquad \bar{v} = v^\dagger \tau_3$
with

$$u(p) = \tfrac{1}{2(mE_p)^{1/2}} \begin{pmatrix} m + E_p \\ m - E_p \end{pmatrix}, \tag{46a}$$

and

$$v(p) = \tfrac{1}{2(mE_p)^{1/2}} \begin{pmatrix} m - E_p \\ m + E_p \end{pmatrix}. \tag{46b}$$

Expression (45) is known as the spectral decomposition of the F.V. propagator from which one identifies the spectrum and corresponding wave functions of the free case:
positive energies $E = E_p = \sqrt{p^2 + m^2}$

$$\Psi_p^{(+)}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{px}} u(p), \tag{47a}$$

and negative energies $E = -E_p = -\sqrt{p^2 + m^2}$

$$\Psi_p^{(-)}(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{-i\mathbf{px}} v(p). \tag{47b}$$

According to F.V. formalism, these two solutions are connected by the charge conjugaison transformation (5). These results coincide exactly with those of the literature[1].

Let us now turn to the related Green's function which plays an important role in physics. It can be deduced from this propagator via the Fourier transformation

$$\mathbb{G}_0(\mathbf{x}_f, \mathbf{x}_i; E) = \int_0^{+\infty} dT \mathbb{K}_0(\mathbf{x}_f, \mathbf{x}_i; E) e^{iET}, \tag{48}$$

where we point out that the integration over T variable goes from 0 to infinity. In fact, this means that the propagation backward in time has been omitted, namely, in F.V formalism the $\theta(-T)$ is replaced by the sign (-) of the matrix τ_3, introduced for an isocharge symmetry.

It is easy to show that the result of integration over T in Eq.(48) yields

$$\mathbb{G}_0(\mathbf{x}_f, \mathbf{x}_i; E) = i\tau(E) \int \frac{d^3p}{(2\pi)^3} \exp\left[i\mathbf{p}(\mathbf{x}_f - \mathbf{x}_i)\right] \tfrac{1}{E^2 - (\mathbf{p}^2 + m^2)}$$

$$-\frac{i}{2m} \delta\left(\mathbf{x}_f - \mathbf{x}_i\right)\left(\tau_3 + i\tau_2\right), \tag{49}$$

where

$$\tau(E) = E + \frac{E^2 + m^2}{2m}\tau_3 + i\frac{E^2 - m^2}{2m}\tau_2. \tag{50}$$

Integrating out the impulse variables, one obtains the following result, according to energy value,

a) for $E > m$:

$$\mathbb{G}_0(\mathbf{x}_f, \mathbf{x}_i; E) = -\frac{i}{4\pi}\frac{e^{i\sqrt{E^2-m^2}|\mathbf{x}_f-\mathbf{x}_i|}}{\mid \mathbf{x}_f - \mathbf{x}_i \mid}\tau(E) - \frac{i}{2m}\delta\left(\mathbf{x}_f - \mathbf{x}_i\right)\left(\tau_3 + i\tau_2\right), \tag{51a}$$

b) for $-m < E < m$:

$$\mathbb{G}_0(\mathbf{x}_f, \mathbf{x}_i; E) = -\frac{1}{4\pi}\frac{e^{-\sqrt{m^2-E^2}|\mathbf{x}_f-\mathbf{x}_i|}}{\mid \mathbf{x}_f - \mathbf{x}_i \mid}\tau(E) - \frac{i}{2m}\delta\left(\mathbf{x}_f - \mathbf{x}_i\right)\left(\tau_3 + i\tau_2\right), \tag{51b}$$

c) for $E < -m$:

$$\mathbb{G}_0(\mathbf{x}_f, \mathbf{x}_i; E) = \frac{i}{4\pi}\frac{e^{-i\sqrt{E^2-m^2}|\mathbf{x}_f-\mathbf{x}_i|}}{\mid \mathbf{x}_f - \mathbf{x}_i \mid}\tau(E) - \frac{i}{2m}\delta\left(\mathbf{x}_f - \mathbf{x}_i\right)\left(\tau_3 + i\tau_2\right). \tag{51c}$$

Let us remark that Green's function contains two parts, one regular times an idempotent matrix $\tau(E)$ and the other irregular $\delta\left(\mathbf{x}_f - \mathbf{x}_i\right)$ times a nilpotent matrix $(\tau_3 + i\tau_2)$. Incidentally, the two parts are essential in the calculations where the potential is present[11]-[**?**]. Now, we go to extract from this 3-dimension Green's function, its radial part which could be useful for applications relative to central potential. For example, it is readily shown that for $E > m$.

$$\mathbb{G}_0^l(r_f, r_i; E) = \frac{\pi}{2\sqrt{r_f r_i}} J_{l+\frac{1}{2}}\left(\sqrt{E^2 - m^2}r_<\right) H^{(1)}_{l+\frac{1}{2}}\left(\sqrt{E^2 - m^2}r_>\right)\tau(E)$$

$$-\frac{i}{2m}\frac{1}{r_f r_i}\delta\left(r_f - r_i\right)\left(\tau_3 + i\tau_2\right), \tag{52}$$

where $r_> = \max(r_f, r_i)$, and $r_< = \min(r_f, r_i)$,and, $J_\nu(x)$ and $H^{(1)}_\nu(x)$ are respectively the Bessel and Hankel functions of order ν.

In obtaining the expression (52), we have used the following relations[13],

$$\frac{e^{ikR}}{R} = \frac{i\pi}{2\sqrt{r_< r_>}}\sum_{l=0}^{\infty}(2l+1)J_{l+\frac{1}{2}}\left(kr_<\right)H^{(1)}_{l+\frac{1}{2}}\left(kr_>\right)P_l(\cos\Theta), \tag{53}$$

where $R = \mid \mathbf{x}_f - \mathbf{x}_i \mid$ and Θ is the angle between $\mathbf{x}_f$ and $\mathbf{x}_i$, and with

$$P_l(\cos\Theta) = \frac{4\pi}{2l+1}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y_l^m(\theta_f,\varphi_f)Y_l^{m*}(\theta_i,\varphi_i) \tag{54}$$

$$\delta\left(\mathbf{x}_f - \mathbf{x}_i\right) = \frac{1}{r_f r_i}\delta\left(r_f - r_i\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y_l^m(\theta_f,\varphi_f)Y_l^{m*}(\theta_i,\varphi_i). \tag{55}$$

C. Constant magnetic field

For simplicity, we choose the direction of magnetic field along the z axis and the gauge is fixed as

$$\mathbf{A} = (0, Bx, 0), \tag{56}$$

where B is the strength of the field.

Thus, the related propagator is given by

$$\mathcal{K}_B(f,i;T) = \exp(\frac{\overline{\eta}_b\eta_b + \overline{\eta}_a\eta_a}{2})\int \mathcal{D}^3x\mathcal{D}^3p\mathcal{D}\overline{\eta}\mathcal{D}\eta$$

$$\exp\left\{i\int_0^T dt\left[\frac{i}{2}\left(\overline{\eta}\dot{\eta} - \dot{\overline{\eta}}\,\eta\right) + \mathbf{p}\dot{\mathbf{x}} - \overline{\eta}\left(\tfrac{(\mathbf{p}-e\mathbf{A})^2}{2m}\left(\tau_3 + i\tau_2\right) + m\tau_3\right)\eta\right]\right\}. \tag{57}$$

Fixing the paths $(\overline{\eta},\eta)$, this expression can be rearranged as follows

$$\mathcal{K}_B(f,i;T) = \exp(\frac{\overline{\eta}_b\eta_b + \overline{\eta}_a\eta_a}{2})\int \mathcal{D}\overline{\eta}\mathcal{D}\eta\exp\left\{i\int_0^T dt\left[\exp i\int_0^T dt\frac{i}{2}\left(\overline{\eta}\dot{\eta} - \dot{\overline{\eta}}\,\eta\right)\right.\right.$$

$$\left.\left. +\mathbf{p}\dot{\mathbf{x}} - m\overline{\eta}\tau_3\eta\right]\right\}\mathcal{K}_\eta(\mathbf{x}_f,\mathbf{x}_i;T), \tag{58}$$

where the kernel $\mathcal{K}_\eta(f,i;T)$ corresponds to the form of the non-relativistic propagator of the magnetic field problem and is given as

$$\mathcal{K}_\eta(\mathbf{x}_f,\mathbf{x}_i;T) = \int \mathcal{D}^3x\mathcal{D}^3p\exp\left\{i\int_0^T dt\left[\mathbf{p}\dot{\mathbf{x}} - \overline{\eta}\left(\tfrac{(\mathbf{p}-e\mathbf{A})^2}{2m}\left(\tau_3 + i\tau_2\right)\right)\eta\right]\right\}. \tag{59}$$

The expression of this latter propagator could be easily evaluated and the result is given as

$$\mathcal{K}_\eta(f,i;T) = \int \frac{dp_z}{2\pi}\frac{dp_y}{2\pi}e^{ip_z(z_f - z_i) + ip_y(y_f - y_i)}$$

$$\times \left[\sum_{n=0}^{\infty} \Phi_n(x_f - p_y/m\omega)\Phi_n^*(x_i - p_y/m\omega) e^{-iE_n \int_0^T dt \eta^\dagger (\tau_3 + i\tau_2) e^{i\lambda}\eta} \right], \tag{60}$$

where

$$E_n = \frac{p_z^2}{2m} + \left(n + \frac{1}{2}\right)\omega; \qquad \omega = \frac{eB}{m}, \tag{61}$$

and $\Phi_n(x)$ are the oscillator wave functions

$$\Phi_n(x) = \left(\tfrac{1}{2^n n! \sqrt{\pi}}\right)^{1/2} \left(\tfrac{m\omega}{4}\right)^{1/2} e^{-\frac{m\omega}{2}x^2} H_n\left(\sqrt{m\omega}x\right), \tag{62}$$

$H_n(x)$ are Hermite polynomials.

At this level, it remains the evaluation of path integrals over $(\overline{\eta}, \eta)$ variables. This could be done using the previous canonical transformation of the free case in which the following modifications occur:

$$\frac{p^2}{2m} \to \frac{p_z^2}{2m} + \left(n + \frac{1}{2}\right)\omega = E_n \tag{63a}$$

$$S(p) \to S(n, p_z) = \tau_1 \theta(n, p_z), \tag{63b}$$

with

$$\theta(n, p_z) = -\frac{i}{2} \tanh^{-1}\left(\frac{E_n}{E_n + m}\right). \tag{64}$$

Therefore, the corresponding Hermitian Hamiltonian gets the following diagonal form:

$$\mathcal{H}(n, p_{z)=}\tau_3 \sqrt{p_z^2 + m^2 + 2m(n + 1/2)\omega}. \tag{65}$$

Proceeding as previously, one obtains the final form of the F.V. propagator of magnetic field problem

$$\mathbb{K}_B(f, i; T) = \int \frac{dp_z}{2\pi}\frac{dp_y}{2\pi} e^{ip_z(z_f - z_i) + ip_y(y_f - y_i)} \sum_{n=0}^{\infty} \Phi_n(x_f - p_y/m\omega)\Phi_n^*(x_i - p_y/m\omega)$$

$$\times \left[e^{-i\mathcal{E}_n T} u_n(p_z)\bar{u}_n(p_z) - e^{i\mathcal{E}_n T} v_n(p_z)\bar{v}_n(p_z)\right], \tag{66}$$

where

$$\mathcal{E}_n = \sqrt{p_z^2 + m^2 + 2m(n + 1/2)\omega} \tag{67}$$

$$u_n(p_z) = \frac{1}{2(m\mathcal{E}_n)^{1/2}} \begin{pmatrix} m + \mathcal{E}_n \\ m - \mathcal{E}_n \end{pmatrix} \tag{68a}$$

$$v_n(p_z) = \frac{1}{2(m\mathcal{E}_n)^{1/2}} \begin{pmatrix} m - \mathcal{E}_n \\ m + \mathcal{E}_n \end{pmatrix}, \tag{68b}$$

and the functions $\Phi_n(x)$ are given by expression (62).

Accordingly, one will identify the positive and negative solutions as

positive solutions: $E = \mathcal{E}_n = \sqrt{p_z^2 + m^2 + 2m(n+1/2)\omega}$

$$\Psi^{(+)}_{n,p_y,p_z}(\mathbf{x}) = \frac{1}{2\pi} e^{ip_y y + ip_z z} \cdot \Phi_n(x - p_y/m\omega) \cdot u_n(p_z), \tag{69a}$$

negative solutions: $E = -\mathcal{E}_n = -\sqrt{p_z^2 + m^2 + 2m(n+1/2)\omega}$

$$\Psi^{(-)}_{n,p_y,p_z}(\mathbf{x}) = \frac{1}{2\pi} e^{-ip_y y - ip_z z} \cdot \Phi_n(x + p_y/m\omega) \cdot v_n(p_z). \tag{69b}$$

We also notice in this case that these solutions are connected by charge conjugation transformation as required by the F.V. formalism, and accordingly the center of the motion has been inverted due to this conjugation of the charge. These results coincide exactly with those of literature[14].

4. Conclusion

In this paper, we have constructed a path integral formalism for Feshbach-Villars equation. The description of charge symmetry has naively been copied from the fermionic Schwinger model of the spin. The propagator is then expressed in an enlarged space. The exact calculations have been done in the cases of free particle and magnetic field interaction with the help of the well-known Flody-Wouthuysen transformation. In the case of free particle, we have also given the related Green's function and a polar decomposition is then deduced. It has been noticed that the F.V. Green's function is equal to Klein-Gordon Green's function times a matrix containing the potential plus a singular part. This latter fact is the feature of the F.V. Green's function in general case. In all cases we have extracted from the spectral decomposition of the propagator the energy spectrum and the normalized wave functions and verify that the positive and negative solutions are connected by charge conjugation transformation

Finally, let us signal that the same problems in the case of Feshbach-Villars equation for spin $\frac{1}{2}$ are under consideration.

References

[1] H. Feshbach and F. Villars. *Rev. Mod. Phys.* **30** (1958) 24

[2] J. D. Bjorken and S.D. Drell 1964 Relativistic Quantum Mechanics. (New York, 1964).

[3] E. Durand "Mecanique Quantique; tome II: Spin et Relativité" Ed. Masson (1976).

[4] L. Chetouani and T.F. Hammann, *Nuovo Cimento* **B 92** (1986) 27; **B 92** (1986) 106; *Phys. Rev.* **A 34** (1986) 4737; *Nuovo Cimento* **B 98** (1987) 1; *J. Math. Phys.* **28** (1987) 598.

[5] J. Schwinger *Phys. Rev.* **82** (1951) 664.

[6] R. P. Feynman *Phys. Rev.* **80** (1950) 440.

[7] L. S. Schulman "Techniques and Applications of Path Integration" (John Wiley and sons, New York 1981).

[8] T. Boudjedaa, L. Chetouani, L. Guechi and T.F. Hammann, *Physica Scripta* **46** (1992) 289; *Nuovo Cimento* **B109** (1994) 209, *Physica Scripta* **52** (1995) 9. B. Bentag, L. Chetouani, L. Guechi and T.F. Hammann, *Nuovo Cimento* **B 111** (1996) 99. L. Chetouani, L. Guechi, A. Lecheheb, T.F. Hammann and A. Messouber, *Physica* **A 234** (1996) 529. B. Bentag, T. Boudjedaa and L. Chetouani *Ann. Phys.* **9** (2000) 103-124.

[9] J. Schwinger "Quantum Theory of Angular Momentum" in Collection of Reprints and Originals Papers; edited by L. C. Biedenharn and H. Vandam Perspectives in Physics; Academic Press. New York and London (1965).

[10] T. Boudjedaa, A Bounames, L. Chetouani, T.F. Hammann and Kh. Nouicer *J. Math. Phys.* **36 (4)** (1995) 1602; *Physica Scripta* **54** (1995) 225; *Physica Scripta* **56** (1997) 545.

[11] T. Boudjedaa, L. Chetouani, M.Merad, *Chinese Journal of Physics* **38(6)** (2000) 1019

[12] T. Boudjedaa, L. Chetouani, M.Merad, *Il Nuovo Cimento* **B 114(11)** (1999) 1261.

[13] I. S. Grandshteyn and I. M. Ryzhik, "Tables of Integrals, Series and Products" (Academic Press. New York (1980)).

[14] K. M. Case *Phys. Rev.* **95** 1323 (1954).