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Calculation of the Potential and Electric Flux Lines for Parallel Plate Capacitors with Symmetrically Placed Equal Lengths by Using the Method of Conformal Mapping

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Abstract

The classical problem of the parallel-plate capacitors has been investigated by a number of authors, including Love [1], Langton [2] and Lin [3]. In this paper, the exact equipotentials and electric flux lines of symmetrically placed two thin conducting plates are obtained using the Schwarz- Cristoffel transformation and the method of conformal mapping. The coordinates $\{x, y\}$ in the z-plane corresponding to the constant electric flux lines and equipotential lines are obtained after very detailed and cumbersome calculations. The complete field distribution is given by constructing the family of lines of electric flux and equipotential.

1. Introduction

The problem of a parallel-plate capacitor with different configurations has a very long history. Love [1] used a standard procedure of conformal mapping method to solve this problem but he only obtained the ratio of the plate length to the plate distances, which is not enough to obtain the complete field distribution of the equal plates. Langton [2] and Lin [3] gave detailed calculation of symmetrically placed unequal plates also using the method of conformal mapping, and latter reduces the problem to the equal plates and gives the numerical results for electric flux lines, but he neither gives numerical results nor the plot for equipotential lines. Electric flux lines and equipotential lines for equal plates are also obtained to compare with the results of the Ordinary Finite Difference Method and the Coarse-Grid Finite-Difference Method. It was found that latter gives much better agreement with the conformal mapping method.

In spite of these studies, the numerical results and the plot of the equipotential are not obtained in the above given references. Therefore, the purpose of the present work is to study the electric flux lines and the lines of the constant potential in great detail for equal plates using the Schwarz- Cristoffel transformation and the method of conformal mapping. The exact results for the coordinates $\{x, y\}$ in the z-plane corresponding to the constant electric flux lines and equipotential lines are obtained explicitly. Also, the combined plot of to the constant electric flux lines and equipotential lines are given.

2. Method of Conformal Mapping for two Symmetrically Placed Thin Conducting Plates

Figure 1a shows two parallel thin conducting plates of equal length at right angles to the plane of the page. In Figure 1b shows the half of the whole field on the upper half of the z-plane to be investigated. The Schwarz-Christoffel [6] equation is used to transform the boundaries in the z-plane (Figure 1b) to the t-plane (Figure 1c), where the two conducting plates lie on the real axis of the t-plane, separated by equal distances from the imaginary axis. In Figure 1d, the t-plane is transformed into a rectangle in the w-plane. The Schwarz-Cristoffel equation gives the transformation of the plates from the z-plane to the t-plane as

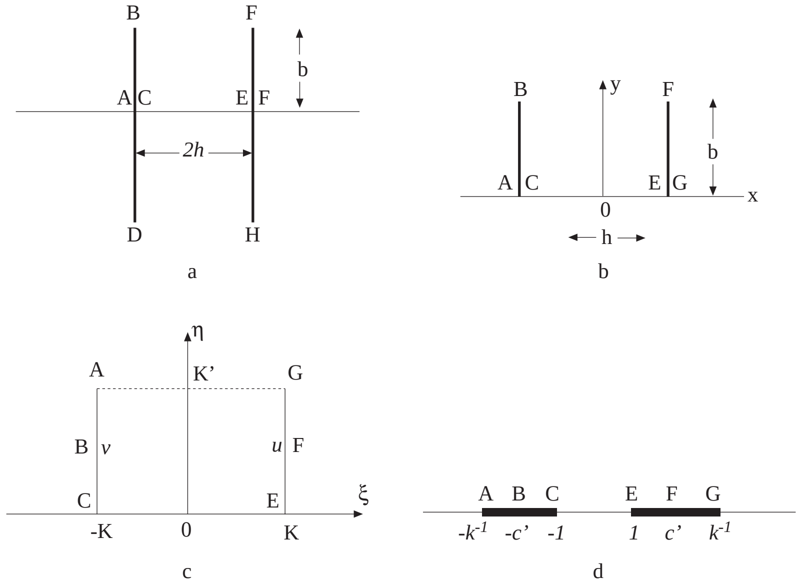

Figure 1. **a.** Parallel-plate capacitor with symmetrically placed equal plates. The lines BD and FG represents the plates of the capacitors. **b.** The half-capacitor is represented on the upper half of the complex z-plane, **c.** The plates are transformed from z-plane to the t-plane, where they are represented as strips, **d.** The strips are transformed into a rectangle in the w-plane. The values of B and F corresponds to v and u respectively.

$$\frac{dz}{dt} = \frac{A(t^2 - c'^2)}{[(t + \frac{1}{k})(t - \frac{1}{k})(t + 1)(t - 1)]^{\frac{1}{2}}}, \tag{1}$$

where c' is a constant and corresponds to the mid-point of the plates in the t-plane and it is calculated in Appendix as

$$c' = \frac{1}{k}\sqrt{\frac{E'}{K'}},$$

where k is the modulus of the elliptic function, $E' = E(k')$ is the complete elliptic integral of the second kind with modulus $k', K' = K(k')$ is the complete elliptic integral of the first kind with modulus k', and k' is called the complimentary modulus and is equal to $\sqrt{(1 - k^2)}$.

In the w-plane, we may write out the uniform field by complex potential $W = U + iV$, where U is the electric potential and V is the electric flux of the plates, and

$$W = wU_o\frac{1}{K} \tag{2}$$

so that two electrodes are at U_o and - U_o, where $K = K(\text{k})$ is the complete elliptic integral of first kind of modulus k. The Schwarz-Cristoffel transformation from the t-plane to w-plane is

$$\frac{dw}{dt} = \frac{B}{[(t + \frac{1}{k})(t - \frac{1}{k})(t + 1)(t - 1)]^{\frac{1}{2}}}. \tag{3}$$

Writing the integral form of Eq. (3) and carrying out the integral, we obtain the transformation from t-plane to the w-plane as

$$t = sn(w, k), \tag{4}$$

where $sn(w, k)$ is the Jacobian elliptic function of w. In order to find the transformation from the z-plane to the w-plane we make use of the equation,

$$\frac{d}{t} = \frac{\partial w}{\partial t}\frac{d}{dw} \tag{5}$$

and the derivative of $t = sn(w, k)$ with respect to w (taken from Whittaker and Watson [7]):,

$$\frac{dsn(w, k)}{dw} = [(1 - sn^2(w, k))(1 - k^2sn^2(w, k))]^{\frac{1}{2}}. \tag{6}$$

Combining Eqs. (3), (4) and (5), one obtains

$$\frac{dz}{dt} = [(1 - sn^2(w,k))(1 - k^2 sn^2(w,k))]^{-\frac{1}{2}} \frac{dz}{dw}. \tag{7}$$

Inserting Eq. (4) into Eq. (1), and combining the resulting equation into Eq. (7), the transformation from z-plane to the w-plane is obtained as

$$\frac{dz}{dw} = -Ak[sn^2(w,k) - c'^2]. \tag{8}$$

Equation (8) can be written in integral form:

$$z = -Ak[\int_0^w sn^2(w,k)dw - c'^2 \int_0^w dw]. \tag{9}$$

In order to calculate the integrals we make use of the equations in Ref. [6],

$$Z(w,k) = E(w,k) - w\frac{E}{K}, \tag{10}$$

where $Z(w,k)$ is the Jacobi Zeta-function, $E(w,k)$ is the elliptic integral of the second kind of w and E is the complete elliptic integral of second kind. The first integral in Eq. (9) is given as

$$\int_0^w sn^2(w',k)dw' = \frac{1}{k^2}[w - E(w,k)]. \tag{11}$$

Eqs. (10) and (11) into Eq. (9) we obtain the transformation from the z-plane to the w-plane as

$$z = \frac{A}{k}[Z(w,k) + w\frac{1}{KK'}\{EK' - KK' + E'K\}]. \tag{12}$$

It is shown in the Appendix that $EK' - KK' + E'K = \frac{\pi}{2}$, inserting this equation into Eq. (12), this transformation can be written as

$$z = \frac{A}{k}[Z(w,k) + w\frac{\pi}{2KK'}]. \tag{13}$$

To find the constant A, we further specify the transformation by requiring that point E, located at $z = h$ in the z-plane, be transformed into the point $w = K$ in the w-plane. One can get

$$h = \frac{A}{k}[Z(K,k) + \frac{\pi}{2K'}]. \tag{14}$$

We also require that point G at $z = h$ in the z-plane transform into $w = K + iK'$ in the w-plane. Then

$$h = \frac{A}{k}[Z(K + iK', k) + (K + iK')\frac{\pi}{2KK'}]. \tag{15}$$

It is also shown in the Appendix that

$$Z(K,k) = 0, \text{and } Z(K + iK', k) = -i\frac{\pi}{2K}.$$

By inserting the first of these equations into Eq. (14) or the second into Eq. (15), we obtain the constant A:

$$A = \frac{2hkK'}{\pi}. \tag{16}$$

Inserting A in Eq. (16) into Eq. (13), we obtain the final form of the transformation from the z-plane to the w-plane:

$$z = \frac{2hK'}{\pi}[Z(w,k) + w\frac{\pi}{2KK'}]. \tag{17}$$

Eq. (17) can be written in another form by using Eq. (10):

$$z = \frac{2hK'}{\pi}[E(w,k) - (\frac{E'}{K'} - 1)w]. \tag{18}$$

We find the transformation from the z-plane to the t-plane by integrating Eq. (1) and it is given as

$$z = -\frac{2hkK'}{\pi}k\int_0^t \frac{t^2 dt}{[(1-t^2)(1-k^2t^2)]^{\frac{1}{2}}} + \frac{2hE'}{\pi}\int_0^t \frac{dt}{[(1-t^2)(1-k^2t^2)]^{\frac{1}{2}}}. \tag{19}$$

The first integral is the elliptic integral of the third kind and is equal to

$D(\varphi,k) = \frac{F(\varphi,k)-E(\varphi,k)}{k^2}$. The second integral is the elliptic integral of the first kind $F(\varphi,k)$, where $\varphi = \text{ArcSin } t$. Combining all these in Eq. (19) we obtain the final form of the transformation from z-plane to the t-plane:

$$z = \frac{2h}{\pi}K'[E(\varphi,k) + (\frac{E'}{K'} - 1)F(\varphi,k)]. \tag{20}$$

In order to find the ratio of the plate length to the plate distances, we consider the point F, which corresponds to $z = h + ib$ in the z-plane and to $w = K + iu$ in the w-plane. Then using Eq. (17), it is obtained

$$h + ib = \frac{2hK'}{\pi}[Z(K+iu,k) + \frac{\pi}{2KK'}(K+iu)] \tag{21}$$

so that

$$\frac{b}{h} = \frac{2K'}{\pi}[-iZ(K+iu,k) + \frac{u\pi}{2KK'}]. \tag{22}$$

Likewise, from Eq. (18), we obtain another useful relationship for the ratio b/h

$$Im[\frac{2K'}{\pi}(E(K+iu,k) - (\frac{E'}{K'} - 1)(K+iu))] = 1; \tag{23}$$

$$Re[\frac{2K'}{\pi}(E(K+iu,k) - (\frac{E'}{K'} - 1)(K+iu))] = \frac{b}{h}. \tag{24}$$

In order to calculate the electric flux lines, Eq. (8) should be taken under consideration:

$$\frac{dz}{dw} = -\frac{2hK'}{\pi}[k^2 sn(w,k) - \frac{E'}{K'}]. \tag{25}$$

In the complex z-plane, $z = x + iy$, and the chain rule for the partial derivation

$$\frac{d}{dz} = \frac{\partial x}{\partial z}\frac{d}{dx} + \frac{\partial y}{\partial z}\frac{d}{dy} = \frac{d}{dx} - i\frac{d}{dy}, \tag{26}$$

and

$$\frac{dw}{dz} = K\frac{dW}{dz}. \tag{27}$$

Let $U = \xi K$ be the electric potential and $V = \eta K$ be the electric flux. Applying Eq. (26) to $W = U + iV$ and inserting into Eq. (27),

$$\frac{dw}{dz} = K[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}](U + iV), \tag{28}$$

carrying out the calculation in Eq. (28) and using the Cauchy-Riemann conditions, which are

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \text{ and } \frac{\partial V}{\partial x} = \frac{\partial U}{\partial y} \text{ it is easily obtained that} \\ \frac{dw}{dz} &= 2K[\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}]. \end{aligned} \tag{29}$$

Inserting Eq. (29) into Eq.(25) and after some mathematical manipulations, the components of the electric flux lines are obtained as

$$E_x - iE_y = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \frac{\pi}{2h}[KE'\{1 - k^2\frac{K'}{E'}sn^2(w,k)\}]^{-1}, \tag{30}$$

where E_x is the real part and E_y is the imaginary part of the right hand side of Eq. (30) .

In order to plot the electric flux lines and equipotential lines, we use $w = \xi + i\eta$ (as in Figure 1.d) in Eq. (18). Then

$$z = \frac{2hK'}{\pi}[E(w,k) - (\xi + i\eta)(K' - E')\frac{1}{KK'}]. \tag{31}$$

As $E(w,k) = E_r(w,k) + iE_i(w,k)$ can be written as the combination of the real and imaginary parts, then the real part of the Eq. (31) gives us the x-coordinate of the point in the z-plane:

$$x = \frac{2hK'}{\pi}[E_r(w,k) + \xi(K' - E')\frac{1}{KK'}]; \tag{32}$$

and the imaginary part, the y-coordinate of the point, in the z-plane:

$$y = \frac{2hK'}{\pi}[E_i(w,k) + \eta(K' - E')\frac{1}{KK'}]. \tag{33}$$

In order to find the electric flux lines, from Eqs. (32) and (33), η is kept constant and ξ is varied from 0 to K, thus obtaining the values of $\{x, y\}$ which lie on the lines of constant electric flux emanating from the plates. The values of $\{x, y\}$ are given in Table 1 for the electric flux lines. To find the potential we do exactly the opposite of the above procedure, in that ξ is kept constant and η is varied from 0 to K', thus obtaining the values of $\{x, y\}$ which lie on the lines of constant electric potential for which the values of $\{x, y\}$ are given in Table 2. In Figure 2, the combined plot of the electric flux lines and constant potential lines are given.

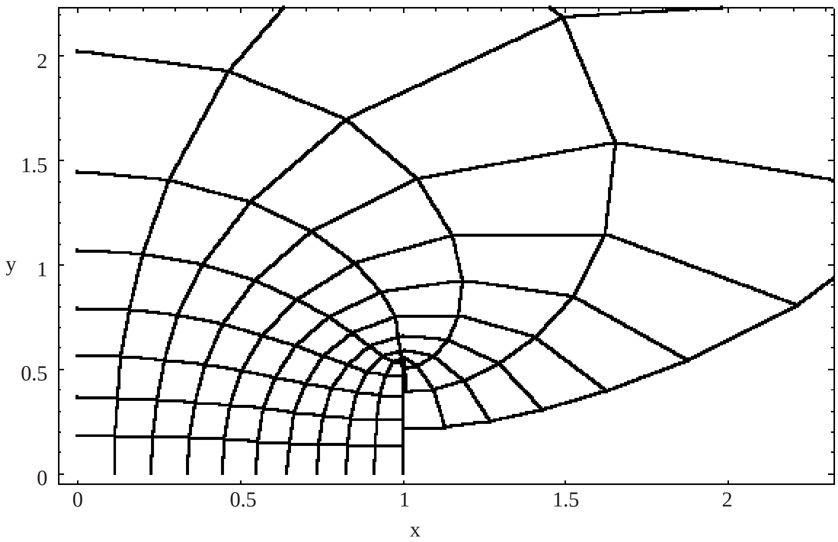

Figure 2. The electric flux lines and the lines of the constant potential. The electric flux lines are labeled with the values of $\frac{\eta}{K'}$ from .0 to .9 and the lines of the constant potential with the values of $\frac{\xi}{K}$ from 0.0 to 1.0. The quarter plate is represented with a thick line.

In order to obtain the electric flux lines and the lines of the constant potential one has to calculate the value of k, the modulus. To do this one can use Eq. (A.9) in Appendix, since we are allowed to determine the length of the plate $2b$ and the distance between the plates $2h$, and can insert the ratio b/h in this equation and solve for k.

To determine the value for u corresponding to point F, the value of c' from (A.5) and the value of k from (A.9) can be inserted into (A.11) and taking the inverse of the Jacobi function $sn(K + iu, k)$. One can check the tables for elliptic functions, but it is easier to solve these equations using Mathematica.

Table 1. The electric flux lines for which η is kept constant and ξ is varied from .0 K to .9 K . The values {x, y} shows the places of the each constant electric flux lines in space.

η= 0.1 K'	η= 0.2 K'	η= 0.3 K'	η= 0.4 K'
{ x , y }	{ x , y }	{ x , y }	{ x , y }
{0.000000, 0.179310}	{0.000000, 0.364908}	{0.000000, 0.564967}	{0.000000, 0.792325}
{0.116982, 0.178038}	{0.123241, 0.361990}	{0.135753, 0.559483}	{0.158681, 0.782322}
{0.232204, 0.174372}	{0.243968, 0.353599}	{0.267357, 0.543793}	{0.309772, 0.753962}
{0.344116, 0.168735}	{0.360013, 0.340756}	{0.391352, 0.520001}	{0.447297, 0.711674}
{0.451556, 0.161756}	{0.469819, 0.324956}	{0.505448, 0.491092}	{0.567856, 0.661423}
{0.553858, 0.154177}	{0.572579, 0.307922}	{0.608699, 0.460367}	{0.670707, 0.609328}
{0.650893, 0.146759}	{0.668255, 0.291375}	{0.701394, 0.430949}	{0.757204, 0.560670}
{0.743043, 0.140202}	{0.757482, 0.276851}	{0.784784, 0.405466}	{0.830012, 0.519456}
{0.831130, 0.135088}	{0.841436, 0.265587}	{0.860780, 0.385920}	{0.892424, 0.488412}
{0.916319, 0.131845}	{0.921675, 0.258475}	{0.931684, 0.373673}	{0.947928, 0.469208}
{1.000000, 0.130735}	{1.000000, 0.256046}	{1.000000, 0.369507}	{1.000000, 0.462717}

η= 0.5 K'	η= 0.6 K'	η= 0.7 K'	η= 0.8 K'
{ x , y }	{ x , y }	{ x , y }	{ x , y }
{0.000000, 1.069890}	{0.000000, 1.443550}	{0.000000, 2.020300}	{0.000000, 3.119920}
{0.200950, 1.051040}	{0.283484, 1.404920}	{0.464371, 1.927640}	{0.954784, 2.819670}
{0.386494, 0.998465}	{0.531031, 1.300590}	{0.825423, 1.694080}	{1.496540, 2.185710}
{0.545685, 0.922417}	{0.721798, 1.157990}	{1.047240, 1.409220}	{1.656580, 1.587260}
{0.674030, 0.835508}	{0.853451, 1.005990}	{1.153230, 1.142170}	{1.624740, 1.147150}
{0.772666, 0.749145}	{0.935523, 0.865374}	{1.183970, 0.923084}	{1.524350, 0.849228}
{0.846121, 0.671708}	{0.981230, 0.747279}	{1.172440, 0.756699}	{1.408810, 0.652571}
{0.900178, 0.608421}	{1.002570, 0.655890}	{1.139400, 0.637591}	{1.296650, 0.524904}
{0.940548, 0.562081}	{1.008740, 0.591694}	{1.096290, 0.558477}	{1.192170, 0.445451}
{0.972330, 0.533966}	{1.006300, 0.553810}	{1.048900, 0.513436}	{1.094290, 0.401978}
{1.000000, 0.524557}	{1.000000, 0.541306}	{1.000000, 0.498827}	{1.000000, 0.388139}

η= 0.9 K'
{ x , y }
{0.000000, 6.336070}
{2.923190, 4.458800}
{3.119650, 2.353520}
{2.649150, 1.311480}
{2.212780, 0.807717}
{1.880420, 0.542316}
{1.629100, 0.391700}
{1.432590, 0.302602}
{1.271250, 0.250230}
{1.130770, 0.222506}
{1.000000, 0.213815}

Table 2. The lines of the constant potential for which ξ is kept constant and η is varied from 0.0 K' to 0.9 K'. The values {x, y} shows the places of the each constant electric potential lines in space.

ξ =0.1 K	ξ =0.2 K	ξ =0.3 K	ξ =0.4 K
{ x , y }	{ x , y }	{ x , y }	{ x , y }
{0.115080,000000.0}	{0.228620,0.000000}	{0.339255,0.000000}	{0.445946,0.000000}
{0.116982,0.178038}	{0.232204,0.174437}	{0.344116,0.168735}	{0.451556,0.161756}
{0.123241,0.361990}	{0.243968,0.353599}	{0.360013,0.340756}	{0.469819,0.324956}
{0.135753,0.559483}	{0.267357,0.543793}	{0.391352,0.520001}	{0.505448,0.491092}
{0.158681,0.782322}	{0.309772,0.753962}	{0.447297,0.711674}	{0.567856,0.661423}
{0.200950,1.051040}	{0.386494,0.998465}	{0.545685,0.922417}	{0.674030,0.835508}
{0.283484,1.404920}	{0.531031,1.300590}	{0.721798,1.157990}	{0.853451,1.005990}
{0.464371,1.927640}	{0.825423,1.694080}	{1.047240,1.409220}	{1.153230,1.142170}
{0.954784,2.819670}	{1.496540,2.185710}	{1.656580,1.587260}	{1.624740,1.147150}
{2.923190,4.458800}	{3.119650,2.353520}	{2.649150,1.311480}	{2.212780,0.807717}

ξ =0.5 K	ξ =0.6 K	ξ =0.7 K	ξ =0.8 K
{ x , y }	{ x , y }	{ x , y }	{ x , y }
{0.548078,0.000000}	{0.645507,0.000000}	{0.738544,0.000000}	{0.827909,0.000000}
{0.553858,0.154177}	{0.650893,0.146759}	{0.743043,0.140202}	{0.831130,0.135088}
{0.572579,0.307922}	{0.668255,0.291375}	{0.757482,0.276851}	{0.841436,0.265587}
{0.608699,0.460367}	{0.701394,0.430949}	{0.784784,0.405466}	{0.860780,0.385920}
{0.670707,0.609328}	{0.757204,0.560670}	{0.830012,0.519456}	{0.892424,0.488412}
{0.772666,0.749145}	{0.846121,0.671706}	{0.900178,0.608421}	{0.940548,0.562081}
{0.935523,0.865374}	{0.981230,0.747279}	{1.002570,0.655890}	{1.008740,0.591694}
{1.183970,0.923084}	{1.172440,0.756699}	{1.139400,0.637591}	{1.096290,0.558477}
{1.524350,0.849228}	{1.408810,0.652571}	{1.296650,0.524904}	{1.192170,0.445451}
{1.880420,0.542316}	{1.629100,0.391700}	{1.432590,0.302602}	{1.271250,0.250230}

ξ =0.9 K	ξ =1.0 K
{ x , y }	{ x , y }
{0.914641,0.000000}	{1.000000,0.000000}
{0.916319,0.131845}	{1.000000,0.130735}
{0.921675,0.258475}	{1.000000,0.255605}
{0.931684,0.373673}	{1.000000,0.369507}
{0.947928,0.469208}	{1.000000,0.462717}
{0.972330,0.533966}	{1.000000,0.524557}
{1.006300,0.553810}	{1.000000,0.541306}
{1.048900,0.513436}	{1.000000,0.498827}
{1.094290,0.401978}	{1.000000,0.388139}
{1.130770,0.222506}	{1.000000,0.213815}

3. Results and Summary

Conformal mapping is a very powerful method, but it is mathematically cumbersome to deal with. The electric flux lines obtained by Lin [3] and the present results in this paper has a very good agreement. As it can be seen from Figure 2, the lines of the constant potential cuts the electric flux lines at right angles. In Figure 2, only a quarter of the parallel plate is shown due to symmetry. We see that the equipotential lines are very closely spaced around the plates while spacing increases away from the plates; and the electric flux lines intersect the plates at right angles in the middle of the capacitor and away from the midpoint edge effects can be clearly seen. The ability to obtain such results and the tabulated data for the electric flux lines and the constant electric potential lines are very important, in applications such as semiconductor device theory where it can be used in the electrostatic case at t=0, and then time-development of devices can be obtained such as the Avalanche Photodiode.

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APPENDIX A

In order to calculate c', we consider differential Eq. (1) which gives the transformation from the z-plane to the t-plane, and consider the points E and G, given in Figures 1b and 1c. The point E and G both correspond to $z = h$ in the z-plane; but point E corresponds to $t = 1$, and point G corresponds to $t = 1/k$ in the t-plane. Writing Eq. (1) in integral form and carrying out the integration from point E to G, we get

$$\int_E^G dz = Ak \int_E^G \frac{(t^2 - c'^2)dt}{[(k^2t^2 - 1)(t^2 - 1)]^{\frac{1}{2}}}. \tag{A.1}$$

Since the points E and G corresponds to the same value h, the left hand side of the above integral vanishes and (A.1) takes the form

$$0 = Ak\{\int_1^{\frac{1}{k}} \frac{t^2 dt}{[(k^2t^2 - 1)(t^2 - 1)]^{\frac{1}{2}}} - c'^2 \int_1^{\frac{1}{k}} \frac{dt}{[(k^2t^2 - 1)(t^2 - 1)]^{\frac{1}{2}}}\}. \tag{A.2}$$

The first integral in (A.2) is calculated as follows: setting $\lambda = \text{ArcSin}\sqrt{\frac{(1-k^2t^2}{(1-k^2}}$ and differentiating λ with respect to t, we obtain the integration operator dt in terms of λ and $d\lambda.t = 1$ corresponds to $\lambda = \pi/2$, and $t = 1/k$ to $\lambda = 0$ Inserting these values into (A.2), we obtain the first integral in (A.2) as

$$-\frac{i}{k^2}\int_0^{\frac{\pi}{2}} [1 - (1 - k^2)\sin^2\lambda]^{\frac{1}{2}} d\lambda = -\frac{i}{k^2} E(\pi/2, k'). \tag{A.3}$$

Using the same procedure we find the Second Integral in (A.2) as

$$-i\int_0^{\frac{\pi}{2}} \frac{d\lambda}{[1 - (1 - k^2)\sin^2\lambda]^{\frac{1}{2}}} = -iK'. \tag{A.4}$$

Inserting the results (A.3) and (A.4) into (A.2), the constant c' is equal to

$$c' = \frac{1}{k}\sqrt{\frac{E'}{K'}}. \tag{A.5}$$

Some other useful equations also can be obtained which are used in obtaining the values {x, y}. The transformation from z-plane to t-plane takes the point E from $z = h$

in the z-plane to t =1 in the t-plane in Figures 1b and 1c, respectively. Putting these values in Eq. (20), we have

$$\frac{\pi}{2} = K'[E(\text{ArcSin}1, k) + (\frac{E'}{K'} - 1)F(\text{ArcSin}1, k)], \tag{A.6}$$

where $E(\text{ArcSin}1, k) = E(\pi/2, k) = E$ and $F(\text{ArcSin}1, k) = F(\pi/2, k) = K$. Inserting these into Eq. (A.6), we obtain

$$EK' - KK' + E'K = \frac{\pi}{2}. \tag{A.7}$$

The point F corresponds to $z = h + ib$ and $t = c'$ in the z-plane and t-plane, respectively. Putting these values again in Eq. (20), we have

$$\text{Re}\,[\frac{2}{\pi}K'E(\text{ArcSin}c', k) + (\frac{E'}{K'} - 1)F(\text{ArcSin}\,c', k)] = 1 \tag{A.8}$$

$$Im[\frac{2}{\pi}K'E(\text{ArcSin}\,c', k) + (\frac{E'}{K'} - 1)F(\text{ArcSin}\,c', k)] = \frac{b}{h} \tag{A.9}$$

Now referring to Figs. 3 and 4, the point E corresponds to $t = 1$ and $w = K$ in the $t-$ and w-planes respectively, then Eq. (4) gives us

$$sn(K, k) = 1. \tag{A.10}$$

The point F corresponds to $t = c'$ and $w = K + iu$,

$$c' = sn(K + iu, k). \tag{A.11}$$

Finally the point G corresponds to $t = 1/k$ and $w = K + iK'$, inverse of the modulus k is

$$\frac{1}{k} = sn(K + iK', k) \tag{A.12}$$

Referring to Figures 1b and 1d, the point E corresponds to $z = h$ and $w = K$ in the z- and w-planes. Inserting these values into the Eq. (13) and arranging, we have

$$Z(K, k) = 0. \tag{A.13}$$

The point G corresponds to $z = h$ and $w = K + iK'$, and thus

$$Z(K + iK', k) = -i\frac{\pi}{2K}. \tag{A.14}$$

All equations shown here are used throughout the paper as indicated.