Turk J Phy
25 (2001) , 87 – 95.

Zeros of Scattering Amplitude for a Spin-Dependent Gaussian Potential

Nilufar Farhat HOSSAIN
Department of Mathematics
Bangladesh University of Engineering & Technology, Dhaka-BANGLADESH
Hasna BANU
Department of Mathematics
King Saud University, Riyadh-SAUDI ARABIA
Amal HALDER
Department of Mathematics, Dhaka University, Dhaka-BANGLADESH

Received 18.11.1999

Abstract

In the proposed model, polarization in the scattering amplitude for a Gaussian potential including spin is considered. The problem of scattering amplitude is formulated in two different ways: the first and second Born approximations along with their high energy limits and the eikonal approximation including Wallace corrections. A comparison of the numerical zeros of the scattering amplitude computed in the above two approximations is studied in the complex momentum transfer plane. An analysis of the zero trajectories shows that the contribution of the spin part in the potential is quite significant at high energies whereas at low energies the spin-part has no appreciable effect.

Key Words: Scattering amplitude, Born approximation, Eikonal approximation, Wallace correction, Gaussian potential.

1. Introduction

The recent study [1-5] of the scattering of particles by Gersten, Barrelet and others has widened largely the possibilities of employing the complex zeros of the scattering amplitude. Gersten [1-3] discussed the possibility of reconstructing the scattering amplitude with the aid of the zeros and has shown that the treatment of the ambiguities in phase shift analysis is greatly simplified, if one considers the complex zeros of the scattering amplitude in the cosine of the scattering angle plane. De Shalit [6] pointed out that the polarization patterns in direct nuclear reactions can be explained simply by the position of the complex zeros. This idea has been employed by Buhring [7] to analyze the slow electrons elastically scattered from heavy atoms. Barrelet [8] analyzed the employment of the zeros of the scattering amplitudes in reconstructing the scattering of spin-$\frac{1}{2}$ and spin-0 particle scattering. He investigated the π^+p scattering data and has shown that the analysis of the complex zeros has many advantages including the analysis of resonance position and identifying their spin. Carter [9] used the zeros of differential cross-section for the process spin $(\frac{1}{2}+\frac{1}{2}) \rightarrow (0+0)$ using the reaction $\bar{p}p \rightarrow \pi^-\pi^+$ and found evidence for the existence of three dominant meson spin states and a detailed fit to the data obtained supported a resonance interpretation of these states.

In the elastic scattering of intermediate energy protons by spin-zero nuclei the strong spin-orbit force gives rise to interesting polarization effects. The polarization is quite large so that the spin-flip amplitude must be significant. To reproduce this, a spin-orbit potential of Gaussian shape is studied in this paper. The first and second Born approximation to the general off-shell scattering amplitude for the scattering from a potential with a spin-orbit part as well as a central component are formulated. The high energy limits of these amplitudes on energy shell are then obtained. A reliable estimate for the scattering amplitudes to first and second order in the potential at high energies with small scattering angles provided by eikonal series are also derived. Corrections to the eikonal approximation suggested by Wallace [10,11] and evaluated by Waxman et al. [12] for a velocity dependent and spin-orbit potential have been included and compared with the high energy Born amplitudes. We then find the zeros of the amplitudes obtained in the above two approaches. The behaviour of the zero trajectories obtained from the different approximation methods for our model potential are then studied in the complex momentum transfer plane which turn out to be the best to study the nature of the zero trajectories.

2. Formulation of scattering amplitude with spin

The formal expression [13] for the scattering matrix in the simplest case of scattering of spin-half particles by a spin-zero target is given by

$$\hat{F}(\mathbf{k}, \cos\theta) = F(\mathbf{k}, \cos\theta) + \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}G(\mathbf{k}, \cos\theta), \tag{1}$$

where $\boldsymbol{\sigma}$ is the Pauli spin vector and $\hat{\mathbf{n}}$ is the axial unit vector perpendicular to the scattering plane. The differential cross-section for a transition from $(\mathbf{k}, \nu)$ to $(\mathbf{k}', \nu')$ states is formulated as

$$\frac{d\sigma_\nu}{d\Omega} = |F|^2 + |G|^2 + (F^\star G + FG^\star)\hat{\mathbf{n}}\cdot\mathbf{P}_i, \tag{2}$$

where $\mathbf{P}_i = < \chi_\nu|\boldsymbol{\sigma}|\chi_\nu >$ is the initial polarization vector, $\mathbf{k}$ and $\mathbf{k}'$- the incident and final momenta and χ_ν and $\chi_{\nu'}$ are the initial and final spin eigenvectors, respectively. We observe that, in general, the above expression depends not only on the momentum and the scattering angle but also on the spin orientation of the incident particles, or more exactly, on the angle between the normal $\hat{\mathbf{n}}$ to the scattering plane and the direction of polarization $\mathbf{P}_i$. An interesting particular case is that we choose the axis of quantization along the vector $\hat{\mathbf{n}}$ perpendicular to the scattering plane. For example, if we write $\chi_{\nu=\frac{1}{2}}$ and $\chi_{\nu=-\frac{1}{2}}$ for the spinors describing incident particles respectively polarized parallel and antiparallel to $\hat{\mathbf{n}}$, the differential cross section then reduces to

$$\frac{d\sigma_{\pm\frac{1}{2}}}{d\Omega} = |F \pm G|^2 \tag{3}$$

and we get a quite simple structure for the 2×2 matrix $\hat{F}$ in an explicit form

$$\hat{F} = \begin{pmatrix} F+G & 0 \\ 0 & F-G \end{pmatrix} \tag{4}$$

where no spin-flip is allowed.

3. First and Second Born approximation for a Gaussian potential

Considering the non-relativistic spin-zero and spin-half scattering, we take a simple model of spin-dependent Gaussian potential of the form

$$V(r) = V_c(r) + \boldsymbol{\sigma} \cdot \mathbf{L} V_s(r) = V_0 e^{-\alpha^2 r^2/2} + \boldsymbol{\sigma} \cdot \mathbf{L} V_2 e^{-\xi^2 r^2/2} \tag{5}$$

The spin non-flip and spin-flip parts of the first Born approximation to the scattering amplitude for this potential are given by

$$F_{B1}(\mathbf{k}, \mathbf{k}') = -\frac{\mu V_0}{\hbar^2} \frac{\sqrt{2\pi}}{\alpha^3} e^{-q^2/2\alpha^2} \tag{6}$$

$$G_{B1}(\mathbf{k}, \mathbf{k}') = -|\mathbf{k} \times \mathbf{k}'| \; \frac{i\mu V_2}{\hbar^2} \frac{\sqrt{2\pi}}{\xi^5} e^{-q^2/2\xi^2}, \tag{7}$$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the momentum transfer. In the second Born approximation for the same potential, we get the spin non-flip and spin-flip parts of the scattering amplitude as

$$\begin{aligned} F_{B2}(\mathbf{k}, \mathbf{k}') = {} & (\frac{\mu V_0}{\hbar^2 \alpha^2})^2 \frac{e^{-q^2/4\alpha^2}}{K} [\sqrt{\pi} Q_{\alpha 1}(K) + \frac{i\pi}{\alpha} E_{\alpha 1}(K)] \\ & + (\frac{\mu V_2}{2\hbar \xi^2})^2 e^{-q^2/4\xi^2} e^{-K^2/\xi^2} \\ & \times [4(\mathbf{k} \cdot \mathbf{k}') \rho'_\xi(K^2) - (\mathbf{k} \times \mathbf{k}')^2 \rho''_\xi(K^2)] \end{aligned} \tag{8}$$

$$\begin{aligned} G_{B2}(\mathbf{k}, \mathbf{k}') = {} & |\mathbf{k} \times \mathbf{k}'| (\frac{\mu \eta}{\hbar^2})^2 \frac{i\hbar V_0 V_2}{2\alpha^3 \xi^3} [e^{-k^2/2\xi^2} e^{-k'^2/2\alpha^2} \rho'_\eta(M^2) \\ & + e^{-k^2/2\alpha^2} e^{-k'^2/2\xi^2} \rho'_\eta(N^2) \\ & - i (\frac{\mu V_2}{2\hbar \xi^2})^2 e^{-q^2/4\xi^2} e^{-K^2/\xi^2} \, 2\, \rho'_\xi(K^2)], \end{aligned} \tag{9}$$

where

$$K = |\mathbf{K}| = |\frac{\mathbf{k}+\mathbf{k}'}{2}|, \qquad \eta^2 = \frac{2\alpha^2\xi^2}{\alpha^2+\xi^2}$$
$$Q_{\alpha 1}(K) = D(\frac{k_0+K}{\alpha}) - D(\frac{k_0-K}{\alpha}), \quad E_{\alpha 1}(K) = e^{-(k_0-K)^2/\alpha^2} - e^{-(k_0+K)^2/\alpha^2}$$
$$\rho_\xi(K^2) = \frac{e^{K^2/\xi^2}}{K^2}[\sqrt{\pi}Q_{\xi 1}(K) + \frac{i\pi}{2}E_{\xi 1}(K)]$$
$$\mathbf{M} = \frac{\eta^2}{2\alpha^2\xi^2}(\alpha^2\mathbf{k}+\xi^2\mathbf{k}'), \qquad \mathbf{N} = \frac{\eta^2}{2\alpha^2\xi^2}(\xi^2\mathbf{k}+\alpha^2\mathbf{k}') \tag{10}$$

$D(z) = e^{-z^2}\int_0^z e^{x^2}dx$ being the Dawson's integral [14] and k_0 is the magnitude of the momentum in the intermidiate stage. Also, $\rho'_\xi(K^2)$ and $\rho''_\xi(K^2)$ are the first and second order derivatives of $\rho_\xi(K^2)$, respectively, and $\rho'_\eta(M^2)$ and $\rho'_\eta(N^2)$ are obtained by just replacing ξ by η and K^2 by M^2 and N^2, respectively, in $\rho'_\xi(K^2)$.

4. High energy limit of the Gaussian amplitudes

For high energy limit $ka >> 1$,which is the short wave length condition, 'a'-being the range of the potential, we consider the amplitudes on-shell, where $|\mathbf{k}| = |\mathbf{k}'| = k_0$ leading to the condition $k_0^2 >> q^2$ with $q^2 = 2k_0^2(1-\cos\theta)$. The spin non-flip amplitude and the spin flip amplitudes in the second Born approximation then become

$$F_{B2H}(q,k_0) = (\frac{\mu V_0}{\hbar^2\alpha^2})^2 e^{-q^2/4\alpha^2}\{\frac{\sqrt{\pi}\alpha}{4k_0^2}(1-\frac{q^2}{2\alpha^2}) + \frac{i\pi}{2k_0}\} + (\frac{\mu V_2}{2\hbar\xi^2})^2 e^{-q^2/4\xi^2}$$
$$\times[\frac{2i\pi k_0}{\xi^2}(1-\frac{q^2}{4\xi^2}) + \frac{4\sqrt{\pi}}{\xi}(\frac{3}{4} - \frac{7q^2}{16\xi^2} + \frac{q^4}{32\xi 4})] \tag{11}$$

$$G_{B2H}(q,k_0) = k_0^2\sin\theta \;\; [\frac{i\hbar V_0 V_2}{(\alpha\xi)^5}(\frac{\mu\eta^3}{\hbar^2})^2 e^{-q^2/\{2(\alpha^2+\xi^2)\}}$$
$$\{\frac{i\pi}{2\eta^2 k_0} + \frac{\sqrt{\pi}}{\eta k_0^2}(\frac{3}{4} - \frac{\eta^2 q^2}{8\alpha^2\xi^2})\}$$
$$+\frac{1}{2i}(\frac{\mu V_2}{\hbar^2\xi^2})^2 e^{-q^2/4\xi^2}\{\frac{i\pi}{2\xi^2 k_0} + \frac{\sqrt{\pi}}{\xi k_0^2}(\frac{3}{4} - \frac{q^2}{8\xi^2}\}]. \tag{12}$$

In the evaluation of these amplitudes we keep the dominant terms and neglected the terms of order lower than $1/k_0^2$. The spin non-flip and spin-flip amplitudes in the first Born approximation for high energy limit are the same as we obtained earlier for the simple first Born approximation, so that we have

$$F_{B1H}(q) = -\frac{\mu V_0}{\hbar^2}\frac{\sqrt{2\pi}}{\alpha^3}e^{-q^2/2\alpha^2} \tag{13}$$

$$G_{B1H}(q,k_0) = k_0^2 \sin\theta \;\; [-\frac{i\mu V_2}{\hbar^2}\frac{\sqrt{2\pi}}{\xi^5}e^{-q^2/2\xi^2}]. \tag{14}$$

5. Eikonal approximation for Gaussian potential

The impact parameter representation for the spin non-flip and spin-flip amplitudes are given by [13]

$$F_E(q,k_0) = -ik_0\int_0^\infty bdbJ_0(qb)\Gamma_f(b) \tag{15}$$

$$G_E(q,k_0) = k_0^2 \sin\theta[-\frac{i}{q}\int_0^\infty bdbJ_1(qb)\Gamma_g(b)], \tag{16}$$

where the profile functions $\Gamma_g(b)$ and $\Gamma_f(b)$ are

$$\Gamma_g(b) \simeq -e^{i\bar{\chi}(b)}\sin[\Delta\chi(b)] \;\; ; \;\; \Gamma_f(b) \simeq 1 - e^{i\bar{\chi}(b)}\sin[\Delta\chi(b)] + i\frac{\Gamma_g(b)}{2k_0 b} \tag{17}$$

with

$$\bar{\chi}(b) = \delta_{l_+} + \delta_{l_-}; \;\; \Delta\chi(b) = \delta_{l_+} - \delta_{l_-}; \;\; \delta_{l_\pm} = -\frac{\mu}{\hbar^2 k_0}\int_{-\infty}^{\infty} V_{l_\pm}(r)dz \tag{18}$$

and

$$V_{l_+} = V_c(r) + lV_s(r); \quad J = l + \frac{1}{2} = l_+ \, , \tag{19}$$

$$V_{l_-} = V_c(r) - (l+1)V_s(r); \quad J = l + \frac{1}{2} = l_- \tag{20}$$

The impact parameter 'b' has the quantal definition $k_0 b = l + \frac{1}{2}$, 'l' being the partial waves and $\delta_{l_\pm}$- the partial wave phase shifts. If we consider Wallace-Waxman [13] expansion for correction in the eikonal phases we see that to the lowest order correction we get the spin non-flip amplitude to first and second order in V as

$$F_{E1}(q) = - \;(\frac{\mu V_0}{\hbar^2})\frac{\sqrt{2\pi}}{\alpha^3}e^{-q^2/2\alpha^2} \tag{21}$$

$$F_{E2}(q,k_0) = (\frac{\mu V_0}{\hbar^2})^2\frac{\pi i}{2k_0\alpha^4}e^{-q^2/4\alpha^2} + (\frac{\mu V_2}{\hbar^2})^2\frac{k_0\pi i}{2\xi^6}(1-\frac{q^2}{4\xi^2})e^{-q^2/4\xi^2}$$
$$-(\frac{\mu V_2}{\hbar^2})^2\frac{\pi i}{8k_0\xi^4}e^{-q^2/4\xi^2} \tag{22}$$

and the spin-flip amplitudes to first and second order in V as

$$G_{E1}(q,k_0) = k_0^2 \sin\theta[-i(\frac{\mu V_2}{\hbar^2})\frac{\sqrt{2\pi}}{\xi^5}e^{-q^2/2\xi^2}] \tag{23}$$

$$G_{E2}(q,k_0) = k_0^2 \sin\theta[-(\frac{\mu}{\hbar^2})^2\frac{V_0 V_2 \pi\eta^4}{2k_0\alpha^5\xi^5}e^{-q^2/2(\alpha^2+\xi^2)} + (\frac{\mu V_2}{\hbar^2})^2\frac{\pi}{4k_0\xi^6}e^{-q^2/4\xi^2}]. \tag{24}$$

Comparing these eikonal amplitudes with those obtained in the Born approximation method, we observe that, to the zeroth order Wallace-Waxman correction, the eikonal result for the first order potential strength exactly reproduces the first order Born amplitude at high energy. But for the second order potential strength, the eikonal result for the non-spin part reproduces only the imaginary part of the second order Born amplitude at high energy and the eikonal result for the spin part reproduces only the real part of the second order Born amplitude for high energy. We therefore consider the first correction term in the Wallace-Waxman [13] expansion and expand $\Gamma_g(b)$ and $\Gamma_f(b)$ to second order in V and have the spin-flip amplitudes to first and second order in V in the following form:

$$G_{E1}(q,k_0) = k_0^2 \sin\theta\ \ [-i\frac{\mu V_2}{\hbar^2}\frac{\sqrt{2\pi}}{\xi^5}e^{-q^2/2\xi^2}] \tag{25}$$

$$\begin{aligned} G_{E2}(q,k_0) = {} & k_0^2 \sin\theta[i(\frac{\mu}{\hbar^2})^2\frac{V_0V_2}{k_0^2}\frac{\sqrt{\pi}\eta^5}{\alpha^5\xi^5}(\frac{3}{4}-\frac{\eta^2q^2}{8\alpha^2\xi^2})e^{-q^2/2(\alpha^2+\xi^2)} \\ & - i(\frac{\mu V_2}{\hbar^2})^2\frac{\sqrt{\pi}}{4k_0^2\xi^5}(\frac{3}{2}-\frac{q^2}{4\xi^2})e^{-q^2/4\xi^2} - (\frac{\mu}{\hbar^2})^2\frac{V_0V_2\pi}{k_0}\frac{\eta^4}{2\alpha^5\xi^5}e^{-q^2/2(\alpha^2+\xi^2)} \\ & + (\frac{\mu V_2}{\hbar^2})^2\frac{\pi}{4k_0\xi^6}e^{-q^2/4\xi^2}] \end{aligned} \tag{26}$$

and the spin non-flip amplitude to first and second order in V as

$$F_{E1}(q) = -\frac{\mu V_0}{\hbar^2}\frac{\sqrt{2\pi}}{\alpha^3}e^{-q^2/2\alpha^2} \tag{27}$$

$$\begin{aligned} F_{E2}(q,k_0) = {} & (\frac{\mu V_0}{\hbar^2\alpha^2})^2 e^{-q^2/4\alpha^2}[\frac{i\pi}{2k_0}+\frac{\sqrt{\pi}\alpha}{4k_0^2}(1-\frac{q^2}{2\alpha^2})] \\ & + (\frac{\mu V_2}{\hbar^2\xi^2})^2 e^{-q^2/4\xi^2}[\frac{i\pi k_0}{2\xi^2}(1-\frac{q^2}{4\xi^2}) - \{\frac{i\pi}{8k_0}+\frac{\sqrt{\pi}\xi}{16k_0^2}(1-\frac{q^2}{2\xi^2})\} \\ & + \frac{\sqrt{\pi}}{\xi}\ \ (\frac{3}{4}-\frac{7q^2}{16\xi^2}+\frac{q^2}{32\xi^4})] \end{aligned} \tag{28}$$

These equations show that the eikonal result completely reproduces the second order Born result at high energies in addition with the term containing $1/k_0$ in the non-spin part. Thus we observe that a Wallace correction as an expansion in the eikonal phase shift function forces the second order eikonal amplitude to reproduce all the real and dominant terms of the second order Born amplitude in the high energy limit.

6. Zeros of the Gaussian amplitudes

Since the functions $F+G$ and $F-G$ are simply connected by the relation $F(-\theta)+G(-\theta)=F(\theta)-G(\theta)$, it is sufficient to formulate the zeros of $F+G$ only. On energy shell we have $|\mathbf{k}|=|\mathbf{k}'|=k_0$ which gives $\boldsymbol{\sigma}\cdot(\mathbf{k}\times\mathbf{k}')=\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}k_0^2\sin\theta$. To obtain the zeros of scattering amplitudes in each of the three approximations we introduce the Mandelstam variable $-q^2=t=t_r+it_i$ in the s-channel. For simplicity we take $\alpha=\xi$ which gives $\eta^2=\alpha^2=\xi^2$ and $\mathbf{M}=\mathbf{N}=\mathbf{K}$. For the zeroth order Wallace correction the zeros of the amplitude from the eikonal approximation do not give us a clear picture of the zero trajectories except at very high energy. So we combine this pure eikonal solutions with the first and second order Born solutions in the high energy limit. Using the expression for $-q^2$ from eikonal approximation, in the zero solutions of the high energy Born amplitude, we have the zeros for eikonal-Born amplitude. In each of the three cases a solution $t=(t_r,t_i)$ can be sought out by using computer minimization routine. We note that the function $t(k_0)$ has an infinite number of sheets due to the logarithmic cut and the principal sheet corresponds to the first zero trajectory. Working with the natural units $\hbar=c=1$ and using computer minimization programs we calculate the principal or first zero trajectories in the C.M system in all three cases for a range of the momentum k, between $0.20\,\text{fm}^{-1}$ to $20.0\,\text{fm}^{-1}$. A small repulsive potential $V_0=0.95\text{fm}^{-1}$ for the central part is taken throughout. The potential strength of the spin orbit part V_s is taken as 0.095fm^{-1} and for simplicity, the shape parameter α and ξ both are taken as 0.99fm^{-1}.

7. Discussion and conclusion

Due to the presence of the spin-orbit part in the potential $V(r)$ we have the zeros in pairs. The computer program is so designed that it calculates simultaneously the zeros of the scattering amplitude for the central potential only with $V_s=0$ as well as the zeros of the scattering amplitude for the potential which includes both the central part and the spin-orbit part. The zero trajectories obtained for $F+G$ in all the three approximations together with the zero trajectories obtained for the zero spin cases are shown in Figure 1 in the complex momentum transfer plane. In Figure we observe that we have three different curves, for each of the three approximations: the bold one for the zero spin cases and the thinner and the dotted one for the spin-dependent cases. First, we observe that the zero spin curves though remain far apart from each other at low energies, become close at high energies. We also observe that in each of the three cases the spin-dependent curves converge slowly towards the corresponding zero spin curves with the increase of energy. After a certain stage, the sets of spin-dependent curves cross the corresponding zero spin curves and then diverge. The values of the momentum k at the crossing points lie between 1.85 and 2.0fm^{-1}. After diverging, the first set of spin-dependent curves (the thinner lines) converge at a point below the zero spin curves and the second set of spin-dependent curves (the dotted lines) converge at a point above the zero spin curves. This analyses that the contribution of the spin-orbit part in the potential is quite significant at high energies whereas at low energies the spin-orbit part has no appreciable effect.

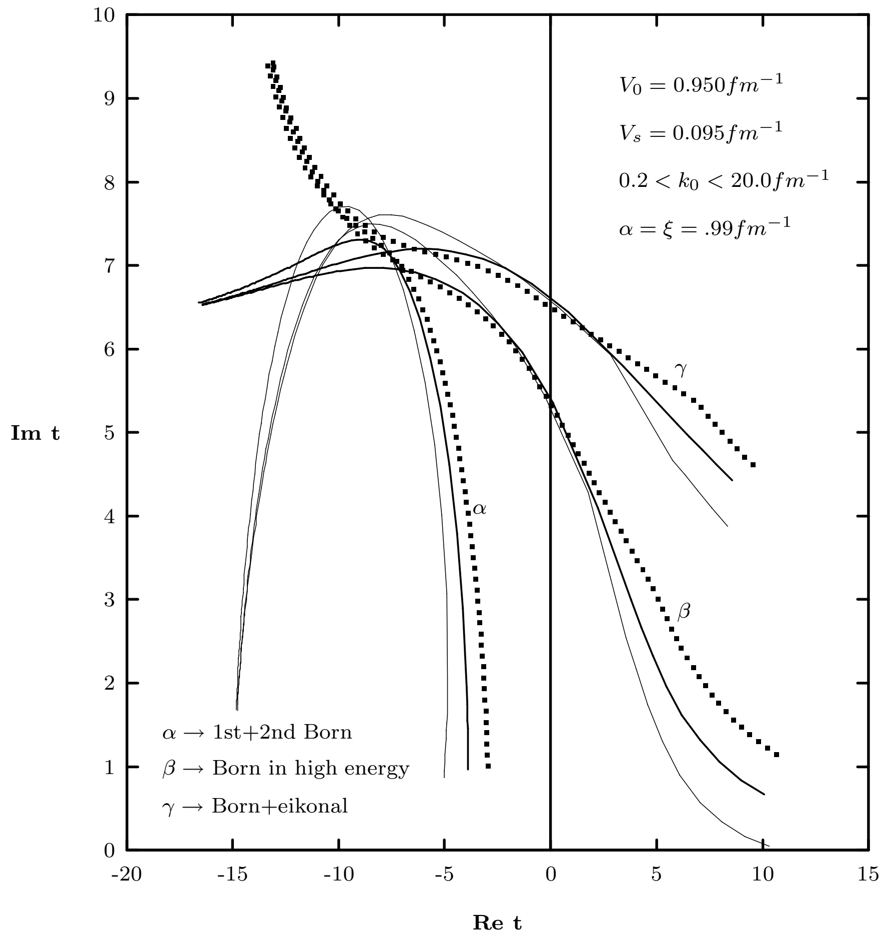

Figure 1. Comparison of the Zero trajectories in the complex-t plane for a spin dependent Gaussian potential with the non-spin part as V_0 and the spin-orbit part as V_s in three different approximation methods.

To conclude, we observe that the behaviour of the zero trajectories we obtain for the spin-dependent Gaussian potential is similar to those of the corresponding spin-independent potential except perhaps for large values of the momentum k. The behaviour of the zero trajectories also shows that the contribution of spin part in the potential is of little significance in the range of energy 140Mev to 550Mev. But for energy higher than

550Mev, it is essential to take account of the spin part in the potential. Also the zero solutions for the Born amplitude in the high energy limit and the zero solutions for the eikonal plus Born amplitude both become close to the numerical Born solutions at high energies as expected.

References

[1] . Gersten, *Nucl. Phys.,* **A** 219 (1974) 317.

[2] . Gersten, *Journal De Physique,* **C** 2 (1985) 471.

[3] . Gersten, *Nucl. Phys.,* **B** 12 (1969) 537.

[4] . E. Bowcock et al., *J. Phys. G: Nucl. Phys.,* 11 (1985) 1145.

[5] . Höhler, I. S. Stefanescu, *Z. Physik C. Particles and Fields,* 2 (1979) 253.

[6] . de-Shalit, in preludes in theoretical physics (North-Holland, Amsterdam. 1966).

[7] . Buhring, *Z. Phys.,* 208 (1968) 286.

[8] . Barrelet, *Nuovo Cimento,* **A** 8 (1972) 331.

[9] . Carter, *Phys. Lett.,* **B** 67 No.1 (1977) 122.

[10] . J. Wallace, *Phys. Rev. Lett.,* 27 No.8 (1971) 622.

[11] . J. Wallace, *Phys. Rev.,* **D** 8 No.6 (1973) 1846.

[12] . Waxman et al., *Phys. Rev.,* **C** 24 No.2 (1981) 578.

[13] . J. Joachain, *Quantum Collision Theory* (North Holland, Amsterdam. 1983) p.490.

[14] . Abramowitz and I. A. Stegun, *Handbook of Mathemetical Functions,* (Constable and Company, U.S.A. 1972) p.297.