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Consistency of the Born Approximation for Aharonov-Bohm Scattering with Massless spin-$\frac{1}{2}$ Particles

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Abstract

The second order contribution for the relativistic scattering of massless spin-$\frac{1}{2}$ particles from an infinitely long solenoid, in the context of covariant perturbation theory, is calculated and shown to vanish. Thus, Born approximation is consistent for this case as well.

1. Introduction

It is a well-known fact that a pure Chern-Simons (CS) gauge field creates an Aharonov-Bohm (AB)-like interaction when particles of conventional statistics are coupled to it [1]. In the context of Galilean field theory, the AB problem can be reconsidered as an arbitrary scattering process by restricting the attention to the N-body sector allowing one to derive a Schroedinger equation for N-body problems [2]. Thus the two particle sector of field theory is formally equivalent to the conventional AB Schroedinger equation [3].

The failure of Born approximation for the AB scattering amplitude, when applied to Schroedinger equation, has been known for some time [4-5]. The source of this failure can be traced to the integral equation satisfied by the lowest partial wave amplitude which contains a quadratic interaction term in the flux parameter. Hence, in the first Born approximation this partial wave does not exist.

In literature there are several works which address this problem. For the spinless case the problem is solved in [6-7]. AB scattering to spin-$\frac{1}{2}$ particles was considered in the context of Dirac equation formalism in [8], and using covariant perturbation theory in [9]. In both of these works, it was shown that Born approximation gives the correct result by demonstrating that this amplitude agrees with the corresponding term in the series expansion of the exact amplitude. In [10] and [11] the consistency of the spin-$\frac{1}{2}$ AB problem was considered in the framework of equivalent Galilean gauge field theory from a different point of view.

When looking at the consistency of the spin-$\frac{1}{2}$ AB problem from a general perspective, one should note that, in the series expansion of the exact amplitude, which is proportional to $\sin \pi\alpha$ (where $\alpha = -\frac{e\phi}{2\pi}$ and ϕ is the magnetic flux carried by the solenoid), $\mathrm{O}(\alpha^2)$ term is missing, to lowest order. However, the demonstration of the full consistency of the Born approximation requires that, aside from the agreement on $\mathrm{O}(\alpha)$ term, the $\mathrm{O}(\alpha^2)$ contribution should vanish as well. This was already done in [10] for the two particle sector in the context of Galilean field theory.

In this work our aim is to carry out the second order perturbative analysis of the relativistic scattering of massless spin-$\frac{1}{2}$ particles from an infinitely long solenoid, along the lines of [9]. As it was shown there, that the first order perturbative term agrees with the corresponding one in the series expansion of the exact amplitude, we fully demonstrate the the validity of the Born approximation by calculating the $\mathrm{O}(\alpha^2)$ term contribution to the scattering amplitude in the framework of covariant perturbation theory and verify that it indeed vanishes.

2. Second Order Covariant Perturbation Theory

Our starting point is the well-known S-matrix in the second order which is defined as

$$S_{fi}^{(2)} = \int\int d^4x d^4y \bar{\psi}_f(x)(-ie\gamma^\mu A_\mu(x)) iS_F(x-y)(-ie\gamma^\nu A_\nu(y))\psi_i(y), \tag{1}$$

where $S_F(x-y)$ is the propagator for massless particles and by definition has the following expression:

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{\gamma^\mu p_\mu}{p^2 + i\varepsilon}. \tag{2}$$

In the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, the vector potential A_μ of the solenoid is taken along the 3rd axis and is given as:

$$\begin{aligned} A_1(z) &= -\frac{\phi}{2\pi}\frac{z_2}{z_1^2+z_2^2} \\ A_2(z) &= -\frac{\phi}{2\pi}\frac{z_1}{z_1^2+z_2^2} \\ A_3(z) &= A_0(z) = 0, \end{aligned} \tag{3}$$

where ϕ is the magnetic flux carried by the solenoid. In Eq. (1), ψ_f and ψ_i are the plane wave solutions for massless spin- $\frac{1}{2}$ particles and, in the Bjorken-Drell convention, can be written as:

$$\begin{aligned} \psi_i(z) &= \sqrt{\frac{1}{2E_i(2\pi)^3}}u(p_i,s_i)e^{-ip_{i\mu}z^\mu} \\ \psi_f(z) &= \sqrt{\frac{1}{2E_f(2\pi)^3}}u(p_f,s_f)e^{-ip_{f\mu}z^\mu}. \end{aligned} \tag{4}$$

The Dirac spinors for the polarized initial and final particles can be constructed as:

$$\begin{aligned} u(i) &= \cos\frac{\theta'}{2}e^{\frac{-i\varphi'}{2}}u_+(i)+\sin\frac{\theta'}{2}e^{\frac{i\varphi'}{2}}u_-(i) \\ u(f) &= (\cos\frac{\theta}{2}\cos\frac{\theta'}{2}e^{\frac{-i\varphi'}{2}}+\sin\frac{\theta}{2}\sin\frac{\theta'}{2}e^{\frac{i\varphi'}{2}})u_+(f) \\ &\quad +(\cos\frac{\theta}{2}\sin\frac{\theta'}{2}e^{\frac{i\varphi'}{2}}-\sin\frac{\theta}{2}\cos\frac{\theta'}{2}e^{\frac{-i\varphi'}{2}})u_-(f), \end{aligned} \tag{5}$$

where $u_\pm(i)$ and $u_\pm(f)$ are the helicity eigenstates for the massless initial and final fermions:

$$\begin{aligned} u_+(i) &= \sqrt{E_i}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad u_-(i)=\sqrt{E_i}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \\ u_+(f) &= \sqrt{E_f}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}, \; u_+(i)=\sqrt{E_f}\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}. \end{aligned} \tag{6}$$

After these definitions, we first carry out the spatial integrals in Eq. (1), and obtain

$$\begin{aligned} S_{fi}^{(2)} &= \frac{i}{2(2\pi^3)}\frac{(e^2\phi^2)}{\sqrt{E_iE_f}}\delta(E_f-E_i)\delta(p_{f3}-p_{i3}) \\ &\quad \int d^2p_\perp \frac{N}{(\vec{p_{f_\perp}}^2-\vec{p_\perp}^2)(\overrightarrow{p_f-p})_\perp^2(\overrightarrow{p_i-p})_\perp^2} \end{aligned} \tag{7}$$

with

$$N = (p_i-p)_2(p_f-p)_1\bar{u}_f\gamma^1(\gamma^0E_f-\gamma^3p_1-\gamma^1p_2)\gamma^3u_i +$$

$$(p_i - p)_1(p_f - p)_2\bar{u}_f\gamma^3(\gamma^0 E_f - \gamma^3 p_1 - \gamma^1 p_2)\gamma^1 u_i -$$
$$(p_i - p)_1(p_f - p)_1\bar{u}_f\gamma^1(\gamma^0 E_f - \gamma^3 p_1 - \gamma^1 p_2)\gamma^1 u_i -$$
$$(p_i - p)_2(p_f - p)_2\bar{u}_f\gamma^3(\gamma^0 E_f - \gamma^3 p_1 - \gamma^1 p_2)\gamma^3 u_i \cdot \tag{8}$$

To carry out the integral in Eq. (7), we rewrite the numerator in terms of the polar angle in the $p_\perp$ plane which we define by φ, making use of the energy conservation provided by $\delta(E_f - E_i)$,

$$\begin{aligned}
N &= \alpha + \beta\cos\varphi + \gamma\sin\varphi \\
\alpha &= k^3\{\frac{E_i}{k}\{A\sin\theta - B\cos\theta\} - u^2\{\frac{E_i}{k}B + D\sin\theta + C(1+\cos\theta)\}\} \\
\beta &= k^3\{Cu^3 + u\{D\sin\theta + C\cos\theta + \frac{E_i}{k}\{(1+\cos\theta)B - A\sin\theta)\}\}\} \\
\gamma &= k^3\{Du^3 + u\{C\sin\theta - D\cos\theta - \frac{E_i}{k}\{A(1-\cos\theta) - B\sin\theta\}\}\} \cdot
\end{aligned} \tag{9}$$

In Eq. (9), the parameter u is defined by $u \equiv \frac{p}{k}$ and $\vec{p_i}^2 = \vec{p_f}^2 \equiv k^2$. A, B, C, D are defined in terms of the Dirac matrices and have the following expressions:

$$\begin{aligned}
A &= i\bar{u}_f\gamma^0\Sigma_2 u_i, \quad \text{with} \quad \Sigma_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \\
B &= \bar{u}_f\gamma^0 u_i, \\
C &= \bar{u}_f\gamma^3 u_i, \\
D &= \bar{u}_f\gamma^1 u_i \cdot
\end{aligned} \tag{10}$$

Now, returning back to the integration given in Eq. (7), which we shall denote as I_S from now on, we change the φ integration into a contour integration over the unit circle $|z| = 1$, where $z = e^{i\varphi}$. Then, using the complex integration techniques, we obtain:

$$I_S = \frac{e^{i\theta}}{2ik}\int_0^\infty \frac{du}{u(1-u^2)}\oint_{|z|=1} dz\frac{c_0 + c_1 z + c_2 z^2}{(z^2 + 1 - 2az)(z^2 + e^{2i\theta} - 2aze^{i\theta})}, \tag{11}$$

where $a = \frac{u^2+1}{2u}$ and the coefficients c_0, c_1, c_2 are calculated in terms of the quantities given in Eq. (10) as:

$$\begin{aligned}
c_0 &= \{C + iD\}u^3 + u\{(C - iD)e^{i\theta} + \frac{E_i}{k}(B - iA + (B + iA)e^{i\theta}) \\
c_1 &= \frac{2E_i}{k}\{A\sin\theta - B\cos\theta\} - 2u^2\{\frac{E_i}{k}B - \frac{m}{k}B' + D\sin\theta + C(1+\cos\theta)\} \\
c_2 &= \{C - iD\}u^3 + u\{(C + iD)e^{-i\theta} + \frac{E_i}{k}(B + iA + (B - iA)e^{-i\theta}) \cdot
\end{aligned} \tag{12}$$

Carrying out the z-integration using the Cauchy theorem and changing the variable $u^2 = v$, we get:

$$\begin{aligned} I_S &= \frac{\pi}{k^2}\int_0^\infty dv\varepsilon(v-1) \\ &\times\{\frac{E_iB}{(v-e^{i\theta})(v-e^{-i\theta})} + \frac{(E_iA)\sin\theta + (E_iB)(1-\cos\theta)}{(v-1)(v-e^{i\theta})(v-e^{-i\theta})}\}. \end{aligned} \tag{13}$$

In Eq. (13), we make a change of variable $v = \frac{1}{w}$ in the $(1, \infty)$ interval, and we see that the first integral vanishes. Thus, the remaining integral can be rewritten as:

$$I_S = \frac{\pi T}{k^2}\int_0^\infty dv \frac{\varepsilon(v-1)}{(v-1)(v-e^{i\theta})(v-e^{-i\theta})}. \tag{14}$$

Here the profactor T is given as:

$$\begin{aligned} T &= E_iA\sin\theta + E_iB(1-\cos\theta) \\ &= \bar{u}_f E_i R u_i, \end{aligned} \tag{15}$$

where

$$R = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \quad \text{with} \quad \mathrm{r} = \begin{pmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{pmatrix}. \tag{16}$$

Using the definitions given in Eq. (10), one can show that T vanishes. This result completes our calculation showing that the Born approximation is consistent by the vanishing of the $O(\alpha^2)$ order term.

3. Conclusions and Discussion

The validity of the Born approximation for the relativistic spin-$\frac{1}{2}$ AB scattering problem was shown in [9] by demonstrating that it agrees with the corresponding term in the series expansion of the exact amplitude which is proportional to $\sin\pi\alpha$. However, demanding a complete check of the full consistency of the Born approximation, not only should one have agreement on $O(\alpha)$ terms, but should also demonstrate that the $O(\alpha^2)$ terms vanish. In this work, the consistency of the Born approximation for the relativistic massless spin-$\frac{1}{2}$ particles is checked in the framework of the covariant perturbation theory. It is shown that the $O(\alpha^2)$ contribution vanishes, thus proving the consistency of the Born approximation for the massless spin-$\frac{1}{2}$ problem.

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