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Successive Toroidal Compactifications of a Closed Bosonic Strings*

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Abstract

Successive toroidal compactifications of a closed bosonic string are studied and some Lie groups solutions are derived.

1. Introduction

Our present understanding of the observed fundamental interactions is encompassed, on the one hand, for the strong, weak and electromagnetic interactions by the standard mode and, on the other hand for, the gravitational interaction by Einstein's classical theory of general relativity which, however, can not be consistently quantized.

Although the success of some of the unified grauge theories (based on the point-like quantum fields concept), there are too many arbitrary parameters and some of the outstanding problems like the Higgs, spontaneous symmetry breaking mechanism, Kobayachi-Maskawa matrix etc... are still unsolved.

The discovery in the summer of 1984 by Green and Schwarz [1] of the unique anomaly free open superstring has once again spurred an enormous interest in string theories as candidates for unified quartum theories of all interactions and matter.

As opposed to point-like particles in ordinary field theories, the fundamental constituents of string theories are 1-dimensional objects. A single classical relativistic string

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can vibrate in an infinite set of normal modes, which, when quantized correspond to an infinite set of states with arbitrary high masses and spins.

These theories can be consistently quantized for one specific dimension of space-time only. This critical dimension is 26 for the bosonic string (open or closed) and 10 for the superstring [2], [3]. However, to keep contact with the real world, the extra space-time dimensions have to be compactified. It turns out that there are too many ways to do such a procedure and consequently, the four-dimensional low energy physics is not unique [4]-[12]. Thus, there is still no clear answer to the important problem of compactification and how constact can be made with a realistic phenomenology.

In this paper, and as a toy model, we consider a closed bosonic string and study the effect of successive toroidal compactifications.

In section 2, we present general solutions resulted from various types of an even dimensional tori compactifications. In section 3, we display our results and draw our conclusions.

2. Formalism

The Nambu-Goto action of a closed bosonic string is given by [2], [3];

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma[(x.x')^2 - \dot{x}^2.x'^2] \tag{1}$$

with :

$$x^\mu(\sigma+\pi,\tau) = x^\mu(\sigma,\tau), \tag{2}$$

and σ, τ are the dimensionless world-sheet parameters. Here σ' is the string scale and x'^μ (resp. $\dot{x}^\mu$) means $\frac{\partial x^\mu}{\partial\sigma}$ (resp. $\frac{\partial x^\mu}{\partial\tau}$). The general solution of the equation of motion (in the orthonormal gauge)

$$\ddot{x}^\mu - x''^\mu = 0 \tag{3}$$

which satisfies the boundary condition (2) is :

$$x^\mu(\sigma,\tau) = q^\mu + \alpha' p^\mu + \frac{i}{2}\sum_{n=0}\frac{1}{n}[\alpha_n^\mu \exp -2in(\tau-\sigma) + \tilde{\alpha}_n^\mu \exp -2in(\tau+\sigma)], \tag{4}$$

where q^μ and p^μ are the string center of mass coordinates and the momentum, respectively.

After quantization, the critical dimension is fixed to $D = 26$ and the physical states $|\psi\rangle_{\rm phys}$ are subject to the Virasoro conditions:

$$\begin{aligned} L_n|\psi\rangle_{\rm phys} &= \tilde{L}_n|\psi\rangle_{\rm phys} \qquad n \geq 1 \\ (L_0 - \tilde{L}_0)|\psi\rangle_{\rm phys} &= 0 \\ (L_0 + \tilde{L}_0 - \alpha(0))|\psi\rangle_{\rm phys} &= 0, \end{aligned} \tag{5}$$

where the Virosoro generators L_n and $\tilde{L}_n$ are given by:

$$L_n = \frac{1}{4\sigma'} \sum_{m=-\infty}^{+\infty} \alpha_{n-m}\alpha_m \tag{6}$$
$$\tilde{L}_n = \frac{1}{4\sigma'} \sum_{m=-\infty}^{+\infty} \tilde{\alpha}_{n-m}\tilde{\alpha}_m$$

(here $\sigma(0) = 2$). To get the mass spectrum, one has to apply the following mass operator M^2

$$M^2 = 4[N + \tilde{N} - \alpha(0)] \tag{7}$$

on the physical states $|\psi\rangle_{\text{phys}}$ (we have taken $\frac{1}{2\alpha'} = 1$) with:

$$N|\psi\rangle_{\text{phys}} = \tilde{N}|\psi\rangle_{\text{phys}},$$

where

$$N = \sum_{m=-\infty}^{+\infty} \alpha^{\mu}_{-n}\alpha_{n\mu} \tag{8}$$
$$\tilde{N} = \sum_{m=-\infty}^{+\infty} \tilde{\alpha}^{\mu}_{-n}\tilde{\alpha}_{n\mu}.$$

Now, our compactification program consists of starting from the critical dimension $D = 26$ and then truncating the extra dimensions successively through a various number of tori compactifications.

We remained the reader that an r-dimensional torus T^r is defined as the set R/Γ, where Γ is an r-dimensional lattice generated by a basis $\{\overrightarrow{e_\alpha}, \alpha = \overrightarrow{1, r}\}$. One can be also define a dual lattice Γ^* as

$$\Gamma^* = \{\overrightarrow{\beta} \in R^r / \forall \overrightarrow{\gamma} \in \Gamma, \overrightarrow{\beta} \cdot \overrightarrow{\gamma} \text{ is an integer}\} \tag{9}$$

with a dual basis $\overrightarrow{e^*_\alpha}, \alpha = \overrightarrow{1, r}\}$ such that

$$\overrightarrow{e}^*_\alpha \cdot \overrightarrow{e_\beta} = \delta_{\alpha\beta}. \tag{10}$$

2.1. Method $N = 1$

The first method consists of taking the left and right movers modes as mixed. Thus, the compactified coordinates x can be written as:

$$x^i(\alpha,\tau) = q^i + \alpha' p^i + \frac{i}{2}\sum_{n=0}\frac{1}{n}[\alpha_n^i \exp -2in(\tau-\sigma) + \tilde{\alpha}_n^i \exp -2in(\tau+\sigma)]. \qquad (11)$$

For the compactified coordinates $x^I(I = \overrightarrow{1,r})$ on an r-dimensional torus, one has to identify the points under the translation by $2\pi R_\alpha$ in the $\overrightarrow{e_\alpha}$ direction. Thus:

$$x^I \simeq x^I + \frac{\pi}{\sqrt{2}}\sum_{\alpha=1}^{r} n_\alpha \cdot R_\alpha \cdot e_\alpha^I \qquad (n \in Z) \qquad (12)$$

where r (resp. R_α) is the torus dimension (resp. radius in the α direction) and therefore one can write:

$$x^I(\sigma,\tau) = q^I + \alpha' p^I + 2\ell^I + \frac{i}{2}\sum_{n=0}\frac{1}{n}[\alpha_n^I \exp -2in(\tau-\sigma) + \tilde{\alpha}_n^I \exp -2in(\tau+\sigma)] \qquad (13)$$

with:

$$p^I = \sum_{\alpha=1}^{r}\frac{m_\alpha}{R_{\alpha=1}}\ \frac{e_\alpha^{I*}}{||\overline{e}_\alpha^*||}, \qquad (m_\alpha \in Z) \qquad (14)$$

and ℓ are the winding numbers which have the following expression:

$$\ell^I = \sum_{\beta=1}^{r} m_\beta R_\beta \frac{e_\beta^I}{||e_\beta||}. \qquad (15)$$

Now, after "n" compactification, the mass operator M takes the form:

$$M^2 = 4[N + \tilde{N} - 2\sum_{k=1}^{r}\sum_{I=1}^{r_k}(\frac{(p^I)^2}{4} + \ell^{I^2})] \qquad (16)$$

(here r_k is the dimension of the k^{th} torus ($\sum_{p=1}^{n} r_p = 22$)). With:

$$N|\psi\rangle_{\text{phys}} = (\tilde{N} + \sum_{k=1}^{n}\sum_{I=1}^{r_k}\ell^I p^I)|\psi\rangle_{\text{phys}}, \qquad (17)$$

where

$$N = \sum_{m=1}^{+\infty}\sum_{k=1}^{n}\sum_{I=1}^{r_k}\alpha_{-m}^{i}\alpha_{m}^{i} + \sum_{m=1}^{+\infty}\sum_{k=1}^{n}\sum_{I=1}^{r_k}\alpha_{-m}^{I}\alpha_{m}^{I} \tag{18}$$

$$\tilde{N} = \sum_{m=1}^{+\infty}\sum_{k=1}^{n}\sum_{I=1}^{r_k}\tilde{\alpha}_{-m}^{i}\tilde{\alpha}_{m}^{i} + \sum_{m=1}^{+\infty}\sum_{k=1}^{n}\sum_{I=1}^{r_k}\tilde{\alpha}_{-m}^{I}\tilde{\alpha}_{m}^{I}.$$

With a self dual lattice and orthonormal basis, eqs (16) and (17) become:

$$M^2 = (N + \tilde{N} - 2) + \sum_{k=1}^{n}\sum_{I=1}^{r_k} + (\frac{m_\alpha^2}{R_\alpha^2} + 4n_\alpha^2 R_\alpha^2) \tag{19}$$

$$N|\psi\rangle_{\text{phys}} = (\tilde{N} + \sum_{k=1}^{n}\sum_{I=1}^{r_k} n_\alpha m_\alpha)|\psi\rangle_{\text{phys}}.$$

It is to be noted that one can characterize the quartum physical states $|\psi\rangle_{\text{phys}}$ by the quantum numbers n_α and m_α. Now, it is easy to show, that for $R_\alpha = \frac{1}{\sqrt{2}}(\forall\alpha = \overline{1, r_p}; p = \overline{1, n})$, the number of the vectorial physical massless states and the quantum numbers n_α and m_α is (see APPENDIX A):

$$\Omega = 4\sum_{p=1}^{n}(r_p^2 + 11) \tag{20}$$

and

$$\Sigma = 2\sum_{p=1}^{n} r_p = 44, \tag{21}$$

respectively. However, for at least one $R_\alpha = \frac{1}{\sqrt{2}}$, the number of the physical vectorial massless states becomes 44.

2.2. Method $N = 2$

In this method the left and the right movers of the closed string $x^I(\sigma - \tau)$ and $x^I(\sigma + \tau)$, respectively, are treated independently. In this case, the compactified coordinates can be written as:

$$x^I(\sigma - \tau) = q^I + p^I(\tau - \sigma) + \frac{i}{2}\sum_{n=0}\frac{1}{n}\alpha_n^I \exp -2in(\tau - \sigma) \tag{22}$$

$$x^I(\sigma + \tau) = \tilde{q}^I + \tilde{p}^I(\tau + \sigma) + \frac{i}{2}\sum_{n=0}\frac{1}{n}\tilde{\alpha}_n^I \exp -2in(\tau + \sigma),$$

where the string center of mass momentum p^I and $\tilde{p}^I$ are given on the dual lattice Γ^* by:

$$\begin{aligned} p^I &= \sum_{\alpha=1}^{r} \frac{m_\alpha}{R_\alpha} \frac{e_\alpha^{I*}}{||e_\alpha^*||} \\ \tilde{p}^I &= \sum_{\alpha=1}^{r} \frac{\tilde{m}_\alpha}{R_\alpha} \frac{e_\alpha^{I*}}{||e_\alpha^*||}. \end{aligned} \qquad (m_\alpha \tilde{m}_\alpha \in Z) \tag{23}$$

It is to be noted that in this case, the winding numbers ℓ^I and $\tilde{\ell}^I$ are related to the momenta p^I and $\tilde{p}^I$ by the relations:

$$\begin{aligned} \ell^I &= -\tfrac{1}{2} p^I \\ \tilde{\ell}^I &= \tfrac{1}{2} \tilde{p}^I. \end{aligned} \tag{24}$$

This means that the lattice Γ and its dual Γ^* have a non zero intersection. Now, the mass shell condition (16) leads to the relation

$$M^2 = -p^{i2} = 2(N + \tilde{N} - 4) + \sum_{k=1}^{n} \sum_{\beta,\alpha=1}^{r_k} [\frac{g^*_{\alpha\beta}}{R_\alpha R_\beta}(m_\alpha m_\beta + \tilde{m}_\alpha \tilde{m}_\beta)]. \tag{25}$$

where $g^*_{\alpha\beta}$ is the dual lattice metric. Moreover, the Varisoro condition $(2-5-b)$ implies that:

$$(N + \frac{1}{4}\sum_{k=1}^{n} \sum_{\beta,\alpha=1}^{r_k} \frac{g^*_{\alpha\beta}}{R_\alpha R_\beta} m_\alpha m_\beta)\psi\rangle\rangle_{phys} = (\tilde{N} + \frac{1}{4}\sum_{k=1}^{n} \sum_{\beta,\alpha=1}^{r_k} \frac{g^*_{\alpha\beta}}{R_\alpha R_\beta} \tilde{m}_\alpha \tilde{m}_\beta)|\phi\rangle_{phys}. \tag{26}$$

It is important to mention that, if $R_\alpha = \frac{1}{\sqrt{2}}$, the massless vectorial states belong to the adjoint representation of the tensorial product $G \otimes G$, where G is the simply laced Lie group of rank $r = 22$ and with a Cartan matrix $g_{\alpha\beta}$. Now, if the lattice Γ is even and integer, i.e.

$$\forall \overrightarrow{\beta}, \overrightarrow{\gamma} \in \Gamma \longrightarrow \overrightarrow{\beta} \cdot \overrightarrow{\gamma} \qquad \text{is an integer;}$$

$$\forall \overrightarrow{\gamma} \in \Gamma \longrightarrow \overrightarrow{\gamma}^2 \qquad \text{is integer and even;}$$

the momenta p^I and $\tilde{p}^I$ are identified with the weight vectors of the Lie group G. Now, if we characterize the vectorial physical states by the quartum numbers m_α and $\tilde{m}_\alpha$, we can show that for $R_\alpha^2 = R^2 =$ an integer or half integer the number Ω of these independent states is [see APPENDIX B]

$$\Omega_2 = 44 + \sum_{p=1}^{n} \frac{2^{s_{p+1}}}{(r_p - S_p)! Q_1! Q_2! \dots Q_{t_p}!} \tag{27}$$

(n is the number of successive compactifications) where, for the p^{th} compactification, r_p, S_p and Q_{t_p} are the dimension of the compactified space, the number of the non zero quantum numbers (m_α and n_α) and the degeneracy of the t_p^{th} quantum number respectively. However, if at least one of the R is not an integer of half an integer, the number, of the physical states becomes $\Omega_2' = 44$.

3. Results and Conclusions

To get an idea and keep our results transparent, we have considered compactifications on an even dimensional tori. Tables 1 and 2 display various types of compactifications and the rank and order of the resulted Lie groups with both methods 1 and 2 with $R = \frac{1}{\sqrt{2}}$ and 1, respectively. It is important to notice that the results depend on:

Table 1. Display the rank and order of the Lie groups coming from various types of Tori compactifications with the use of the first method and $R = \frac{1}{\sqrt{2}}$.

Type of compactification	**rank**	**order**
T^{22}	44	234212
$T^2 \otimes T^{20}$	44	174882
$T^4 \otimes T^{18}$	44	223644
$T^6 \otimes T^{16}$	44	123868
$T^8 \otimes T^{14}$	44	42172
$T^{10} \otimes T^{12}$	44	29116
$T^2 \otimes T^2 \otimes T^{18}$	44	115046
$T^2 \otimes T^4 \otimes T^{16}$	44	79758
$T^2 \otimes T^6 \otimes T^{14}$	44	47806
$T^2 \otimes T^8 \otimes T^{12}$	44	29630
$T^2 \otimes T^{10} \otimes T^{10}$	44	23742
$T^4 \otimes T^4 \otimes T^{14}$	44	46998
$T^4 \otimes T^6 \otimes T^{12}$	44	23254
$T^4 \otimes T^8 \otimes T^{10}$	44	15254
$T^6 \otimes T^6 \otimes T^{10}$	44	15126
$T^6 \otimes T^8 \otimes T^8$	44	12758
$T^2 \otimes T^4 \otimes T^4 \otimes T^{12}$	44	29408
$T^2 \otimes T^4 \otimes T^6 \otimes T^{10}$	44	12512
$T^2 \otimes T^4 \otimes T^8 \otimes T^8$	44	12832
$T^2 \otimes T^2 \otimes T^2 \otimes T^{16}$	44	89110
$T^2 \otimes T^2 \otimes T^4 \otimes T^{14}$	44	54640
$T^2 \otimes T^2 \otimes T^6 \otimes T^{12}$	44	32112
$T^2 \otimes T^2 \otimes T^8 \otimes T^{10}$	44	21400
$T^2 \otimes T^6 \otimes T^6 \otimes T^8$	44	11698
$T^4 \otimes T^4 \otimes T^4 \otimes T^{10}$	44	16820
$T^4 \otimes T^4 \otimes T^6 \otimes T^8$	44	11124
$T^4 \otimes T^6 \otimes T^6 \otimes T^6$	44	7732
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^{14}$	44	64602

Table 1. Continue

Type of compactification	**rank**	**order**
$T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^{12}$	44	38898
$T^2 \otimes T^2 \otimes T^2 \otimes T^6 \otimes T^{10}$	44	23898
$T^2 \otimes T^2 \otimes T^2 \otimes T^8 \otimes T^8$	44	19074
$T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^{10}$	44	21706
$T^2 \otimes T^2 \otimes T^4 \otimes T^6 \otimes T^8$	44	13970
$T^2 \otimes T^2 \otimes T^6 \otimes T^6 \otimes T^6$	44	11066
$T^2 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^8$	44	12530
$T^2 \otimes T^4 \otimes T^4 \otimes T^6 \otimes T^6$	44	9674
$T^4 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^6$	44	8250
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^{12}$	44	45424
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^{10}$	44	26866
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^6 \otimes T^8$	44	18034
$T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^8$	44	24612
$T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^6 \otimes T^6$	44	612916
$T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^6$	44	15990
$T^2 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^4$	44	9464
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^{10}$	44	30662
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^8$	44	18590
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^6$	44	12598
$T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^4$	44	10510
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^8$	44	25910
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^6$	44	19616
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^4$	44	16458
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^6$	44	23976
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4$	44	21848
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4$	44	28530
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2$	44	36180

a) The choice of the method:

In fact, it is clear from tables 1 and 2 that for the same type of compactification, the resulted Lie groups obtained with the first method are totally different from the second one. For example, a compactification on T^{22} gives with the first method, the folowing possible Lie groups : $SO(5)\otimes SO(60)\otimes U(12)$; $SO(58)\otimes SO(22)\otimes U(4)$; $SO(14)\otimes SO(14)\otimes SO(61)$; $SO(56)\otimes SO(5)\otimes SO(29)$; $SO(63)\otimes U(8)\otimes U(5)$; $SO(44)\otimes SO(44)$; $SO(58)\otimes SO(16)\otimes E7$. However with the second method one gets: $SO(51)\otimes SO(36)\otimes U(1)$; $SO(36)\otimes SO(3)\otimes SO(51)$; $SO(45)\otimes SO(45)$. As a second example, the ten successive tori compactifications $T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4$ lead to no solutions for the first method and $SO(11) \otimes SO(58) \otimes U(12)$; $SO(47) \otimes SO(40) \otimes U(1)$; $SO(57) \otimes SO(24) \otimes U(4)$; $SO(59) \otimes SU(3) \otimes SO(14)$; $SO(3) \otimes SO(37) \otimes SO(51)$; $SO(58) \otimes SO(11) \otimes U(5)$; $SO(48) \otimes SO(36) \otimes G2$; $SO(58) \otimes SU(5) \otimes U(11)$ for the second one.

Table 2. The same as Table 1 but with the use of the second method and $R = 1$

Type of compactification	**rank**	**order**
T^{22}	44	9624428
$T^2 \otimes T^{20}$	44	6168104
$T^4 \otimes T^{18}$	44	6154058
$T^6 \otimes T^{16}$	44	1405082
$T^8 \otimes T^{14}$	44	568380
$T^{10} \otimes T^{12}$	44	243836
$T^2 \otimes T^2 \otimes T^{18}$	44	3841396
$T^2 \otimes T^4 \otimes T^{16}$	44	1821254
$T^2 \otimes T^6 \otimes T^{14}$	44	656222
$T^2 \otimes T^8 \otimes T^{12}$	44	317942
$T^2 \otimes T^{10} \otimes T^{10}$	44	189190
$T^4 \otimes T^4 \otimes T^{14}$	44	782618
$T^4 \otimes T^6 \otimes T^{12}$	44	305290
$T^4 \otimes T^8 \otimes T^{10}$	44	122554
$T^6 \otimes T^6 \otimes T^{10}$	44	111282
$T^6 \otimes T^8 \otimes T^8$	44	127674
$T^2 \otimes T^4 \otimes T^4 \otimes T^{12}$	44	425734
$T^2 \otimes T^4 \otimes T^6 \otimes T^{10}$	44	161392
$T^2 \otimes T^4 \otimes T^8 \otimes T^8$	44	91560
$T^2 \otimes T^2 \otimes T^2 \otimes T^{16}$	44	1283546
$T^2 \otimes T^2 \otimes T^4 \otimes T^{14}$	44	1041106
$T^2 \otimes T^2 \otimes T^6 \otimes T^{12}$	44	426592
$T^2 \otimes T^2 \otimes T^8 \otimes T^{10}$	44	186860
$T^2 \otimes T^6 \otimes T^6 \otimes T^8$	44	66458
$T^4 \otimes T^4 \otimes T^4 \otimes T^{10}$	44	156310
$T^4 \otimes T^4 \otimes T^6 \otimes T^8$	44	60072
$T^4 \otimes T^6 \otimes T^6 \otimes T^6$	44	37170
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^{14}$	44	1356836
$T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^{12}$	44	572866
$T^2 \otimes T^2 \otimes T^2 \otimes T^6 \otimes T^{10}$	44	228514
$T^2 \otimes T^2 \otimes T^2 \otimes T^8 \otimes T^8$	44	134170
$T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^{10}$	44	224942

Table 2. Continue

Type of compactification	**rank**	**order**
$T^2 \otimes T^2 \otimes T^4 \otimes T^6 \otimes T^8$	44	93598
$T^2 \otimes T^2 \otimes T^6 \otimes T^6 \otimes T^6$	44	51502
$T^2 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^8$	44	84654
$T^2 \otimes T^4 \otimes T^4 \otimes T^6 \otimes T^6$	44	9674
$T^4 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^6$	44	33934
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^{12}$	44	790064
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^{10}$	44	313838
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^6 \otimes T^8$	44	170800
$T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^8$	44	153804
$T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^6 \otimes T^6$	44	76606
$T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^6$	44	6226
$T^2 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^4$	44	49194
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^{10}$	44	430974
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^8$	44	178496
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^6$	44	89848
$T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^4 \otimes T^4$	44	68048
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^8$	44	253958
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^6$	44	130820
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4 \otimes T^4$	44	97954
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^6$	44	187536
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4 \otimes T^4$	44	141310
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^4$	44	198350
$T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2 \otimes T^2$	44	277918

b) Type and number of compactifications:

Each type and number of successive compactifications gives different results. In fact, the type $T^2 \otimes T^{20}$ (for eample) leads to the following Lie groups: $SO(12) \otimes SO(28) \otimes SO(48)$; $SO(38) \otimes SO(42) \otimes U(4)$; $SO(8) \otimes SO(33) \otimes SO(49)$; $SO(55) \otimes U(2) \otimes U(15)$; $SO(48) \otimes SO(28) \otimes E6$; $SO(48) \otimes SO(26) \otimes E7$; However, the type $T^2 \otimes T^2 \otimes T^6 \otimes T^{12}$ gives $SO(20) \otimes U(15) \otimes U(19)$; $\otimes SO(13) \otimes U(17) \otimes U(21)$; $SO(25) \otimes U(14) \otimes U(18)$.

c) Tori radius:

The results of the successive compactifications depend strongly on the choice of the radius of the compactified tori. For example the first method gives for $R = \frac{1}{\sqrt{2}}(\forall \alpha = \overline{1, r}; k = \overline{1, n})$, a number of $44 + 2\sum_{k=1}^{n} r_k^2$ vectorial physical states which can form the irreducible representation of a Lie group. However, for at least $R = \frac{1}{\sqrt{2}}$, this number is reduced to 44 and leads to different Lie groups solutions.

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Appendix A

The possible physical vectorial states are: $\alpha^i_{-t}|0\rangle, \tilde{\alpha}^i_{-1}|0\rangle, \alpha^i_{-t}\prod_{s=1}^{q}\tilde{\alpha}^{I_s}_{-u_s}|0\rangle$, $\tilde{a}^i_{-t}\prod_{s=1}^{q}\alpha^{I_s}_{-u_s}|0\rangle, \tilde{\alpha}^i_{-t}\prod_{s=1}^{q}\tilde{\alpha}^{I_s}_{-u_s}|0\rangle$ and $\alpha^i_{-t}\prod_{s=1}^{q}\alpha^{I_s}_{-u_s}|0\rangle$ with t, u and $q \in N^*$. For the states $|\psi\rangle_{\text{phys}}$ of the form $\alpha^i_{-t}|0\rangle$ and $\tilde{\alpha}^i_{-t}|0\rangle$ and by imposing

$$M^2|\psi\rangle_{\text{phys}} = 0 \tag{A-1}$$

and

$$N|\psi\rangle_{\text{phys}} = (\tilde{N}\sum_{p=1}^{n}\sum_{\alpha=1} n_\alpha m_\alpha)|\psi\rangle_{\text{phys}} \tag{A-2}$$

one gets

$$\sum_{p=1}^{n}\sum_{\alpha=1}^{r_p}(\frac{m_\alpha^2}{4R_\alpha^2} + n_\alpha^2 R_\alpha^2) = 2 - t \tag{A-3}$$

and

$$\sum_{p=1}^{n}\sum_{\alpha=1}^{r_p} n_\alpha m_\alpha = \pm t,$$

where n denotes the number of tori compactifications ($+$ and $-$ signs are for the case $\alpha^i_{-t}|0\rangle$ and $\tilde{\alpha}^i_{-t}|0\rangle$ respectively). This implies that $t = 1$ and $R = (2)^{-1/2}$ and one of the n_α and m_α are equal to ± 1 (For the others, $n_\alpha = m_\beta$ if $\alpha \# \beta$). Thus, the $4\sum_{p=1}^{n} r_p^2$ states can be written as:

$|1, 0, \ldots, 0; 1, 0, \ldots, 0\rangle, |0, 1, 0, \ldots, 0; 0, 1, 0, \ldots, 0\rangle, \ldots,$
$|-1, 0, \ldots, 0; -1, 0, \ldots, 0\rangle|0, -1, 0, \ldots, 0; 0, -1, 0 \ldots, 0\rangle$
$|1, 0, \ldots, 0; -1, 0, \ldots, 0\rangle, |0, 1, 0, \ldots, 0; 0, -1, 0, \ldots, 0\rangle, \ldots,$
and
$1 - 1, 0 \ldots, 0; 1, 0, \ldots, 0\rangle, |0, -1, 0, \ldots, 0; , 1, 0 \ldots, 0\rangle, \ldots, .$

It is worth mentioning that the states of the form:

$$\tilde{\alpha}^i_{-t}\prod_{s=1}^{q}\tilde{\alpha}^{I_s}_{-u_s}|0\rangle \text{ and } \alpha^i_{-t}\prod_{s=1}^{q}\alpha^{I_s}_{-u_s}|0\rangle.$$

used with eq. $(A-1)$ and $(A-2)$ can e easily shown to be equivalent to the states $\alpha^i_{-t}|0\rangle$ and $\tilde{\alpha}^i_{-t}|0\rangle$, respectively. For the states of the form $\alpha^i_{-t}\prod_{s=1}^{q}\tilde{\alpha}^{I_s}_{-u_s}|0\rangle; \tilde{\alpha}^i_{-t}\prod_{s=1}^{q}\alpha^{I_s}_{-u_s}|0\rangle$, the conditions $(A-2)$ and $(A-3)$ lead to:

$$2-(t+u_1+u_2+\cdots+u_q)=\sum_{p=1}^{n}\sum_{\alpha=1}^{r_p}(\frac{m_\alpha^2}{4R_\alpha^2}+n_\alpha^2R_\alpha^2)$$

and

$$\sum_{p=1}^{n}\sum_{\alpha=1}^{r_p}n_\alpha m_\alpha=t-(u_1+u_2+\cdots+u_q),$$

which implies that $n_\alpha = m_\alpha = 0(\alpha = \overline{1,r})$. Thus, the number of the physical states is $2\sum_{p=1}^{n} r_p = 44$.

Appendix B

The number of massless vectorial states of the form $\alpha^i_{-t}|0\rangle, \tilde{\alpha}^i_{-t}|0\rangle$ with $t \in N$ can be determined by solving the equations

$$\begin{aligned} m_\alpha &= 0 \qquad \sum_{p=1}^{n}\sum_{\alpha=1}^{r_p} \frac{m_\alpha^2}{R_\alpha^2} = 2 \\ \tilde{m}_\alpha &= 0 \qquad \sum_{p=1}^{n}\sum_{\alpha=1}^{r_p} \frac{\tilde{m}_\alpha^2}{R_\alpha^2} = 2, \end{aligned} \tag{B-1}$$

respectively. Notice that in both cases the solution is the same. Setting $R_\alpha = R(\forall\alpha = \overline{1,r}; p = \overline{1,n})$ we obtain:

$$\sum_{p=1}^{n}\sum_{\alpha=1}^{r_p} m_\alpha^2 = 2R^2 = \sum_{p=1}^{n}\sum_{\alpha=1}^{r_p} \tilde{m}_\alpha^2. \tag{B-2}$$

Now, it is obvious that if R^2 is not an integer or half an integer, eqs. $(B-1)$ and $(B-2)$ have no solutions. In what follows we denote by s the number of the non zero $\tilde{m}_\alpha$'s. As an example, for $R = (2)^{3/2}$, eq. $(B-2)$ becomes

$$\sum_{p=1}^{n}\sum_{\alpha=1}^{r_p} \tilde{m}_\alpha^2 = 16$$

which can be written as:
a) 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1=16
b) 4+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1=16
c) 4+ 4+ 1+ 1+ 1+ 1+ 1+ 1+ 1+ 1=16
d) 4+ 4+ 4+ 1+ 1+ 1+ 1=16
e) 4+ 4+ 4+ 4=16
f) 9+ 1+ 1+ 1+ 1+ 1+ 1+ 1=16
g) 9+ 4+ 1+ 1+ 1=16
In this case, the values of s are respectively 16, 13, 10, 7, 4, 8, 5.

Now, if $R = 1$ (case of our interest) one gets:

$$\sum_{p=1}^{n}\sum_{\alpha=1}^{r_p} \tilde{m}_\alpha^2 = 2$$

which implies that $|\tilde{m}_\alpha| = 1$. So, the degeneracy s of $\tilde{m}_\alpha$ is equal to 2. Thus the number Ω of all possible physical states of the form $|11, 1, 0, \ldots, 0\rangle$, $|11, 0, 1, \ldots, 0\rangle, \ldots$ etc is

$$\Omega = \frac{1}{2}\sum_{p=1}^{n} r_p(r_p - 1) = \sum_{p=1}^{n} \frac{r_p!}{2!(r_p - 1)} \tag{B-3}$$

This result can be found in an equivalent way by taking r number arranged in two and without repetation. Thus, the number of the different physical vectorial states Ω is:

$$\Omega = \sum_{p=1}^{n} \frac{2!C_{rp}^{2}}{2!} = \sum_{p=1}^{n} \frac{r_p!}{2!(r_p-1)!} \tag{B-4}$$

Now, taking into account the positiv and negative values of $\tilde{m}_\alpha$ amounts to multiplying the result by 2^2. Hence, the total number, of states Ω_{tot} is:

$$\Omega_{\text{tot}} = \sum_{p=1}^{n} 2^2 \frac{r_p!}{2!(r_p-2)!}. \tag{B-5}$$

Then, it is clear that for a given s, eq. $(B-6)$ can be generalized to

$$\Omega_{\text{tot}} = \sum_{p=1}^{n} 2^{s_p} \frac{r_p!}{(r_p-s_p)! \prod_{q=1}^{t_p} Q_q!}$$

where Q_q (resp. s_p) is the degeneracy of the q^{th} quartum number (resp. the number of the non zero quantum numbers $\tilde{m}_\alpha$), t_p is the number of the non identical quantum numbers among the s ones for the p^{th} compactification. The factor $\frac{r_p!}{(r_p-s_p)!}$ represents the number of the rearrangements of r_p by s_p numbers. i.e.

$$A_{r_p}^{s_p} = s_p! C_{r_p}^{s_p}.$$

However, if there are some identical non zero quantum numbers, one has to divided by the factor $\prod_{q=1}^{t_p} Q_q!$. Notice that the factor 2^{s_p} comes from the fact that $\tilde{m}_\alpha$ can take both positive and negative values.