Turk J Phy
25 (2001) , 79 – 85.

Multiparticle Equations for Scalar Particles

Valery Nikolaevich KAPSHAI
Department of Physics, Gomel Skorina University,
Sovetskaja str., 104, 246699, Gomel - BELARUS
Aleksandr Ivanovich FURSIN
Department of Physics, Gomel Technical University,
October Avenue, 48, 246746, Gomel - BELARUS

Received 27.01.1999

Abstract

Multiparticle equations for scalar particles are derived in the framework of the Lagrangian formalism of field theory as a consequence of the Dyson-Schwinger equations for the generating functional of Green's functions. The general form of the n-particle equation is determined. The formula for the kernel of the equation is obtained. It makes possible to use the functional methods for the investigation of the equation. It is proven that the equation is correct in the lower orders for two-particle Green's functions.

1. Introduction

Multiparticle equations for Green's functions are used in the description of the bound states of elementary particles and at the same time for the description of processes: bound state scattering, particle scattering on bound states etc. For the first time the two-particle equation has been derived by Bethe and Salpeter [1]. Every attempt to generalize the latter equation for the case of three or more particles leads to essential difficulties.

Nevertheless there is a possibility to describe multiparticle states in the framework of the Lagrangian formalism of field theory [2-3] using functional integration or path integration.

Model-independent derivation of the system of two particle Edwards-Bethe-Salpeter equations and the system of three-particle equations for three-linear scalar field is carried out using Legendre transformations of the generating functional for the Green's functions [4].

The two-particle fermion-antifermion equations have been studied using the Legendre transformations [5]. The general formula of the n-fermion equation has been derived and its kernel structure is investigated in [6]. The formula connecting the kernel of the n-particle equation with the 2(n+1)-point Green's function has been obtained.

In the present paper the n-particle equations for the system of scalar particles interacting with electromagnetic field are derived by the method of functional integration. These equations are obtained using the Dyson-Schwinger equations [7]. The kernels of the equations are expressed in terms of the functional derivatives of n-particle propagators. Calculation of the generating functional for scalar field is carried out taking into account the second order. The derived equation is correct in this order for two-particle Green's functions.

2. Multiparticle equations for scalar field

Let us consider now the Lagrangian of complex scalar fields describing charged spinless particles. In order to satisfy the condition of the local gauge invariance, it is necessary to introduce the electromagnetic field. The total Lagrangian, which is invariant under the gauge transformation and contains the complementary gauge fixing term, is defined by the formula

$$L_{eff} = -\phi^*(\Box + m^2)\phi + \frac{1}{2}A^\mu[g_{\mu\nu} + (\frac{1}{\alpha} - 1)\partial_\mu\partial_\nu]A^\nu - ie\phi^* \overleftrightarrow{\partial}_\mu \phi A^\mu + e^2 A_\mu A^\mu \phi^* \phi, \quad (1)$$

where we use the following notation:

$$f(x) \overleftrightarrow{\partial}_\mu g(x) = f(x)(\partial_\mu g(x)) - (\partial_\mu f(x))g(x). \quad (2)$$

By definition the propagator is the inverse operator to the quadratic term of the Lagrangian. Consequently the following equality holds:

$$L_{eff} = \phi^* S^{-1}\phi + \frac{1}{2}A^\mu D^{-1}_{\mu\nu}A^\nu - ie(\phi^* \overleftrightarrow{\partial}_\mu \phi)A^\mu + e^2 A_\mu A^\mu \phi^*\phi. \quad (3)$$

Let us introduce the sources of fields J_μ, K^*, K and define

$$L_s = J_\mu A^\mu + \phi^* K + K^*\phi. \quad (4)$$

The generating functional of the latter fields is defined as a functional integral:

$$\tilde{Z}[K^*, K, J] = N \int D\phi^* D\phi DA_\mu e^{i\int L_{tot} dx};$$
$$L_{tot} = L_{eff} + L_s. \quad (5)$$

Let us transform (5) to a form which is useful for functional derivation, using the properties of path integral. Then we have

$$\tilde{Z} = N_{exp}(i \int L_{int}(\frac{\delta}{i\delta K(x)}; \frac{\delta}{i\delta K^*(x)}; \frac{\delta}{i\delta J_\mu(x)}))Z, \quad (6)$$

where

$$Z = exp(\frac{i}{2}\int J^{\mu}(x)D_{\mu\nu}(x,y)J^{\nu}(y)dxdy) \times$$
$$\times\, exp(-i\int K^{*}(x)S(x,y)K(y)dxdy), \tag{7}$$

$$L_{int} = -ie(\phi^{*} \overset{\leftrightarrow}{\partial}_{\mu} \phi)A^{\mu} + e^{2}A_{\mu}A^{\mu}\phi^{*}\phi. \tag{8}$$

It is evident that one-particle free propagator is

$$S(x,y) = i\frac{\delta^{2}Z[0]}{\delta K(x)\delta K^{*}(y)}, \tag{9}$$

where zero in square brackets denotes that all sources of the fields must be equal to zero after the calculation of functional derivatives. The exact total one-particle propagator for scalar particles with respect to the Lagrangian of interaction is defined as follows:

$$\tilde{S}(x,y) = i\frac{\delta^{2}\tilde{Z}[0]}{\delta K(x)\delta K^{*}(y)}. \tag{10}$$

The exact n-particle propagator is defined by analogy

$$\tilde{S}_{n}(x_{1},\ldots,x_{n};y_{1},\ldots,y_{n}) = i^{n}\frac{\delta^{2n}\tilde{Z}[0]}{\delta K(x_{1})\ldots\delta K(x_{n})\delta K^{*}(y_{1})\ldots\delta K^{*}(y_{n})}. \tag{11}$$

The Dyson-Scwinger equations for generating functional of Green's functions follow the invariance of functional measure under the translation $\phi^{*} \to \phi^{*} + \delta\phi^{*}$.

In this way the field equation can be derived from the condition

$$\delta_{\phi^{*}}\tilde{Z} = 0, \tag{12}$$

$$\int D\phi^{*}D\phi DA_{\mu}\left(\frac{\delta I_{tot}}{\delta\phi^{*}(x)} + K(x)\right)exp(iI_{tot}) =$$
$$= \left(\frac{\delta I_{tot}}{\delta\phi^{*}(x)}(\frac{\delta}{i\delta K^{*}(x)};\frac{\delta}{i\delta K(x)};\frac{\delta}{i\delta J_{\mu}(x)}) + K(x)\right)\tilde{Z} = 0,$$
$$I_{tot} = \int L_{tot}dx. \tag{13}$$

As the functional derivative of the action function with respect to the field $\phi^{*}(x)$ is equal to

$$\frac{\delta I_{tot}}{\delta\phi^{*}(x)} = -(\Box + m^{2})\phi(x) - 2ie\partial_{\mu}\phi(x)A^{\mu}(x) - ie\phi(x)\partial_{\mu}A^{\mu}(x)$$
$$+ e^{2}A_{\mu}(x)A^{\mu}(x)\phi(x), \tag{14}$$

then (13) implies that:

$$S^{-1}(y_1,z)\frac{\delta\tilde{Z}[K^*,K,J]}{\delta K^*(z)}-\left(2e\frac{\delta}{\delta J_\mu(y_1)}\frac{\partial}{\partial y_1^\mu}+e\frac{\partial}{\partial y_1^\mu}\frac{\delta}{\delta J_\mu(y_1)}+\right.$$
$$\left.+e^2\frac{\delta}{\delta J_\mu(y_1)}\frac{\delta}{\delta J^\mu(y_1)}\right)\frac{\delta\tilde{Z}[K^*,K,J]}{\delta K^*(y_1)}+iK(y_1)\tilde{Z}[K^*,K,J]=0. \quad (15)$$

Here, and later repeating variables denote the integration over the whole spacetime.

Let us calculate the functional derivative of the last identity with respect to 2n-1 sources $K(x_1)\dots K(x_n)K^*(y_2)\dots K^*(y_n)$ at the point where all sources are equal to zero. Thus

$$S^{-1}(y_1,z)\frac{\delta^{2n}\tilde{Z}[0]}{\delta K(x_1)\dots\delta K(x_n)\delta K^*(z)\delta K^*(y_2)\dots\delta K^*(y_n)}-\left(2e\frac{\delta}{\delta J_\mu(y_1)}\frac{\partial}{\partial y_1^\mu}+\right.$$
$$\left.+e\frac{\partial}{\partial y_1^\mu}\frac{\delta}{\delta J_\mu(y_1)}+e^2\frac{\delta}{\delta J_\mu(y_1)}\frac{\delta}{\delta J^\mu(y_1)}\right)\frac{\delta^{2n}\tilde{Z}[0]}{\delta K(x_1)\dots\delta K(x_n)\delta K^*(y_1)\dots\delta K^*(y_n)}+$$
$$+i\sum_{i=1}^{n}\delta(x_i-y_1)\frac{\delta^{2(n-1)}\tilde{Z}[0]}{\delta K(x_1)\dots\delta K(x_{i-1})\delta K(x_{i+1})\dots\delta K(x_n)\delta K^*(y_2)\dots\delta K^*(y_n)}=0. \quad (16)$$

Using (11) we can write

$$S^{-1}(y_1,z)\frac{1}{i^n}\tilde{S}_n(x_1,\dots,x_n;z,y_2,\dots,y_n)=\left(2e\frac{\delta}{\delta J_\mu(y_1)}\frac{\partial}{\partial y_1^\mu}+\right.$$
$$\left.+e\frac{\partial}{\partial y_1^\mu}\frac{\delta}{\delta J_\mu(y_1)}+e^2\frac{\delta}{\delta J_\mu(y_1)}\frac{\delta}{\delta J^\mu(y_1)}\right)\frac{1}{i^n}\tilde{S}_n(x_1,\dots,x_n;y_1,y_2,\dots,y_n)-$$
$$-\frac{i}{i^{n-1}}\sum_{i=1}^{n}\delta(x_i-y_1)\tilde{S}_{2(n-1)}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n;y_2,\dots,y_n)=0. \quad (17)$$

After some transformations we have

$$\tilde{S}_n(x_1,\dots,x_n;z,y_2,\dots,y_n)=S(z,y_1)\left(2e\frac{\delta}{\delta J_\mu(y_1)}\frac{\partial}{\partial y_1^\mu}+\right.$$
$$\left.+e\frac{\partial}{\partial y_1^\mu}\frac{\delta}{\delta J_\mu(y_1)}+e^2\frac{\delta}{\delta J_\mu(y_1)}\frac{\delta}{\delta J^\mu(y_1)}\right)\tilde{S}_n(x_1,\dots,x_n;y_1,y_2,\dots,y_n)+$$
$$+\sum_{i=1}^{n}S(x_i-z)\tilde{S}_{2(n-1)}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n;y_2,\dots,y_n)=0. \quad (18)$$

The expression in the brackets in (18) is a functional operator and a kernel of this equation. It is denoted by $\hat{T}(y_1)$:

$$\hat{T}(y_1) = 2e\frac{\delta}{\delta J_\mu(y_1)}\frac{\partial}{\partial y_1^\mu} + \frac{\partial}{\partial y_1^\mu}e\frac{\delta}{\delta J_\mu(y_1)} + e^2\frac{\delta}{\delta J_\mu(y_1)}\frac{\delta}{\delta J^\mu(y_1)}. \tag{19}$$

With the help of (19) equation (18) can be written as following:

$$\tilde{S}_n = S\hat{T}\tilde{S}_n + \Sigma S\tilde{S}_{2(n-1)}. \tag{20}$$

3. The verification for the case of two scalar particles

The derived equation (20) can be verified for the case of two scalar particles. We have in this case

$$\tilde{S}_2(x_1, x_2; z, y_2) = S(z, y_1)T(y_1)\tilde{S}_2(x_1, x_2; y_1, y_2)+$$

$$S(x_1, z)\tilde{S}(x_2, y_2) + S(x_2, z)\tilde{S}(x_1, y_2). \tag{21}$$

In the zero order, the first term on the right hand side vanishes. Since $S_2(x_1, x_2; z, y_2) = S(x_1, z)S(x_2, y_2) + S(x_2, z)S(x_1, y_2)$, then (18) is correct. In the second order, equation (18) for two-particle propagator can be written as follows:

$$\frac{\delta^4 Z^{(2)}[0]}{\delta K(x_1)\delta K(x_2)\delta K^*(z)\delta K^*(y_2)} = S(z, y_1)\left(-2e\frac{\delta}{\delta J_\mu(y_1)}\frac{\partial}{\partial y_1^\mu}\times\right.$$

$$\times\frac{\delta^4 Z^{(1)}[0]}{\delta K(x_1)\delta K(x_2)\delta K^*(y_1)\delta K^*(y_2)} - e\frac{\partial}{\partial y_1^\mu}\frac{\delta}{\delta J_\mu(y_1)}\frac{\delta^4 Z^{(1)}[0]}{\delta K(x_1)\delta K(x_2)\delta K^*(y_1)\delta K^*(y_2)}$$

$$\left.-e^2\frac{\delta}{\delta J_\mu(y_1)}\frac{\delta}{\delta J^\mu(y_1)}\frac{\delta^4 Z[0]}{\delta K(x_1)\delta K(x_2)\delta K^*(y_1)\delta K^*(y_2)}\right) + iS(x_1, z)\frac{\delta^2 Z^{(2)}[0]}{\delta K(x_2)\delta K^*(y_2)}$$

$$+iS(x_2, z)\frac{\delta^2 Z^{(2)}[0]}{\delta K(x_1)\delta K^*(y_2)}. \tag{22}$$

Let us calculate the generating functional (5) in two lower orders of the perturbation theory with respect to the coupling constant e. In the first order we obtain

$$Z^{(1)} = eK^*(x)\left(S(x, z)\overset{\leftrightarrow}{\partial}{}^{\mu}_{z} S(z, y)\right)K(y)D_{\mu\nu}(z, t)J^\nu(t)Z, \tag{23}$$

where we use the notation (2). By analogy in the second order we have

$$
\begin{aligned}
Z^{(2)} &= ie^2 K^*(x)S(x,z)S(z,y)K(y)D^{\mu\rho}(z,t)J_\rho(t)D_{\mu\sigma}(z_1,t_1)J^\sigma(t_1)Z + \\
&+ e^2 K^*(x)\left(S(x,z)\overset{\leftrightarrow\mu}{\partial_z} S(z,t)\right)\left(\partial_t^\nu S(t,y)\right)K(y)D_{\mu\nu}(z,t)Z - \\
&- e^2 K^*(x)\left(S(x,z)\overset{\leftrightarrow\mu}{\partial_z}\partial_t^\nu S(z,t)\right)S(t,y)K(y)D_{\mu\nu}(z,t)Z - \\
&- e^2 K^*(x)\left(S(x,z)\overset{\leftrightarrow\mu}{\partial_z} S(z,y)\right)K(y)\times \\
&\times K^*(x_1)\left(S(x_1,t)\overset{\leftrightarrow\nu}{\partial_t} S(t,y_1)\right)K(y_1)D_{\mu\nu}(z,t)Z + \ldots. \qquad (24)
\end{aligned}
$$

To obtain the second order approximation for the generating functional, let us write only the terms giving nonzero contribution into (22). For the fourth derivative of (24) we have

$$
\begin{aligned}
&-\frac{\delta^4 Z^{(2)}[0]}{\delta K(x_1)\delta K(x_2)\delta K^*(z)\delta K^*(y_2)} = \\
&= ie^2 S(z,u)\left(\partial_u^\mu S(u,v)\overset{\leftrightarrow\nu}{\partial_v} S(x_1,v)\right)S(x_2,y_2)D_{\mu\nu}(u,v) - \\
&- ie^2\left(\partial_u^\mu S(z,v)\right)\left(S(u,v)\overset{\leftrightarrow\nu}{\partial_v} S(x_1,v)\right)S(x_2,y_2)D_{\mu\nu}(u,v) - \\
&- e^2 S(y_2,u)\left(\partial_u^\mu S(x_1,u)\right)\left(S(z,v)\overset{\leftrightarrow\nu}{\partial_v} S(x_2,v)\right)D_{\mu\nu}(u,v) + \\
&+ \{x_1\leftrightarrow x_2; z\leftrightarrow y_2; x_1\leftrightarrow x_2, z\leftrightarrow y_2\}. \qquad (25)
\end{aligned}
$$

By analogy, we obtain the terms of (22) containing second derivatives:

$$
\begin{aligned}
iS(x_1,z)\frac{\delta^2 Z^{(2)}[0]}{\delta K(x_2)\delta K^*(y_2)} + \{x_1\leftrightarrow x_2\} &= ie^2 S(x_1,z)S(y_2,u)\times \\
&\times\left(\partial_u^\mu S(u,v)\overset{\leftrightarrow\nu}{\partial_v} S(x_2,v)\right)D_{\mu\nu}(u,v) - \\
-ie^2 S(x_1,z)\left(\partial_u^\mu S(y_2,u)\right)\left(S(u,v)\overset{\leftrightarrow\nu}{\partial_v} S(x_2,v)\right)&D_{\mu\nu}(u,v) + \{x_1\leftrightarrow x_2\}. \qquad (26)
\end{aligned}
$$

For the fourth derivative of the functional $Z^{(1)}$, that is of (23), we have

$$
\begin{aligned}
&- S(z,y_1)K(y_1)\frac{\delta^4 Z^{(1)}[0]}{\delta K(x_1)\delta K(x_2)\delta K^*(y_1)\delta K^*(y_2)} = \\
&ie^2\left(S(z,y_1)\overset{\leftrightarrow\mu}{\partial}_{y_1} S(y_1,t)\right)\left(\partial^\nu_t S(x_1,t)\right) S(x_2,y_2)D_{\mu\nu}(y_1,t)+ \\
&+ie^2 S(z,y_1)\left(S(y_2,t)\overset{\leftrightarrow\nu}{\partial}_t S(x_1,t)\right)\left(\overset{\leftrightarrow\mu}{\partial}_{y_1} S(x_2,y_1)\right)D_{\mu\nu}(y_1,t)- \\
&-ie^2\left(S(z,y_1)\overset{\leftrightarrow\mu}{\partial}_{y_1}\partial^\nu_t S(y_1,t)\right)S(x_1,t)S(x_2,y_2)D_{\mu\nu}(y_1,t)+ \\
&\{x_1\leftrightarrow x_2; y_1\leftrightarrow y_2; x_1\leftrightarrow x_2, y_1\leftrightarrow y_2\}.
\end{aligned}
\tag{27}
$$

Substituting (25), (26), (27) into equation (22), collecting similar terms, renaming integration variables when it is necessary, we can see that the equation for the two-particle propagator is correct in the second order.

The main result of this paper is the derivation of equation (18) which permits to use functional methods for investigation of the kernel of the n-particle equation. We are planning to develop further this approach in a future publication.

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